11 | Tietze Extension Theorem

The main goal of this chapter is to prove the following fact which describes one of the most useful properties of normal spaces:

11.1 Tietze Extension Theorem (v.1). Let $X$ be a normal space, let $A \subseteq X$ be a closed subspace, and let $f : A \to [a, b]$ be a continuous function for some $[a, b] \subseteq \mathbb{R}$. There exists a continuous function $\tilde{f} : X \to [a, b]$ such that $\tilde{f}|_A = f$.

The main idea of the proof is to use Urysohn Lemma 10.1 to construct functions $\tilde{f}_n : X \to [a, b]$ for $n = 1, 2, \ldots$ such that as $n$ increases $\tilde{f}_n|_A$ gives ever closer approximations of $f$. Then we take $\tilde{f}$ to be the limit of the sequence $\{\tilde{f}_n\}$. We start by looking at sequences of functions and their convergence.

11.2 Definition. Let $X, Y$ be a topological spaces and let $\{f_n : X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ converges pointwise to a function $f : X \to Y$ if for each $x \in X$ the sequence $\{f_n(x)\} \subseteq Y$ converges to the point $f(x)$.

11.3 Note. If $\{f_n : X \to Y\}$ is a sequence of continuous functions that converges pointwise to $f : X \to Y$ then $f$ need not be continuous. For example, let $f_n : [0, 1] \to \mathbb{R}$ be the function given by $f_n(x) = x^n$. Notice that $f_n(x) \to 0$ for all $x \in [0, 1)$ and that $f_n(1) \to 1$. Thus the sequence $\{f_n\}$ converges pointwise to the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$$

The functions $f_n$ are continuous but $f$ is not.

11.4 Definition. Let $X$ be a topological space, let $(Y, \rho)$ be a metric space, and let $\{f_n : X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ converges uniformly to a function $f : X \to Y$ if
for every \( \varepsilon > 0 \) there exists \( N > 0 \) such that
\[
g(f(x), f_n(x)) < \varepsilon
\]
for all \( x \in X \) and for all \( n > N \).

11.5 Note. If a sequence \( \{f_n\} \) converges uniformly to \( f \) then it also converges pointwise to \( f \), but the converse is not true in general.

11.6 Proposition. Let \( X \) be a topological space and let \((Y, \rho)\) be a metric space. Assume that \( \{f_n : X \to Y\} \) is a sequence of functions that converges uniformly to \( f : X \to Y \). If all functions \( f_n \) are continuous then \( f \) is also a continuous function.

Proof. Let \( U \subseteq Y \) be an open set. We need to show that the set \( f^{-1}(U) \subseteq X \) is open. If suffices to check that each point \( x_0 \in f^{-1}(U) \) has an open neighborhood \( V \) such that \( V \subseteq f^{-1}(U) \). Since \( U \) is an open set there exists \( \varepsilon > 0 \) such \( B(f(x_0), \varepsilon) \subseteq U \). Choose \( N > 0 \) such that \( g(f(x), f_N(x)) < \frac{\varepsilon}{3} \) for all \( x \in X \), and take \( V = f_N^{-1}(B(f_N(x_0), \frac{\varepsilon}{3})) \). Since \( f_N \) is a continuous function the set \( V \) is an open neighborhood of \( x_0 \) in \( X \). It remains to show that \( V \subseteq f^{-1}(U) \). For \( x \in V \) we have:
\[
g(f(x), f(x_0)) \leq g(f(x), f_N(x)) + g(f_N(x), f_N(x_0)) + g(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]
This means that \( f(x) \in B(f(x_0), \varepsilon) \subseteq U \), and so \( x \in f^{-1}(U) \).

11.7 Lemma. Let \( X \) be a normal space, \( A \subseteq X \) be a closed subspace, and let \( f : A \to \mathbb{R} \) be a continuous function such that for some \( C > 0 \) we have \( |f(x)| \leq C \) for all \( x \in A \). There exists a continuous function \( g : X \to \mathbb{R} \) such that \( |g(x)| \leq \frac{1}{3}C \) for all \( x \in X \) and \( |f(x) - g(x)| \leq \frac{2}{3}C \) for all \( x \in A \).

Proof. Define \( Y := f^{-1}([-C, -\frac{1}{2}C]) \), \( Z := f^{-1}([\frac{1}{2}C, C]) \). Since \( f : A \to \mathbb{R} \) is a continuous function these sets are closed in \( A \), but since \( A \) is closed in \( X \) the sets \( Y \) and \( Z \) are also closed in \( X \). Since \( Y \cap Z = \emptyset \) by the Urysohn Lemma 10.1 there is a continuous function \( h : X \to [0, 1] \) such that \( h(Y) \subseteq \{0\} \) and \( h(Z) \subseteq \{1\} \). Define \( g : X \to \mathbb{R} \) by
\[
g(x) := \frac{2}{3} \left( h(x) - \frac{1}{2} \right)
\]
Proof of Theorem 11.1. Without loss of generality we can assume that \([a, b] = [0, 1]\). For \(n = 1, 2, \ldots\) we will construct continuous functions \(g_n : X \to \mathbb{R}\) such that

1. \(|g_n(x)| \leq \frac{1}{2} \cdot \left(\frac{3}{2}\right)^{n-1}\) for all \(x \in X\);
2. \(|f(x) - \sum_{i=1}^{n} g_i(x)| \leq \left(\frac{3}{2}\right)^n\) for all \(x \in A\).

We argue by induction. Existence of \(g_1\) follows directly from Lemma 11.7. Assume that for some \(n \geq 1\) we already have functions \(g_1, \ldots, g_n\) satisfying (i) and (ii). In Lemma 11.7 take \(f\) to be the function \(f - \sum_{i=1}^{n} g_i\) and take \(C = \left(\frac{3}{2}\right)^n\). Then we can take \(g_{n+1} := g\) where \(g\) is the function given by the lemma.

Let \(\tilde{f}_n := \sum_{i=1}^{n} g_n\) and let \(\tilde{f} := \sum_{i=1}^{\infty} g_n\). Using condition (i) we obtain that the sequence \(\{\tilde{f}_n\}\) converges uniformly to \(\tilde{f}\) (exercise). Since each of the functions \(\tilde{f}_n\) is continuous, thus by Proposition 11.6 we obtain that \(\tilde{f}\) is a continuous function. Also, using (ii) be obtain that \(\tilde{f}(x) = f(x)\) for all \(x \in A\) (exercise).

Here is another useful reformulation of Tietze Extension Theorem:

11.8 Tietze Extension Theorem (v.2). Let \(X\) be a normal space, let \(A \subseteq X\) be a closed subspace, and let \(f : A \to \mathbb{R}\) be a continuous function. There exists a continuous function \(\tilde{f} : X \to \mathbb{R}\) such that \(\tilde{f}|_A = f\).

Proof. It is enough to show that for any continuous function \(g : A \to (-1, 1)\) we can find a continuous function \(\tilde{g} : X \to (-1, 1)\) such that \(\tilde{g}|_A = g\). Indeed, if this holds then given a function \(f : A \to \mathbb{R}\) let \(g = hf\) where \(h : \mathbb{R} \to (-1, 1)\) is an arbitrary homeomorphism. Then we can take \(\tilde{f} = h^{-1}\tilde{g}\).

Assume then that \(g : A \to (-1, 1)\) is a continuous function. By Theorem 11.1 there is a function \(g_1 : X \to [-1, 1]\) such that \(g_1|_A = g\). Let \(B := g_1^{-1}((-1, 1])\). The set \(B\) is closed in \(X\) and \(A \cap B = \emptyset\) since \(g_1(A) = g(A) \subseteq (-1, 1)\). By Urysohn Lemma 10.1 there is a continuous function \(k : X \to [0, 1]\) such that \(B \subseteq k^{-1}(\{0\})\) and \(A \subseteq k^{-1}(\{1\})\). Let \(\tilde{g}(x) := k(x) \cdot g_1(x)\). We have:

1) if \(g_1(x) \in (-1, 1)\) then \(g(x) \in (-1, 1)\)
2) if \(g_1(x) \in \{-1, 1\}\) then \(x \in B\) so \(\tilde{g}(x) = 0 \cdot g_1(x) = 0\)

It follows that \(\tilde{g} : X \to (-1, 1)\). Also, \(\tilde{g}\) is a continuous function since \(k\) and \(g_1\) are continuous. Finally, if \(x \in A\) then \(\tilde{g}(x) = 1 \cdot g_1(x) = g(x)\), so \(\tilde{g}|_A = g\).

Tietze Extension Theorem holds for functions defined on normal spaces. It turns out the function extension property is actually equivalent to the notion of normality of a space:

11.9 Theorem. Let \(X\) be a space satisfying \(T_1\). The following conditions are equivalent:

1) \(X\) is a normal space.
2) For any closed sets $A, B \subseteq X$ such that $A \cap B = \emptyset$ there is a continuous function $f: X \to [0, 1]$ such that such that $A \subseteq f^{-1}({0})$ and $B \subseteq f^{-1}({1})$.

3) If $A \subseteq X$ is a closed set then any continuous function $f: A \to \mathbb{R}$ can be extended to a continuous function $\bar{f}: X \to \mathbb{R}$.

**Proof.** The implication 1) $\Rightarrow$ 2) is the Urysohn Lemma 10.1 and 2) $\Rightarrow$ 1) is Proposition 9.15. The implication 1) $\Rightarrow$ 3) is the Tietze Extension Theorem 11.8. The proof of implication 3) $\Rightarrow$ 1) is an exercise.

\[\square\]

### Exercises to Chapter 11

**E11.1 Exercise.** Prove implication 3) $\Rightarrow$ 1) of Theorem 11.9.

**E11.2 Exercise.** Let $X$ be a normal space, let $A \subseteq X$ be a closed subspace, and let $f: A \to \mathbb{R}$ be a continuous function.

a) Assume that $g: X \to \mathbb{R}$ is a continuous function such that $f(x) \leq g(x)$ for all $x \in A$. Show that there exists a continuous function $F: X \to \mathbb{R}$ satisfying $F|_A = f$ and $F(x) \leq g(x)$ for all $x \in X$.

b) Assume that $g, h: X \to \mathbb{R}$ are a continuous function such that $h(x) \leq f(x) \leq g(x)$ for all $x \in A$ and $h(x) \leq g(x)$ for all $x \in X$. Show that there exists a continuous function $F': X \to \mathbb{R}$ satisfying $F'|_A = f$ and $h(x) \leq F'(x) \leq g(x)$ for all $x \in X$.

**E11.3 Exercise.** Recall that if $X$ is a topological space then a subspace $Y \subseteq X$ is a called a retract of $X$ if there exists a continuous function $r: X \to Y$ such that $r(x) = x$ for all $x \in Y$. Let $X$ be a normal space and let $Y \subseteq X$ be a closed subspace of $X$ such that $Y \cong \mathbb{R}$. Show that $Y$ is a retract of $X$.

**E11.4 Exercise.** Let $X$ be topological space. Recall from Exercise 10.3 that a set $A \subseteq X$ is a $G_\delta$-set if there exists a countable family of open sets $U_1, U_2, \ldots$ such that $A = \bigcap_{n=1}^{\infty} U_n$.

a) Show that if $X$ is a normal space and $A \subseteq X$ is a closed $G_\delta$-set then there exists a continuous function $f: X \to [0, 1]$ such that $A = f^{-1}({0})$.

b) Show that if $X$ is a normal space and $A, B \subseteq X$ are closed $G_\delta$-sets such that $A \cap B = \emptyset$ then there exists a continuous function $f: X \to [0, 1]$ such that $A = f^{-1}({0})$ and $B = f^{-1}({1})$. 