MILLION BUCK PROBLEMS
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Upon publication of Apostolos Doxiadis' new novel, *Uncle Petros & Goldbach's Conjecture*, the publishers, Faber and Faber in Britain and Bloomsbury Publishing USA, have offered $1,000,000 to individual(s) who solve Goldbach's Conjecture. On May 10, The Clay Mathematical Sciences Institute has inaugurated a $7,000,000 Millenium Prize, a million dollar award for the solution of each of seven famous problems. Contrary to belief, this publicity stunt has precedence in Mathematics. This article is a result of my personal review of the history of a few famous unsolved problems whose statements can be understood by a person with an undergraduate mathematics degree or less.

When I was a student, the Burnside Problem, the Simple Odd Group Conjecture (1963), and the Continuum Hypothesis had just been resolved but the Riemann Hypothesis, the Four Color Map Problem, Fermat's Last Theorem, the Bieberbach Conjecture, the Poincaré conjecture, and the Goldbach Conjecture were all famous open problems. Ten years later, the Four Color Problem and the Alexandrov Conjecture were solved. In twenty years the Bieberbach Conjecture was proved. Thirty years later Fermat's Last Theorem is gone and just a few of the aforementioned problems remain, although others surface. A solution to any of these problems brings "fame" and occasionally one of the major Mathematical prizes such as the $145,000 Steele Prize, the $50,000 Wolf Prize, a special gold medal (along with $15,000) called The Fields Medal, informally known as the "Nobel Prize of Mathematics" or what I call the real "Nobel Prize" for mathematicians, the Royal Swedish Academy of Sciences' $500,000 Crafoord Prize.

The seven problems who solutions will bring $1,000,000 each from the Clay Mathematical Institute (http://www.claymath.org) are the Poincaré Conjecture and the Riemann Hypothesis, both discussed below, and the P versus NP problem, the Hodge Conjecture, the Yang-Mills Existence and Mass Gap, the Navier-Stokes Existence and Smoothness; and the Birch and Swinnerton-Dyer Conjecture. The problems are accompanied with articles written by Stephen Cook, Pierre Deligne, Enrico Bombieri, Charles Fefferman; and Andrew Wiles.

Attaching monetary value to mathematics questions is not new. In 1908 German industrialist Paul Wolfskehl established a prize of 10,000DM (approximatey $1,000,000 at the time) for a proof of Fermat's Last Theorem. Unfortunately inflation diminished the prize value so that in 1997 Wiles collected just $50,000; however, the Royal Swedish Academy of Sciences also awarded Wiles the Schock Prize and he received the Prix Fermat from the Université Paul Sabatier. DeBranges was awarded the Ostrowski Prize for proving a much stronger conjecture than the Bieberbach Conjecture.

"The Prince of Problem Solvers and the Monarch of Problem Posers," the late Paul Erdös, who won the $50,000 Wolf Mathematics Prize, was famous for offering cash prizes for those mathematicians who solved certain of his problems. These prizes ranged from $10,000 for what he called "a hopeless problem" in number theory to $25 for something that he considered not particularly difficult but still tricky, proposed in the middle of a lecture. Since Erdös' 1996 death, other mathematicians have continued this practice. Now a corporation offers one million dollars and an Institute offers more.

Fields Medals have not been awarded to persons over the age of forty. Concerning solutions of famous problems, some Fields Medal were awarded to:

Selberg (1950) for his work on the Riemann Hypothesis; Cohen, (1966), for his resolution of the Continuum Hypothesis; Smale (1966) for his work on the Generalized Poincaré conjecture for $n>4$; Thompson (1970) for his part in the solution of the Odd Simple Group Conjecture; Bombieri (1974) for his work on the Bieberbach Conjecture; Faltings (1986) for his solution of Mordell's Conjecture; Freedman (1986) for his work on the Generalized Poincaré conjecture for n=4; Borcherds (1998) for his solution of the Monstrous Moonshine Conjecture.

Perhaps via "fame" a solution will bring to some a modest fortune. The unsolved problems below (Goldbach's Conjecture, The Kolakoski sequence, The 3x+1 Problem, Schanuel's Conjectures, Box Product Problem, Odd Perfect Number Problem, Riemann Hypothesis, Twin primes conjecture, Lost in a Forest Problem, Palindrome Problem, The Poincaré Conjecture) all have simple statements. Some of these problems (the Riemann Hypothesis and the Poincaré Conjecture) are usually taken to have more value to the field than others. However, there have been lesser problems which were not resolved by simply pushing the existing techniques further than others had done, but rather by introducing highly original ideas which were to lead to many developments. I, therefore, call them all million buck problems because I believe (the techniques involved in) their resolution will be worth at least one million dollars to Mathematics.

1. **Goldbach's Conjecture:** On June 7, 1742, Christian Goldbach wrote a letter to L. Euler suggesting every even integer is the sum of two primes and this is unproved still today, though it is known to be true for all numbers up to $4\cdot 10^{13}$. The closest approximation to a solution to Goldbach's Conjecture is Chen-Jing Run's recent result that every "sufficiently large" even number is of the form $p + qr$, where p,q,r are primes. For the **$1,000,000** prize, Faber and Faber in Britain and Bloomsbury Publishing USA, have issued a stringent set of requirements, which include publishing the solution to Goldbach's Conjecture. Contestants have until March 2002 to submit their applications and March 2004 to publish the solution. If there is a winner, the prize will be awarded by the end of 2004.

A still unsolved consequence of Goldbach's Conjecture is the <u>odd Goldbach Conjecture</u>, "every odd integer greater than five is the sum of three primes." This has been shown to be true for odd integers greater than $10^{7000000}$ and will probably fall when proper computing power is devoted to it.

2. **Beal's Conjecture**: This is a generalization of Fermat's Last Theorem. If $A^x + B^y = C^z$, where A, B, C, x, y and z are positive integers and x, y and z are all greater than 2, then A, Band C must have a common factor. Andrew Beal is a banker and an amateur mathematician yet he offers **$75,000** for the resolution of this conjecture which was first announced in 1997. The prize committee consists of Charles Fefferman, Ron Graham, amd R. Daniel Mauldin and the funds are held in trust by the American Mathematical Society.

3. **Schanuel's Two Conjectures** (not to be confused with the Schanuel Lemma or the Ax-Schanuel Theorem)**:** In he early 1960s, Schanuel made two conjectures about the algebraic behavior of the complex exponential function. Stephen Schanuel offers **$2,000**, $1,000 each, for the published resolution of the conjectures in his lifetime. <u>The Schanuel Conjecture</u> is the following *independence property* ($\mathbf{C}, e^z$): If $z_1, z_2, \dots, z_n$ in $\mathbf{C}$ are complex numbers linearly independent over the rationals, then some n of the $2n$ numbers $z_1, z_2, \dots, z_n, e^{z_1}, e^{z_2}, \dots e^{z_n}$ are algebraically independent. <u>The Converse Schanuel Conjecture</u> says that there is nothing more to be said. Explicitly, let F be a countable field of characteristic zero and $E : F \to F$ a homomorphism from the additive multiplicative group whose kernel is cyclic. The conjecture is that if (F,E) has the independence property, then there is a homomorphism of fields $h : F \to \mathbf{C}$ such that $h(E(x)) = e^{h(x)}$. Either of the two conjectures would imply, for example, algebraic independence of e and π. [For the first take $z_1 = 1$, $z_2 = \pi i$; for the second, one must construct (F,E) with an

element p such that $E(ip) = -1$ and so that $E(1),p$ are algebraically independent.] On the other hand, we don't even know whether $e + \pi$ is rational.

4. **The Kolakoski sequence:** Consider the sequence of ones and twos

$$\sigma = \langle 12211212212211211221211212211211212212211212212112112212211 2\rangle.$$

A <u>block</u> of σ is a maximal constant subsequence. We consider the blocks and their lengths. For example, beginning from the left, the first block <1> has length 1. The second block <22> has length 2. The third block <11> has length 2. Continue in this fashion and notice that the sequence $\lambda = \langle 1221121221\ldots\rangle$ of block lengths is an initial segment of σ. The Kolakoski sequence is the (unique) infinite sequence σ of ones and twos, beginning with 1, for which the sequence λ of block lengths satisfies $\lambda=\sigma$. Chris Kimberling (see http://cedar.evansville.edu/~ck6/index.html) promises a prize of **$200** to the first person to publish a solution of all five problems below (He says chances are if you solve one, you'll see how to solve the others). Considering the last 4 questions as one, makes the Kolakoski sequence questions interesting:

i. Is there a formula for the nth term of σ?

ii. If a string (e.g., 212211) occurs in σ, must it occur again?

iii. If a string occurs in σ, must its reversal also occur? (112212 occurs)

iv. If a string occurs in σ, and all its 1s and 2s are swapped, must the new string occur? (121122 occurs)

v. Does the limiting frequency of 1s in σ exist - and is it 1/2?

5. **The Box Product Problem:** Given countably infinite many copies of the interval [0,1], the typical (Tychonov) product topology on their product is topologically a copy of the Hilbert Cube. Give it Urysohn's 1923 box product topology instead (so open sets are unions of products of arbitrary open intervals). The Box Product Problem asks, "Is the box product topology on the product of countably infinitly many copies of the real line normal?" In other words, can disjoint closed sets be separated by disjoint open sets? In 1994, the answer was shown to be NO to the corresponding problem for uncountably many copies, but in 1972 Mary Ellen Rudin showed that the continuum hypothesis implies YES. What is known about the problem is no different whether the real line is replaced by such related spaces as the closed interval [0,1] or the convergent sequence and its limit (the space $X=\{2^{-n}: n \quad \mathbf{N}\} \quad \{0\} \quad \mathbf{R}$) and is related to combinatorial questions in Set Theory. Scott Williams offers (with appeal to <u>A Hitch-Hikers Guide to the Galaxy</u>) a **$42** prize to the person who settles the box product problem in his lifetime.

6. **The Collatz' 3x+1 Conjecture:** Because it is easy to program your computer to look for solutions, many youngsters (and adults) have played with the 3x+1 problem: On the positive integers define the function $F(x)=3x+1$, if x is odd and $F(x)=x/2$ if x is even. Iterations of F lead to the sequences <1, 4, 2, 1>, <3, 10, 5, 16,…, 1>, and <7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10,…, 1>. The 3x+1 conjecture, stated in 1937 by Collatz, is "For each integer x, applying successive iterations of F, eventually yields 1." During Thanksgiving vacation in 1989 I programmed my desktop computer to verify the conjecture by testing integers in their usual order. After 3 days it verified the first 500,000 integers satisfied the 3x+1 conjecture. Currently, the conjecture has been verified for all numbers up to $5.6 \cdot 10^{13}$, but not by me.

For fun, consider the different conclusions to three slightly different versions of this problem obtained by exchanging 3x+1 for one of 3x-1, 3x+3, or 5x+1.

7. **Odd Perfect Number Problem:** Does there exist a number that is perfect and odd? A given number is perfect if it is equal to the sum of all its proper divisors. This question was first posed by Euclid and is still open. Euler proved that if N is an odd perfect number, then in the prime power decomposition of N, exactly one exponent is congruent to 1 mod 4 and all the other exponents are even. Using computers it has been shown that there are no odd perfect numbers $<10^{300}$.

8. **Riemann Hypothesis:** This is the most famous open problem in mathematics. In his 1859 paper *On the Number of Primes Less Than a Given Magnitude*, Bernhard Riemann (1826-1866) extended the zeta function, defned by Euler as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for s>1, to be defined for every complex number. Riemann noted that his zeta function had trivial zeros at s=-2, -4, -6, ... and that the set of nontrivial zeros were symmetric about the line Re(s) = 1/2. The Riemann hypothesis says all nontrivial zeros are on this line; i.e., they have real part 1/2.

9. **Twin primes conjecture**: A twin prime is an integer p such that both p+1 and p-1 are prime numbers. The first five twin primes are 4, 6,12, 18, and 30. The Twin Prime conjecture states there are infinitely many twin primes. It is known there are 27,412,679 twin primes $<10^{10}$. Currently, the largest known twin prime is $2{,}409{,}110{,}779{,}845 \cdot 2^{60000}$, which has 18072 digits.

10. **The Poincaré Conjecture:** Henri Poincaré said, "Geometry is the art of applying good reasoning to bad drawings." For a positive integer n, an n-manifold is a Hausdorff topological space with the property that each point has a neighborhood homeomorphic to n-space $\mathbf{R}^n$. The manifold is simply connected if each loop in it can be deformed to a point (not possible if it, like a doughnut, has a hole). The Generalized Poincaré Conjecture says each simply connected compact n-manifold is homeomorphic to the n-sphere. Near the end of the 19th century, Poincaré conjectured this for n=3, and the Generalized Poincaré Conjecture has been solved in all cases except n=3.

11. **Palindrome Problem:** A palindrome is a phrase or word which is the same if you reverse the position of all the letters. A integer palindrome has the same property; e.g., 121. Here is an algorithm which one might think leads to a palindrome: Given an integer x, let x* be the reverse of n's digits, and F(x)=x+x*. Now iterate the process. Considering sequences of iterations of F, we have <29, 29+92=121> and <176, 176+671=847, 1595, 7546, 14003, 44044>. The examples show that iterations of 29 and 176 lead, respectively, to palindromes 121 and 44044. The Palindrome problem is "Given any integer x, do iterations of F lead to a palindrome." This is unsolved *even in the case x = 196*.

12. **Lost in a Forest Problem:** In 1956 R. Bellman asked the following question: Suppose that I am lost without a compass in a forest whose shape and dimensions are precisely known to me. How can I escape in the shortest possible time? Limit answers to this question for certain two-dimensional forests; planar regions. For a given region, choose a path to follow and determine the initial point which requires the maximum time to reach the outside. Then minimize the maximum time over all paths. For many plane regions the answer is known: circular disks, regular even sided polygonal regions, half-plane regions (with known initial distance), equilateral triangular regions. However, for some regions, for regular odd-sided polygonal regions in general and triangular regions in particular, only approximates to the answer are known.

This article is dedicated to John Isbell. Concerning this article, I had personal correspondences with William Massey, Mohan Ramachandran, Samuel Schack, and Stephen Schanuel. All errors, however, are mine.

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The extraordinary story of Fermat's Last Theorem: http://www.cs.uleth.ca/~kaminski/esferm03.html

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