TOPOLOGY PROCEEDINGS

Volume 15, 1990

Pages 135–141

http://topology.auburn.edu/tp/

THE PARACOMPACTNESS OF SPACES RELATED TO UNCOUNTABLE BOX PRODUCTS

by

JUDITH ROITMAN AND SCOTT WILLIAMS

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog.edu
ISSN: 0146-4124

THE PARACOMPACTNESS OF SPACES RELATED TO UNCOUNTABLE BOX PRODUCTS

JUDITH ROITMAN AND SCOTT WILLIAMS

Abstract. Let X be a box product of κ many compact metric spaces. We give various models in which X/finite and X/countable are paracompact, as well as related results.

0. Introduction

One of the outstanding open questions in set theoretic topology is the question of the paracompactness of box products. The box product of countably many spaces has been studied extensively (see the survey [Williams$_1$], but the paracompactness of $\square_{\alpha<\omega_1} X_\alpha$ has seemed intractable. In particular there are no known consistency results in any direction on the paracompactness of $\square_{\alpha<\omega_1} X_\alpha$ when each X_α is compact metrizable, or even when each X_α is the convergent sequence $\omega + 1$.[1]

Motivated by this, we look at spaces related to $\square_{\alpha<\omega_1} X_\alpha$ when each X_α is not too far from compact metrizable.

Definition 1. *Let* $X = \square_{\alpha<\kappa} X_\alpha$. *If* $y \in X$ *we define*

(a) $[y] = \{z \in X : \{\alpha : y_\alpha \neq z_\alpha\}$ *is finite* $\}$

(b) $< y >= \{z \in X : \{\alpha : y_\alpha \neq z_\alpha\}$ *is countable*$\}$.

The space X*/fin is the quotient topology on* X *induced by the equivalence classes* $[y]$. *The space* X*/ctble is the quotient topology on* X *induced by the equivalence classes* $< y >$.

If $Y \subset X$ we define $[Y] = \{[y] : y \in Y\}$. Similarly, $< Y >= \{< y >: y \in Y\}$. A basic open set in X has the form $\prod_{\alpha<\kappa} u_\alpha$

[1] Added in Proof: Brian Lawrence has recently shown in ZFC that $\square_{\alpha<\omega_1} X_\alpha$ is not paracompact, in fact, not normal.

where each u_α is open in X_α. Such a set is called an open box. So a basis for X/fin is all [u] where u is an open box in X, and a basis for X/ctble is all <u> where u is an open box in X. By "basic open set" in, respectively, X, X/fin, or X/ctble, we mean an open box and its respective quotients.

Note that if each K_α is closed in X_α then $[\prod_{\alpha<\kappa} K_\alpha]$ and $< \prod_{\alpha<\kappa} K_\alpha >$ are closed in the respective spaces X/fin and X/ctble.

In what follows we assume that X is the full box product topology on $\prod_{\alpha<\kappa} X_\alpha$ where each X_α is normal Hausdorff, $\kappa > \omega$.

Let us first consider dimension. If some X_α fails to be 0-dimensional, then so does X. But our auxiliary spaces can be 0-dimensional even when no component space is. This was known in the case when $G_{\delta's}$ are open: X/fin when $\kappa = \omega$ (Kunen). While $G_{\delta's}$ may fail to be open in X/fin and in X/ctble in general, 0-dimensionality still holds.

Proposition 2. *For $\beta < \lambda$, let $K_\beta = \prod_{\alpha<\kappa} K_{\alpha,\beta}$ where each $K_{\alpha,\beta}$ is closed in X_α. Then*

(a) if $\lambda = \omega$, $\cup_{\beta<\lambda}[K_\beta]$ is closed in X/fin.

(b) if $\lambda \leq \omega_1$, $\cup_{\beta<\lambda} < K_\beta >$ is closed in X/ctble.

Proof. Let $K = \cup_{\beta<\lambda}[K_\beta]$ (respectively $\cup_{\beta<\lambda} < K_\beta >$). Suppose [y] (respectively $< y >$) $\notin K$. Then for each β there is a set $C_\beta \subset \kappa, |C_\beta| = \lambda$, with $y_\alpha \notin K_{\alpha,\beta}$ for $\alpha \in C_\beta$. Without loss of generality the C_β's are disjoint. Let u be basic open so $u_\alpha \cap K_{\alpha,\beta} = \emptyset$ for $\alpha \in C_\beta$.

Proposition 3. *X/fin and X/ctble are 0-dimensional.*

Proof. For X/fin: let [u] be an basic open set with [x] $\in$ [u]. Without loss of generality, x $\in$ u. By normality of the factors, for each α there is a sequence $\{v_{n,\alpha} : n < \omega\}$ where

$$x_\alpha \in v_{n,\alpha} \subset cl(v_{n,\alpha}) \subset v_{n+1,\alpha} \subset u_\alpha.$$

Let $v_n = \prod_{\alpha<\kappa} v_{n,\alpha}$. Then $\text{cl}(v_n) \subset v_{n+1}$. Let $G = \cup_{n\in\omega}[v_n]$. Clearly $[x] \in G \subset [u]$ and G is open. By proposition 2, G is closed.

The proof for X/ctble is similar, again using only countable unions of closed sets.

Clopen countable increasing unions of basic open sets $\{v_n : n < \omega\}$ where each $cl(v_n) \subset v_{n+1}$, are called basic clopen sets.

1. Using Cohen reals

We generalize the combinatorics of the several proofs that, under various set-theoretic hypotheses, "each countable box product of compact first countable spaces of small weight is paracompact" is consistent. The method also generalizes to X/ctble.

We state our theorems first in terms of Martin's axiom, and then generalize to other forcing extensions. Anyone familiar with forcing will have no trouble generalizing the MA proofs, which are considerably more readable than the direct forcing proofs.

Again, suppose $X=$ the full box product topology on $\prod_{\alpha<\kappa} X_\alpha$, where each X_α is normal Hausdorff.

Theorem 4. *(MA) Suppose κ is regular, $\kappa < 2^\omega$ and each X_α is first countable with cardinality $\leq 2^\omega$. Then X/fin and $X/ctble$ are ultraparacompact, i.e. every open cover has a pairwise disjoint open covering refinement.*

Proof. we give the proof for X/fin:

X has size at most 2^ω. Let $\mathcal{U}$ be an open cover of X/fin by basic open sets.

We show that if

a. $\mathcal{V}$ is a pairwise disjoint collection of basic clopen sets

b. $| \{\mathcal{V} |< 2^\omega$

c. $[x] \notin \cup\mathcal{V}$

then there is $[w]$ basic open, $[w] \cap V = \emptyset$ for all $V \in \mathcal{V}$, and $[x] \in [w] \subset [u]$ for some $[u] \in \mathcal{U}$.

Fix u so $[x] \in [u] \in \mathcal{U}$. We construct w as follows. Let P be the partial order whose element choose, for finitely many α, some element w_α of a fixed countable base of x_α contained in u_α, i.e., an element p of $\mathcal{P}$ is a function from a finite subset of κ onto ω , where we interpret $p(\alpha) = n$ to mean "$w_\alpha =$ the nth element of the base of x''_α. Order is reverse inclusion. P is ccc, in fact it is isomorphic to the forcing adding κ many Cohen reals. If $\bigcup_{n\in\omega}[v_n] \in \mathcal{V}$ then for each n there are infinitely many α with $x_\alpha \notin (v_n)_\alpha$. Fix $p \in P$. We can find infinitely many α for which p has made no decision about w_α. Fix n. Extend p to a condition deciding w_α so $w_\alpha \cap cl(v_n)_\alpha = \emptyset$. Since we can do this uncountably often for each n, we have $[w] \cap \bigcup_{n\in\omega}[v_n] = \emptyset$.

Now find a basic clopen set inside [w] and add it to $\mathcal{V}$ to get a pairwise disjoint cover of [x] $\cup(\bigcup\mathcal{V})$. A straightforward induction (left to the reader) completes the proof for X/fin.

Since $< x >$ separates from each $cl(v_n)$ using uncountably many coordinates, and, for fixed p, $\{\alpha : p$ has made no decision about u_α and $x_\alpha \notin cl(v_{n,\alpha})\}$ is uncountable, a minor adaptation of the proof for X/fin shows that X/ctble is ultraparacompact.

Now let's generalize to models in which MA may fail.

In the proof of Theorem 4, we used κ small and $\mathcal{V}$ small only so that we could enumerate X and invoke MA. Adding κ many Cohen reals will simultaneously separate an arbitrary number of points over disjoint collection $\mathcal{V}$ of basic clopen sets as long as $\mathcal{V}$ sits in the ground model. The technical lemma here is

Lemma 5. *Suppose each factor of X is first countable, $\mathcal{U}$ is an open cover of X/fin (respectively X/ctble). Let P be a partial order adding at least κ many Cohen reals to an inner model M, and suppose $\mathcal{V} \subset M$ is a pairwise disjoint collection of basic clopen sets refining $\mathcal{U}$. Then, in $M^P, \mathcal{V}$ extends to a disjoint collection $\mathcal{W}$ of basic clopen sets refining $\mathcal{U}$ so that, if $Z = \{s \in M : s \notin \bigcup\mathcal{V}$ and $s \in U$ for some $U \in M \cap \mathcal{U}\}$ and $w \in \mathcal{W}$, then $|Z \cap W| \leq 1$.*

Sketch of Proof. Now we interpret the elements of p differently. For $x \in X_\alpha$ fix $\{u_{x,n} : n \in \omega\}$ an open neighborhood basis. If $p \in P$ and $z \in Z$ the neighborhood $u(z,p)$ of z is defined by

$$u(z,p)_\alpha = u_{z_\alpha,p(\alpha)} \text{ if } \alpha \in dom\ p. \text{ Otherwise } u(z,p)_\alpha = X_\alpha.$$

Let G be a generic filter over M in P. G simultaneously adds basic open sets U_z for each $z \in Z$, where we define $U_z = \bigcap_{p\in G} u(z,p)$. Simple generic arguments show that each $U_z \cup \mathcal{V} = \emptyset$, and if z, y are distinct elements of Z then U_z, U_y are disjoint.

Remark. Assume $\mathcal{V} = \emptyset$. We have shown that $[X \cap M]$(*respectively* $< X \cap M >$) is closed discrete. In particular

Corollary 6. *Let P be the forcing adding λ many Cohen reals, $\kappa < \lambda, X \in V^P$, where each X_α is first countable of weight $< \lambda$. The following holds in V^P : If $A \subset X/fin$ (respectively $X/ctble$) and $|A| < \lambda$ then A is closed discrete.*

Proof. By hypothesis, we can fix a basis $\{u_{\beta,\alpha} : \beta < \lambda_\alpha < \lambda\}$ of X_α. Every point in X_α is determined by an ω-sequence of $u_{\beta,\alpha}$'s, i.e. by an ω-sequence from $\lambda_\alpha \times \{\alpha\}$. Thus every point in X is determined by a κ-sequence of these ω-sequences. Thus every subset of X of size $< \lambda$ sits in some inner model to which we are adding κ many Cohen reals.

Let us define the hypothesis which will replace MA in our generalization of theorem 4. It is generalization of the statement "the universe is an extension of an inner model by Cohen reals."

> CCR(λ, δ, μ): (CCR means"cofinal Cohen reals"). There is a sequence $\{N_\alpha : \alpha < \mu\}$ so that
> a. Each N_α is an elementary submodel of $H(\lambda)$ – here $H(\lambda)$ is the collection of all sets hereditarily of size $< \lambda$.
> b. $H(\lambda) \subset \cup_{\alpha<\mu} N_\alpha$.
> c. In $N_{\alpha+1}$ there are δ many Cohen reals over N_α
> d. If $\alpha < \beta$ then $N_\alpha \subset N_\beta$.

For example, the generic extension by ω_2 Cohen reals is a model of CCR($\omega_1, \omega_2, \omega_i$) where i =1,2, and CCR($\omega_2, \omega_1, 2^\omega$) follows from MA and holds in any model obtained by adding, say at least ω_2 Cohen reals to a model of CH.

Theorem 7. *Assume CCR(λ, κ, μ), where $\lambda, \mu, > \kappa$. If each factor of X is first countable with weight $< \lambda$ then X/fin and $X/ctble$ are ultraparacompact.*

Sketch of Proof. By hypothesis, every point in X sits in some inner model to which we are adding κ many Cohen reals. Similarly, every basic set and every basic clopen set also sits in some inner model over which we are adding κ many Cohen reals. Use induction, applying Lemma 6.

2. AVOIDING FORCING

Here we exploit proposition 2b to get results in ZFC or from simple cardinal arithmetic

Theorem 8. *Assume* $2^{\omega_1} = \omega_2, \kappa = \omega_1$, *and suppose each* $|X_\alpha| \leq \omega_2$. *Then* $X/ctble$ *is ultraparacompact.*

Remark. In particular, if each X_α is Lindelőf first countable, then each $|X_\alpha| \leq 2^\omega \leq 2^{\omega_1}$.

Proof of Theorem 8. By hypothesis $|X| \leq \omega_2$. List $X/$ctble as $\{< x(\alpha) >: \alpha < \omega_2\}$. Suppose $\mathcal{U}$ is a basic open cover of $X/$ctble. We will find a pairwise disjoint covering refinement by basic clopen sets. Suppose we have such a refininement $\{\mathcal{U}_\alpha : \alpha < \beta\}$ covering $\{< x(\alpha) >: \alpha < \beta\}$. If$< x(\beta) >$ is already covered, let $\mathcal{U}_\beta = \emptyset$. Otherwise, since each $\mathcal{U}_\alpha$ is a countable union of equivalence classes of closed boxes, $\cup_{\alpha<\beta}\mathcal{U}_\alpha$ is closed, so we can find a basic clopen set containing $< x(\beta) >$ which refines an element of $\mathcal{U}$.

Proposition 2 also tells us about closed discrete sets, no matter what κ is.

Corollary 10. *Every subset of* $X/ctble$ *of size at most* ω_1 *is closed discrete.*

Finally, Proposition 2 gives us information about closures in X.

Corollary 11. *If* $Y \subset X$ *and* $|Y| \leq \omega_1$ *then* $cl_X(Y) \subset < Y >$.

Proof. If $x \in cl_X(Y)$ then $< x > \in cl_{X/ctble}(< Y >) \subset < Y >$.

Corollary 12. *Assume CH,* $\kappa = \omega_1$, *and suppose each* $|X_\alpha| \leq \omega_1$. *If* $Y \subset X$ *and* $|Y| \leq \omega_1$ *then* $|cl_X(Y)| \leq \omega_1$.

REFERENCES

[Kunen$_1$] K. Kunen, *Some comments on box products,* Coll. Math. Soc. Janos Bolyai **10**, Kesthely, 1011-1016.

[Kunen$_1$] K. Kunen, *On paracompactness of box products of compact spaces,* TAMS **240**, 207-216.

[Roitman$_1$] J. Roitman, *Paracompact box products in forcing extensions,* Fund. Math. CII, 219-228.

[Roitman$_2$] ——, *More paracompact box products,* **74** PAMS, 171-176.

[Williams$_1$] S. W. Williams *Box products,* Handbook of Set Theoretic Topology, eds. K. Kunen and J. Vaughan, North Holland, Amsterdam, 1984, 169-200.

[Williams$_2$] ——, *Paracompactness in box products* Proc. of the 12th Winter School on Abstract Analysis, 1984, 313-328.

University of Kansas
Lawrence KS 66045

SUNY, Buffalo
Buffalo, NY 14214