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# Stable decompositions for some symmetric group characters arising in braid group cohomology

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A B S T R A C T

We prove that certain permutation characters for the symmetric group $\Sigma_n$ decompose in a manner that is independent of $n$ for large $n$. This result is a key ingredient in the recent work of T. Church and B. Farb, who obtain a "representation stability" theorem for the character of $\Sigma_n$ acting on the cohomology $\mathrm{H}^p(P_n,\mathbb{C})$ of the pure braid group $P_n$.



## 1. Introduction

Let $\Sigma_n$ denote the symmetric group on $n$ letters. For $\sigma \in \Sigma_n$, one can easily describe [3, 4.1.19] the centralizer $Z(\sigma) := C_{\Sigma_n}(\sigma)$. For certain linear characters $\psi$ of $Z(\sigma)$, the induced character $\mathrm{Ind}_{Z(\sigma)}^{\Sigma_n}\psi$ plays an important role in [4], where the authors study the $\Sigma_n$ action on the cohomology $\mathrm{H}^p(P_n,\mathbb{C})$ of the pure braid group $P_n$.

The easiest example is for $\sigma = (1,2)$ a two-cycle. Here the centralizer is a Young subgroup $\Sigma_2 \times \Sigma_{n-2}$. For the trivial character $\mathbb{C}$, Young's rule [3, 2.8.5] gives:

$$\mathrm{Ind}_{Z(\sigma)}^{\Sigma_n}\mathbb{C} \cong \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} \tag{1.1}$$

provided $n \geqslant 4$.

The decomposition in (1.1) is stable for $n \geqslant 4$. Recently we were asked by Benson Farb if this stability behavior generalizes to other centralizers and other characters. The purpose of this note is to demonstrate this is indeed the case by proving a more general result for arbitrary characters of subgroups $H \leqslant \Sigma_a$ induced up to $\Sigma_{a+b}$. Although the proof is straightforward, the result does not seem to have been noticed before.

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Our affirmative answer (Theorem 2.4) was used in the proof of Theorem 4.1 in [2], stated as Theorem 3.1 below.

## 2. Proof of the stability result

We work only over the complex numbers, and all the character theory we need can be found in [3], in particular the parametrization of irreducible $\mathbb{C}\Sigma_n$ characters by partitions of $n$.

**Definition 2.1.** For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s) \vdash n$, let $\tilde{\mu} = (\mu_1 + 1, \mu_2, \ldots, \mu_s) \vdash n + 1$.

For $\lambda \vdash d$ let $\chi^\lambda$ denote the irreducible character of $\Sigma_d$ corresponding to $\lambda$. Recall for $\lambda \vdash a$ and $\mu \vdash b$ the Littlewood–Richardson rule [3, 2.8.13]:

$$\operatorname{Ind}_{\Sigma_a \times \Sigma_b}^{\Sigma_{a+b}} \left(\chi^\lambda \boxtimes \chi^\mu\right) = \sum_{\rho \vdash a+b} c^\rho_{\lambda,\mu} \chi^\rho$$

where the *Littlewood–Richardson coefficient* $c^\rho_{\lambda,\mu}$ is the number of semistandard tableau of shape $\rho/\lambda$ and content $\mu$ which yield lattice permutations when we read their entries from right to left and downwards.

We use only the following special case, known as Pieri's Rule. To state this result, define a *horizontal strip* to be a skew shape $\mu/\lambda$ with no two boxes in the same column. Recall that $\chi^{(b)}$ is just the trivial $\Sigma_b$ character $\mathbb{C}$. We have:

**Proposition 2.2** *(Pieri's Rule). (See [5, 7.15.7].) Let $\lambda \vdash a$. Then*

$$\operatorname{Ind}_{\Sigma_a \times \Sigma_b}^{\Sigma_{a+b}} \left(\chi^\lambda \boxtimes \chi^{(b)}\right) = \sum_\mu \chi^\mu \tag{2.1}$$

*summed over all partitions $\mu$ such that $\mu/\lambda$ is a horizontal strip of size $b$.*

Next we have the following easy equality:

**Lemma 2.3.** *Suppose $\lambda \vdash a \leqslant b$ and $\mu \vdash a + b$. Then*

$$c^\mu_{\lambda,(b)} = c^{\tilde{\mu}}_{\lambda,(b+1)}.$$

**Proof.** Just use Proposition 2.2. The coefficient $c^\rho_{\lambda,(b+1)}$ is zero or one. It is one precisely when $\rho$ is obtained by adding a box to $b + 1$ distinct columns of the Young diagram of $\lambda$ (including empty columns of $\lambda$ corresponding to boxes added to the first row). Since $\lambda_1 \leqslant a \leqslant b$, any such $\rho$ must have at least one such box added to the first row. So removing a box from the end of the first row gives a bijection between the diagrams corresponding to $c^{\tilde{\mu}}_{\lambda,(b+1)}$ and those corresponding to $c^\mu_{\lambda,(b)}$. □

For $H \leqslant \Sigma_a$ we consider $H \times \Sigma_b$ as a subgroup of $\Sigma_{a+b}$ in the natural way, with $\Sigma_b$ acting on $\{a + 1, \ldots, a + b\}$. We have:

**Theorem 2.4.** *Suppose $H \leqslant \Sigma_a$ and $\psi$ is a character of $H$. Suppose further that $a \leqslant b$. Let*

$$\operatorname{Ind}_{H \times \Sigma_b}^{\Sigma_{a+b}} (\psi \boxtimes \mathbb{C}) = \sum_{\tau \vdash a+b} d_\tau \chi^\tau.$$

*Then*

$$\operatorname{Ind}_{H \times \Sigma_{b+1}}^{\Sigma_{a+b+1}} (\psi \boxtimes \mathbb{C}) = \sum_{\tau \vdash a+b} d_\tau \chi^{\tilde{\tau}}.$$

**Proof.** Just observe that

$$\operatorname{Ind}_{H\times\Sigma_b}^{\Sigma_{a+b}}(\psi\boxtimes\mathbb{C})=\operatorname{Ind}_{\Sigma_a\times\Sigma_b}^{\Sigma_{a+b}}\bigl(\operatorname{Ind}_{H\times\Sigma_b}^{\Sigma_a\times\Sigma_b}(\psi\boxtimes\mathbb{C})\bigr)=\operatorname{Ind}_{\Sigma_a\times\Sigma_b}^{\Sigma_{a+b}}\bigl(\operatorname{Ind}_{H}^{\Sigma_a}(\psi)\boxtimes\mathbb{C}\bigr)$$

so we can apply Lemma 2.3 to each constituent of the character $\operatorname{Ind}_H^{\Sigma_a}\psi$. □

## 3. Application to representation stability

We briefly discuss the application by Church and Farb of the stability Theorem 2.4. Let $P_n$ denote the pure braid group, the quotient of the full braid group $B_n$ by the symmetric group. Let $X_n$ denote the set of ordered $n$-tuples of distinct points in complex $n$-space $\mathbb{C}^n$, which is a hyperplane complement with fundamental group $P_n$. The action of $\Sigma_n$ on $X_n$ gives the cohomology groups $\mathrm{H}^i(P_n;\mathbb{Q})$ the structure of an $\Sigma_n$-module. Church and Farb proved:

**Theorem 3.1.** *(See [2, 4.1].) For each fixed $i\geqslant 0$, the sequence of $\Sigma_n$-representations $\{\mathrm{H}^i(P_n;\mathbb{Q})\}$ is uniformly representation stable.*

For details, including the definition of representation stability and the vast number of other settings where similar behavior arises, see the preprint [2].

## 4. Examples

In this section we give a few examples illustrating Theorem 2.4.

**Example 4.1.** Let $\sigma=(1,2)(3,4)$ so $C_{\Sigma_4}(\sigma)=\langle(1,2),(1,3)(2,4)\rangle\cong D_4$. Recall that $C_{\Sigma_n}(\sigma)\cong C_{\Sigma_4}(\sigma)\times\Sigma_{n-4}$. Let $\psi$ be the linear character having value 1 on $(1,2)$, $-1$ on $(1,3)(2,4)$, and 1 on $\Sigma_{n-4}$. Then an easy computation with Magma [1] gives (for $n\geqslant 7$):

$$\begin{aligned}
&\operatorname{Ind}_{Z(\sigma)}^{\Sigma_4}\psi=\chi^{(3,1)},\\
&\operatorname{Ind}_{Z(\sigma)}^{\Sigma_6}\psi=\chi^{(5,1)}+\chi^{(4,2)}+\chi^{(4,1,1)}+\chi^{(3,3)}+\chi^{(3,2,1)},\\
&\operatorname{Ind}_{Z(\sigma)}^{\Sigma_7}\psi=\chi^{(6,1)}+\chi^{(5,2)}+\chi^{(5,1,1)}+\chi^{(4,3)}+\chi^{(4,2,1)}+\chi^{(3,3,1)},\\
&\operatorname{Ind}_{Z(\sigma)}^{\Sigma_n}\psi=\chi^{(n-1,1)}+\chi^{(n-2,2)}+\chi^{(n-2,1,1)}+\chi^{(n-3,3)}+\chi^{(n-3,2,1)}+\chi^{(n-4,3,1)}.
\end{aligned}$$

**Remark 4.2.** If $\sigma\in\Sigma_a$ has no fixed points and $\psi$ is the trivial character, then the induced character does not actually stabilize until $a=b$. This can be shown by counting double cosets, or applying the Littlewood–Richardson rule. Specifically the $\mathbb{C}\Sigma_{a+b}$ permutation character includes a constituent $\chi^\mu$ where $\mu$ is not of the form $\tilde{\lambda}$ for some $\chi^\lambda$ in the $\mathbb{C}\Sigma_{a+b-1}$ permutation character. For nontrivial $\psi$ the stability may occur sooner. For instance in Example 4.1 above, the character multiplicities stabilized at $n=7$ while Theorem 2.4 only guarantees stability for $n\geqslant 8$.

**Example 4.3.** Let $\sigma=(1,2,3)$ and observe $C_{\Sigma_n}(\sigma)\cong\langle\sigma\rangle\times\Sigma_{n-3}$. Let $\psi$ be either linear character of $Z(\sigma)$ that is faithful on $\langle\sigma\rangle$ and trivial on $\Sigma_{n-3}$. Then we computed (for $n\geqslant 5$):

$$\begin{aligned}
&\operatorname{Ind}_{Z(\sigma)}^{\Sigma_4}\psi=\chi^{(3,1)}+\chi^{(2,2)}+\chi^{(2,1,1)},\\
&\operatorname{Ind}_{Z(\sigma)}^{\Sigma_5}\psi=\chi^{(4,1)}+\chi^{(3,2)}+\chi^{(3,1,1)}+\chi^{(2,2,1)},\\
&\operatorname{Ind}_{Z(\sigma)}^{\Sigma_n}\psi=\chi^{(n-1,1)}+\chi^{(n-2,2)}+\chi^{(n-2,1,1)}+\chi^{(n-3,2,1)}.
\end{aligned}$$

Finally we give a nonlinear example.

**Example 4.4.** For our final example let $\psi$ be the two-dimensional irreducible character for $D_8 \cong C_{\Sigma_4}((1,2)(3,4))$. Then we computed, for $n \geqslant 7$:

$$\begin{aligned}
\mathrm{Ind}_{Z(\sigma)}^{\Sigma_4}\psi &= \chi^{(3,1,1)} + \chi^{(2,1,1)},\\
\mathrm{Ind}_{Z(\sigma)}^{\Sigma_6}\psi &= \chi^{(5,1)} + \chi^{(3,3)} + \chi^{(4,2)} + 2\cdot\chi^{(4,1,1)} + \chi^{(3,1,1,1)} + 2\cdot\chi^{(3,2,1)} + \chi^{(2,2,1,1)},\\
\mathrm{Ind}_{Z(\sigma)}^{\Sigma_7}\psi &= \chi^{(6,1)} + \chi^{(4,3)} + \chi^{(5,2)} + 2\cdot\chi^{(5,1,1)} + \chi^{(4,1,1,1)} + 2\cdot\chi^{(4,2,1)}\\
&\quad + \chi^{(3,2,1,1)} + \chi^{(3,3,1)},\\
\mathrm{Ind}_{Z(\sigma)}^{\Sigma_n}\psi &= \chi^{(n-1,1)} + \chi^{(n-3,3)} + \chi^{(n-2,2)} + 2\cdot\chi^{(n-2,1,1)} + \chi^{(n-3,1,1,1)}\\
&\quad + 2\cdot\chi^{(n-3,2,1)} + \chi^{(n-4,2,1,1)} + \chi^{(n-4,3,1)}.
\end{aligned}$$

Left open is the obvious problem of computing these stable values.

**Problem 4.5.** Given $\sigma \in \Sigma_a$ and a linear character $\psi$ of $C_{\Sigma_a}(\sigma)$, compute the multiplicities in:

$$\mathrm{Ind}_{C_{\Sigma_a}(\sigma)\times\Sigma_a}^{\Sigma_{2a}}\psi.$$

As far as we know Problem 4.5 is unsolved even for $\psi$ the trivial character.

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