Integration

1 Introduction

In this section, we will look at various numerical methods for evaluating Riemann integrals. Recall from Calculus that Riemann integrals are generally written in the form,

$$\int_a^b f(x)dx$$

for some function $f(x)$ over an interval $[a, b]$. Recall that a function, f, is Riemann integrable iff the value of the sum,

$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

where $t_i \in [x_{i-1}, x_i]$, exists as the distances between the points x_i and x_{i-1} approaches zero or the number of points used in the summation (n) approaches infinity. Numerical methods for evaluating integrals fundamentally rely on using the above definition, with slight modifications.

2 Midpoint Rule

The Midpoint Rule is a numerical method that approximates the integral of f by evaluating the function at the midpoint of a finite number of subintervals over the interval that is integrated. Suppose $f(x)$ is a Riemann integrable function of x, and consider a partition of the interval $[a, b]$ into n equal subintervals each with length,

$$h_n = \frac{b-a}{n}.$$

Thus, the interval $[a, b]$ can be divided as follows:

$$a < a + h_n < a + 2h_n < \ldots < a + nh_n$$

and $a + nh_n = b$. Now if we choose the midpoints of each one of the subintervals of $[a, b]$, we have that the sum, S, becomes,

$$S = (f(a + \frac{1}{2}h_n) + f(a + \frac{3}{2}h_n) + \ldots + f(a(n - \frac{1}{2})h_n))h_n.$$

Therefore, the formula to approximate an integral using the Midpoint Rule can be stated as follows:

$$\int_a^b f(x)dx \approx h_n \sum_{i=1}^{n} f(a + (i - \frac{1}{2})h_n).$$

3 Trapezoidal Rule

The Trapezoidal Rule is another numerical method that can be used to approximate the value of an integral. The Trapezoidal Rule is more accurate than the Midpoint Rule since trapezoids are used (as opposed to rectangles) to calculate the area under a curve. Recall that the area of a trapezoid is given by the formula,

$$A = \frac{1}{2}b(d_1 + d_2),$$

where b is the length of the base of the trapezoid, and d_1 and d_2 are the heights of the trapezoid. Let h_n be defined as before. Consider the points $(a + kh_n, f(a + kh_n))$ for $k = 1, 2, \ldots, n$. For each trapezoid enclosed by the points $a + kh_n, f(a + kh_n), a + (k + 1)h_n$, and $f(a + (k + 1)h_n)$, we have an area equal to $\frac{1}{2}h_n(f(a + kh_n) + f(a + (k + 1)h_n))$. Summing up all of the areas of these trapezoids yields the summation formula for the approximation of an integral using the Trapezoidal Rule:

$$\int_a^b f(x)dx \approx \frac{1}{2}h_n f(a) + h_n \sum_{i=1}^{n-1} f(a + ih_n) + \frac{1}{2}h_n f(b).$$

4 Simpson's Rule

Simpson's Rule is another numerical method for finding an approximation of an integral. Unlike the Midpoint Rule and the Trapezoidal Rule, Simpson's Rule approximates the graph of a function f with parabolic arcs. Let $(-h, y_0), (0, y_1)$, and (h, y_2) be three points that lie on a quadratic curve given by the formula $y = Ax^2 + Bx + C$. The area under this curve between $-h$ and h is equal to $\frac{1}{3}h(y_0 + 4y_1 + y_2)$. Let f be a continuous function over the interval $[a, b]$, let $h_n = \frac{(b-a)}{n}$, but now for *even* values of n. Then, for each of the intervals

$$[a, a + 2h_n], [a + 2h_n, a + 4h_n], \ldots, [b - 2h_n, b],$$

f can be approximated with the points,

$$y_0 = f(a), y_1 = f(a + h_n), y_2 = f(a + 2h_n), \ldots, y_n = f(b).$$

This generates Simpson's Rule, which is given by,

$$\int_a^b f(x)dx \approx \frac{1}{3}h_n(f(a)+4f(a+h_n)+2f(a+2h_n)+4f(a+3h_n)+2f(a+4h_n)+\ldots+2f(b-2h_n)+4f(b-h_n)+f(b)).$$