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The Ablowitz-Ladik system on the natural numbers with certain linearizable boundary conditions

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We solve the initial-boundary value problem (IBVP) for the Ablowitz–Ladik system on the natural numbers with certain linearizable boundary conditions. We do so by employing a nonlinear method of images, namely, by extending the scattering potential to all integers in such a way that the extended potential satisfies certain symmetry relations. Using these extensions and the solution of the initial value problem (IVP), we then characterize the symmetries of the discrete spectrum of the scattering problem, and we show that discrete eigenvalues in the IBVP appear in octets, as opposed to quartets in the IVP. Furthermore, we derive explicit relations between the norming constants associated with symmetric eigenvalues, and we identify a new kind of linearizable IBVP. Finally, we characterize the soliton solutions of these IBVPs, which describe the soliton reflection at the boundary of the lattice.

Keywords: solitons; integrable systems; boundary value problems; nonlinear method of images

AMS Subject Classifications: 37K15; 35Q51; 35R10; 35G30

1. Introduction

The inverse scattering transform (IST) has been successfully used to solve initial value problems (IVPs) for a certain class of nonlinear partial differential equations (PDEs) called integrable systems, such as the nonlinear Schrödinger equation (NLS), Korteweg-deVries (KdV) and sine-Gordon equation (see [1,2] and references therein). The IST has also been extended to solve IVPs for integrable discrete nonlinear equations [3,4]. One of such equations is the celebrated Ablowitz–Ladik (AL) system, also known as the integrable discrete NLS:

$$i\dot{q}_n + \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \nu|q_n|^2(q_{n+1} + q_{n-1}) = 0, \tag{1.1}$$

where $q_n = q_n(t)$, $\dot{f} = \partial f/\partial t$ and h is the lattice spacing. As usual the cases $\nu = -1$ and $\nu = 1$ are called, respectively, focusing and defocusing.

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The effort to extend the IST to solve initial-boundary value problems (IBVPs) for nonlinear PDEs also has a long history. In [5], the even and odd extensions of the potential to the whole line were developed to solve IBVPs for the NLS equation on the half-line with homogeneous Dirichlet and Neumann boundary conditions (BCs). This method was generalized to nonlinear sine and nonlinear cosine transforms [6]. However, these approaches only worked for IBVPs with linearizable BCs (namely, for the NLS equation, homogeneous Robin BCs [7–10]). A different method was introduced in [11–15], which is based on the simultaneous spectral transform of both parts of the Lax pair. In [16], we showed that this method can be extended to solve IBVPs for the AL system on the natural numbers. Other interesting approaches have been presented. Bäcklund transformations have been used to identify linearizable BCs for integrable nonlinear PDEs on the half-line [7–10]. It was also shown in [17] that a similar approach can be applied to IBVPs for integrable discrete lattices. An 'elbow scattering' transform was introduced in [18] to solve the IBVP for the KdV equation by formulating an appropriate Gel'fand–Levitan–Marchenko equation. Finally, a rigorous analysis of the problem was also presented in [19–22].

The purpose of this work is to solve IBVPs for the AL system (1.1) on the natural numbers with certain linearizable BCs. In [16], we solved the IBVP for (1.1) when a generic boundary datum $q_0(t)$ is given. As in the continuum case, in general the expression for the solution depends on some unknown boundary data, and their elimination leads to a complicated system of coupled nonlinear differential equations. That is (as in the continuum case), the IBVP in general does not linearize. As in the continuum case, however, a special class of BCs exist for which the IBVPs can be solved as effectively as the IVP via the IST. Such BCs are called *linearizable*. For the AL system, it was shown in [17] that the linearizable BCs are the discrete analogue of homogeneous Robin BCs, namely,

$$q_0(t) - \alpha q_1(t) = 0, \quad \alpha \in \mathbb{R}. \tag{1.2}$$

These same linearizable BCs are also found naturally via spectral analysis of both parts of the Lax pair [16].

Here we consider the IBVP for the AL system (1.1) on the natural numbers with the linearizable BCs obtained by setting $\alpha = 0$ or $\alpha = 1$ in (1.2). The first case is the discrete analogue of homogeneous Dirichlet BCs; the second case is the discrete analogue of homogeneous Neumann BCs. In a recent work on the NLS equation on the half-line with linearizable BCs [23], we characterized the discrete eigenvalues of the scattering problem and soliton solutions. Because of its conceptual similarities to the method of images in electrostatics [24], we referred to the approach we used in [23] as a nonlinear method of images. Here we show how the same approach can be used to solve the IBVPs for the AL system on the natural numbers with linearizable BCs. As in the continuous case, we characterize the discrete spectrum of the associated scattering problem, thereby showing that the discrete eigenvalues appear in octets, as opposed to quartets in the IVP. This ensures that for each soliton there exists a symmetric counterpart typically located beyond the boundary, with equal amplitude and opposite velocity. As a result, the apparent soliton reflection at the boundary is simply the manifestation of the interchanging of the roles between the true and mirror solitons satisfying the BCs.

The outline of this work is the following: In Section 2, we briefly review the solution of the IVP for the AL system via the IST. In Section 3, we then solve the IBVP and we characterize the discrete eigenvalues of the associated scattering problem. In Section 4, we obtain a novel expression in terms of the scattering data for the normalization constant that appears in the direct and inverse scattering problem. Finally, in Section 5, we derive quantitative relations between the norming constants associated to the symmetric eigenvalues, and in Section 6, we use our results to discuss the soliton behaviour. Section 7 ends this work with a few final remarks. Throughout this work, whenever the discrete spectrum is discussed, we implicitly assume $\nu = -1$. After rescaling $t' = t/h^2$ and $q'_n(t) = hq_n(t)$, one can always reduce to a modified AL equation where $h = 1$ effectively. Thus for simplicity we consider the rescaled problem.

2. IVP for the AL system via IST

The IVP for the AL system on all integers was successfully solved via the IST in [3,4] in the case of zero BCs at infinity and in [25,26] in the case of constant-amplitude BCs. The method that we use in Section 3 to solve the IBVP on the naturals with linearizable BCs makes essential use of the IST for the IVP. Hence, here we briefly recall the main steps of the method, thereby introducing much of the machinery that will be used. We refer the reader to Refs. [16,27,28] for all details. As usual, we assume that $q_n(0)$ is given for all $n \in \mathbb{Z}$, and that $q_n(0) \in \ell^1(\mathbb{Z})$ (i.e. $\sum_{n\in\mathbb{Z}} |q_n(0)| < \infty$).

The Lax pair of the AL system is, after factoring out the term $\mathsf{Z}^n e^{-i\omega(z)t\sigma_3}$ from the eigenfunctions,

$$\mu_{n+1} - \mathsf{Z}\mu_n\mathsf{Z}^{-1} = \mathsf{Q}_n\mu_n\mathsf{Z}^{-1}, \quad \dot{\mu}_n + i\omega(z)[\sigma_3, \mu_n] = \mathsf{H}_n\mu_n, \tag{2.1}$$

where μ_n is the matrix eigenfunction, $\omega(z) = -(z - 1/z)^2/2$, $[\mathsf{A}, \mathsf{B}] = \mathsf{AB} - \mathsf{BA}$ and

$$\mathsf{Z} = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}, \quad \mathsf{Q}_n(t) = \begin{pmatrix} 0 & q_n(t) \\ r_n(t) & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2a}$$

$$\mathsf{H}_n(z,t) = i\sigma_3\big(\mathsf{Q}_n\mathsf{Z}^{-1} - \mathsf{Q}_{n-1}\mathsf{Z} - \mathsf{Q}_n\mathsf{Q}_{n-1}\big) = i\begin{pmatrix} -q_n r_{n-1} & zq_n - q_{n-1}/z \\ zr_{n-1} - r_n/z & r_n q_{n-1} \end{pmatrix}, \tag{2.2b}$$

where $r_n(t) = \nu q_n^*(t)$ and the asterisk denotes complex conjugate. As usual, the Jost solutions are defined as the eigenfunctions of the Lax pair (2.1) which reduce to the identity matrix as $n \to \mp\infty$:

$$\begin{aligned} \mu_n^{(1)}(z,t) &= \mathsf{I} + \mathsf{Z}^{-1} \sum_{m=-\infty}^{n-1} \mathsf{Z}^{n-m}\mathsf{Q}_m\mu_m^{(1)}\mathsf{Z}^{m-n}, \\ \mu_n^{(2)}(z,t) &= \mathsf{I} - \mathsf{Z}^{-1} \sum_{m=n}^{\infty} \mathsf{Z}^{n-m}\mathsf{Q}_m\mu_m^{(2)}\mathsf{Z}^{m-n}. \end{aligned} \tag{2.3}$$

Hereafter we denote the columns of the eigenfunctions as $(\mu_n^{(m,L)}, \mu_n^{(m,R)}) =: \mu_n^{(m)}(z,t)$, for $m = 1, 2$. From (2.3), one obtains the analyticity regions for each column as

$$\mu_n^{(1,L)}, \mu_n^{(2,R)}\colon |z| > 1, \quad \mu_n^{(1,R)}, \mu_n^{(2,L)}\colon |z| < 1. \tag{2.4}$$

The first of (2.1) also implies

$$\det \mu_n^{(1)} = \prod_{m=-\infty}^{n-1} (1 - q_m r_m), \quad \det \mu_n^{(2)} = \prod_{m=n}^{\infty} (1 - q_m r_m)^{-1} =: 1/C_n. \tag{2.5}$$

As usual, we assume that $q_n r_n \neq 1$ for $\forall n \in \mathbb{Z}$ and that $C_{-\infty} = \lim_{n\to-\infty} C_n$ is finite. Then $\mu_n^{(1)}$ and $\mu_n^{(2)}$ are both fundamental matrix solutions of the scattering problem (2.1). Thus one can introduce the scattering matrix $\mathsf{A}(z)$ as

$$\mu_n^{(1)}(z,t) = \mu_n^{(2)}(z,t) \mathsf{Z}^n \, \mathrm{e}^{-i\omega(z)\sigma_3 t} \mathsf{A}(z) \, \mathsf{Z}^{-n} \mathrm{e}^{i\omega(z)\sigma_3 t}, \quad |z| = 1. \tag{2.6}$$

With the above definitions, $\mathsf{A}(z)$ is independent of time. It can also be expressed as

$$\mathsf{A}(z) = \mathsf{I} + \mathsf{Z}^{-1} \sum_{n=-\infty}^{\infty} \mathsf{Z}^{-n} \mathrm{e}^{-i\omega(z)\sigma_3 t} \mathsf{Q}_n(t) \mu_n^{(1)}(z,t) \mathsf{Z}^n \mathrm{e}^{i\omega(z)\sigma_3 t}, \tag{2.7}$$

from which it follows that $a_{11}(z)$ and $a_{22}(z)$ can be analytically continued off the unit circle, respectively, into the domains $|z| > 1$ and $|z| < 1$. The reflection coefficients, which will appear in the inverse problem, are nowhere analytic and are only defined on $|z| = 1$:

$$\rho_1(z) = a_{21}(z)/a_{11}(z), \quad \rho_2(z) = a_{12}(z)/a_{22}(z). \tag{2.8}$$

The discrete eigenvalues of the scattering problem (2.1) are the zeros of the analytic scattering coefficients; $a_{11}(z)$ and $a_{22}(z)$. We assume that there are finite number of simple zeros of $a_{22}(z)$ and $a_{11}(z)$. We denote by z_j the discrete eigenvalues corresponding to the zeros of $a_{22}(z)$ for $j = 1, \ldots, J$, with $|z_j| < 1$ and $\mathrm{Im}\, z_j \geq 0$. Similarly, we denote by $\bar{z}_j$ the zeroes of $a_{11}(z)$ for $j = 1, \ldots, \bar{J}$, with $|\bar{z}_j| > 1$ and $\mathrm{Im}\, \bar{z}_j \geq 0$. From the Wronskian representations of the scattering coefficients we have

$$\mu_n^{(1,R)}(z_j,t) = b_j z_j^{2n} \mathrm{e}^{-2i\omega(z_j)t} \mu_n^{(2,L)}(z_j,t). \quad \mu_n^{(1,L)}(\bar{z}_j,t) = \bar{b}_j \bar{z}_j^{-2n} \mathrm{e}^{2i\omega(\bar{z}_j)t} \mu_n^{(2,R)}(\bar{z}_j,t), \tag{2.9}$$

for some complex constants b_j and $\bar{b}_j$. Moreover, (2.9) imply the residue relations:

$$\operatorname*{Res}_{z=z_j} \left[\frac{\mu_n^{(1,R)}}{a_{22}} \right] = K_j z_j^{2n} \mathrm{e}^{-2i\omega(z_j)t} \, \mu_n^{(2,L)}(z_j), \quad \operatorname*{Res}_{z=\bar{z}_j} \left[\frac{\mu_n^{(1,L)}}{a_{11}} \right] = \bar{K}_j \bar{z}_j^{-2n} \mathrm{e}^{2i\omega(\bar{z}_j)t} \, \mu_n^{(2,R)}(\bar{z}_j), \tag{2.10}$$

where

$$K_j = b_j / a'_{22}(z_j), \quad \bar{K}_j = \bar{b}_j / a'_{11}(\bar{z}_j). \tag{2.11}$$

As customary, K_j and $\bar{K}_j$ (or equivalently, b_j and $\bar{b}_j$) are referred to as the norming constants. Hereafter, the prime will denote the derivative of a function with respect to its argument.

When $r_n(t) = \nu q_n^*(t)$, we have the well-known symmetries:

$$\mu_n^{(m,L)}(z,t) = \sigma_\nu \big(\mu_n^{(m,R)}(1/z^*,t)\big), \quad \mu_n^{(m,R)}(z,t) = \nu\sigma_\nu \big(\mu_n^{(m,L)}(1/z^*,t)\big)^*.$$

We can thus write the scattering matrix in terms of just two coefficients as

$$\mathsf{A}(z) = \begin{pmatrix} a^*(1/z^*) & b(z) \\ \nu b^*(1/z^*) & a(z) \end{pmatrix}. \tag{2.12}$$

The above symmetries also imply

$$J = \bar{J}, \quad \bar{z}_j = 1/z^*, \quad \bar{b}_j = -b_j^*, \quad \bar{K}_j = (z_j^*)^{-2} K_j^*. \tag{2.13}$$

An additional symmetry also exists for the scattering coefficients [27]:

$$a(z) = a(-z), \quad b(z) = -b(-z). \tag{2.14}$$

Together, (2.12) and (2.14) imply that discrete eigenvalues appear as quartets: $\{\pm z_j, \pm \bar{z}_j\}_{j=1}^{J}$, with $\bar{z}_j = 1/z_j^*$ for all $j = 1, \ldots, J$. Also, (2.14) implies that the norming constants at $z = \pm z_j$ are identical, and so are those at $z = \pm\bar{z}_j$.

From (2.6), one can define the inverse problem in terms of the following matrix Riemann–Hilbert problem (RHP):

$$\mathsf{M}_n^+(z,t) - \mathsf{M}_n^-(z,t) = \mathsf{M}_n^+(z,t)\mathsf{J}_n(z,t), \tag{2.15a}$$

where, introducing the normalization matrix $\mathsf{C}_n(t) = \mathrm{diag}(1, C_n(t))$, the sectionally meromorphic matrices $\mathsf{M}_n^\pm$ are

$$\mathsf{M}_n^+(z,t) = \mathsf{C}_n \left(\mu_n^{(2,L)}(z,t),\ \frac{\mu_n^{(1,R)}(z,t)}{a(z)} \right), \tag{2.15b}$$

$$\mathsf{M}_n^-(z,t) = \mathsf{C}_n \left(\frac{\mu_n^{(1,L)}(z,t)}{a^*(1/z^*)},\ \mu_n^{(2,R)}(z,t) \right), \tag{2.15c}$$

and

$$\mathsf{J}_n(z,t) = \begin{pmatrix} \rho_1(z)\rho_2(z) & z^{2n}\mathrm{e}^{-2i\omega(z)t}\rho_2(z) \\ -z^{-2n}\mathrm{e}^{2i\omega(z)t}\rho_1(z) & 0 \end{pmatrix}. \tag{2.15d}$$

Thanks to the presence of C_n, it is $\mathsf{M}_n^-(z,t) \to \mathsf{I}$ as $z \to \infty$, while

$$\mathsf{M}_n^+(z,t) = \begin{pmatrix} 1/C_n & 0 \\ 0 & C_n \end{pmatrix} + \begin{pmatrix} 0 & q_{n-1} \\ r_n & 0 \end{pmatrix} z + O(z^2) \quad \text{as } z \to 0. \tag{2.16}$$

The solution of the RHP is simply found using Cauchy projectors after subtracting out the pole contributions:

$$\mathsf{M}_n(z,t) = \mathsf{I} + \sum_{j=1}^{J} \left(\frac{1}{z \mp z_j} \operatorname*{Res}_{z=\pm z_j} [\mathsf{M}_n^-(z)] + \frac{1}{z \mp \bar{z}_j} \operatorname*{Res}_{z=\pm\bar{z}_j} [\mathsf{M}_n^+(z)] \right) + \frac{1}{2\pi i} \int_{|\zeta|=1} \mathsf{M}_n^+(\zeta,t) \frac{\mathsf{J}_n(\zeta,t)}{\zeta - z} \,\mathrm{d}\zeta. \tag{2.17}$$

Computing the asymptotic behaviour of M_n^+ as $z \to 0$ and comparing with (2.16) then yields the reconstruction formula for the potential as

$$q_n(t) = -2\sum_{j=1}^{J} K_j z_j^{2n} \mathrm{e}^{-2i\omega(z_j)t} \mu^{(2)}_{n+1,11}(z_j, t) + \frac{1}{2\pi i}\int_{|z|=1} z^{2n}\mathrm{e}^{-2i\omega(z)t}\rho_2(z)\mu^{(2)}_{n+1,11}(z,t)\,\mathrm{d}z. \tag{2.18}$$

In the reflectionless case with $\nu = -1$, that is, when $\rho(z) = 0$ for all $|z| = 1$, (2.17) and (2.18) yield the pure soliton solutions of the AL system:

$$q_n(t) = \sum\nolimits_{j=1}^{J} Z_j, \tag{2.19a}$$

where $\mathbf{Z} = (Z_1, \ldots, Z_J)^T$ solves the algebraic system of equations

$$(\mathsf{I} - \mathsf{G})\mathbf{Z} = \mathbf{y}, \tag{2.19b}$$

with $\mathsf{G} = (G_{j,m})$ and $\mathbf{y} = (y_1, \ldots, y_J)^T$, and where $y_j = -2z_j^{2n}\,\mathrm{e}^{-2i\omega(z_j)t}K_j$ and

$$G_{j,m} = K_j z_j^{2n}\mathrm{e}^{-2i\omega(z_j)t}\sum_{p=1}^{J}\left(\frac{1}{z_j - \bar z_p} - \frac{1}{z_j + \bar z_p}\right)\left(\frac{1}{\bar z_p - z_m} + \frac{1}{\bar z_p + z_m}\right)\bar K_p\,\bar z_p^{-2(n-1)} z_m^2 \mathrm{e}^{2i\omega(\bar z_p)t}, \tag{2.19c}$$

for all $j, m = 1, \ldots, J$. For a single quartet (i.e. for $J = 1$), one recovers the well-known one-soliton solution of the AL system:

$$q_n^{(1s)}(t) = \mathrm{e}^{i[n\beta + wt + \varphi]}\sinh\alpha\,\mathrm{sech}[\alpha(n - vt - \delta)], \tag{2.20}$$

where $z_1 = \mathrm{e}^{(\alpha + i\beta)/2}$, and

$$v = (2/\alpha)\sinh\alpha\sin\beta, \quad w = 4(\cosh\alpha\cos\beta - 1), \quad \varphi = \pi + \arg K_1,$$
$$\delta = \big[\log(\sinh\alpha) - \log|K_1|\big]/\alpha.$$

3. IBVPs for the AL system with certain linearizable BCs

We now consider IBVPs for the AL system on the natural numbers with certain kinds of linearizable BCs. Note first that from the n-part of the Lax pair (2.1) we have

$$\mu_n(z) = (\mathsf{Z} + \mathsf{Q}_n)^{-1}\mu_{n+1}(z)\mathsf{Z}.$$

Letting $w_n(z,t) = C_n(t)\mu_n(z,t)$, and noting $\det(\mathsf{Z} + \mathsf{Q}_n) = 1 - q_n r_n$, we have that w_n solves

$$w_n - \mathsf{Z}^{-1}w_{n+1}\mathsf{Z} = -\mathsf{Q}_n w_{n+1}\mathsf{Z}. \tag{3.1}$$

Replacing n by $-n$ and z by $1/z$, and letting $\psi_n(z,\ t) = w_{-n+1}(1/z, t)$, we then have

$$\psi_{n+1} - \mathsf{Z}\psi_n\mathsf{Z}^{-1} = -\mathsf{Q}_{-n}\psi_n\mathsf{Z}^{-1}. \tag{3.2}$$

We will use (3.2) in Sections 3.1 and 3.2, and (3.1) in Section 3.3.

3.1. *Odd extension*

We consider first the IBVP with a homogeneous Dirichlet BC given; namely, $q_0(t) = 0$. We introduce the odd extension of the potential as

$$\mathsf{Q}_n^{\text{ext}}(t) = \mathsf{Q}_n(t)\Theta(n) - \mathsf{Q}_{-n}(t)\Theta(-n-1), \tag{3.3}$$

which is the discrete analogue of the odd extension in the continuum case [5,23]. Hereafter, $\Theta(n)$ is the Heaviside theta function, defined as $\Theta(n) = 1$ for $n \geq 0$ and $\Theta(n) = 0$ for $n < 0$. Obviously, $\mathsf{Q}_n^{\text{ext}}(t)$ satisfies the BC at $n = 0$. We then define the Jost solutions via (2.3) with $\mathsf{Q}_n(t)$ replaced by $\mathsf{Q}_n^{\text{ext}}(t)$, and we use the IST for the IVP to solve to the IBVP. It should be clear that the symmetries of the IVP are still valid. Moreover, we show next that the following symmetries apply for the Jost solutions and the scattering data:

$$\mu_n^{(1)}(z,t) = C_{-n+1}(t)\,\mu_{-n+1}^{(2)}(1/z,t), \quad \mathsf{A}(1/z) = C_{-\infty}\,\mathsf{Z}\mathsf{A}^{-1}(z)\mathsf{Z}^{-1}, \tag{3.4}$$

where

$$C_{-\infty} = \prod_{m=-\infty}^{\infty}(1 - q_m r_m). \tag{3.5}$$

Note the shift $-n \to -n+1$ in the first of (3.4), which disappears in the continuum limit.

To prove the first of (3.4) we use (3.2) and the fact that $\mathsf{Q}_n^{\text{ext}}(t)$ is odd: $\mathsf{Q}_{-n}^{\text{ext}}(t) = -\mathsf{Q}_n^{\text{ext}}(t)$. That is, if $\mu(z,t)$ solves the Lax pair, so does $\psi_n(z,t)$. Moreover, if $\mu(z,t) = \mu_n^{(2)}(z,t)$ in (2.1), we have $\psi_n \to \mathsf{I}$ as $n \to -\infty$. That is, $\psi_n(z,t)$ satisfies the same BCs of $\mu_n^{(1)}(z,t)$. By uniqueness, we thus have $\psi_n(z,t) = \mu_n^{(1)}(z,t)$, which proves the first of (3.4).

To prove the second of (3.4), note that (2.6) and the first of (3.4) imply

$$C_n(t)C_{-n+1}(t) = \mathsf{Z}^{-1}\mathsf{A}(1/z)\mathsf{Z}\mathsf{A}(z). \tag{3.6}$$

However, the odd extension of the potential implies, for all $n \geq 0$,

$$C_{-n+1}(t) = \prod_{m=1}^{n-1}(1 - q_m r_m)\prod_{m=-\infty}^{0}(1 - q_m^{\text{ext}}\, r_m^{\text{ext}}). \tag{3.7}$$

Hence $C_n(t)C_{-n+1}(t) = C_{-\infty}$, and the second of (3.4) then follows.

In particular, using $\det\mathsf{A} = C_{-\infty}$, the second of (3.4) yields

$$a(z) = a^*(z^*), \quad b(1/z) = -z^2 b(z). \tag{3.8}$$

Equations (3.8) will provide the key to characterize the soliton solutions to the IBVP. An obvious consequence is that discrete eigenvalues appear in symmetric octets (as opposed to quartets in the IVP):

$$\pm z_j, \quad \pm z_j^*, \quad \pm \bar{z}_j, \quad \pm \bar{z}_j^*, \tag{3.9}$$

where $\bar{z}_j = 1/z_j^*$ as before, with $|z_j| < 1$ and $\mathrm{Im} z_j \geq 0$ for all $j = 1, \ldots, J$.

3.2. *Even extension*

Similarly to Section 3.1, we now introduce the even extension of the potential as

$$\mathsf{Q}_n^{\mathrm{ext}}(t) = \mathsf{Q}_n(t)\Theta(n) + \mathsf{Q}_{-n}(t)\Theta(-n-1). \tag{3.10}$$

The corresponding symmetries of the eigenfunctions and the scattering matrix are

$$\mu_n^{(1)}(z,t) = C_{-n+1}(t)\,\sigma_3\mu_{-n+1}^{(2)}(1/z,t)\sigma_3, \quad \mathsf{A}(1/z) = C_{-\infty}\,\mathsf{Z}\sigma_3\mathsf{A}^{-1}(z)\sigma_3\mathsf{Z}^{-1}. \tag{3.11}$$

To prove (3.11), we multiply (3.2) by σ_3 both from the left and the right, we use the property $\mathsf{Q}_{-n}^{\mathrm{ext}}(t) = \mathsf{Q}_n^{\mathrm{ext}}$ and we note that $\sigma_3\mathsf{B} = \mathsf{B}\sigma_3$ and $\sigma_3\mathsf{B} = -\mathsf{B}\sigma_3$ if B is a diagonal matrix and an off-diagonal matrix, respectively. We then obtain that the matrix function $\sigma_3\psi_{n+1}(z,t)\sigma_3$ is a solution of the Lax pair in (2.1). Also, if $\mu_n = \mu_n^{(2)}$ in (2.1), $\sigma_3\psi_{n+1}\sigma_3 \to \mathsf{I}$ as $n \to -\infty$. Hence, by uniqueness, we conclude that $\sigma_3\psi_{n+1}(z,t)\sigma_3 = \mu(z,t)$, thereby obtaining the first of (3.11). The second of (3.11) follows in a similar way as before.

From the second of (3.11) follow the symmetries of the scattering coefficients:

$$a(z) = a^*(z^*), \quad b(1/z) = z^2 b(z). \tag{3.12}$$

These imply that the symmetry (3.9) of the discrete spectrum also holds for the even extension of the potential. Note, however, that $\mathsf{Q}_n^{\mathrm{ext}}(t)$ does not satisfy the BC (1.2) for any value of α. Instead, it satisfies the BC $q_1(t) - q_{-1}(t) = 0$ for a modified IBVP posed on $n \in \mathbb{N}_0$ instead of $n \in \mathbb{N}$. As far as we know, this kind of IBVP had not been previously studied in the literature.

3.3. *Even-shift extension*

Motivated by the results of Section 3.2, to solve the IBVP for the AL system with the BC (1.2) with $\alpha = 1$ we introduce the following shifted version of the even extension:

$$\mathsf{Q}_n^{\mathrm{ext}}(t) = \mathsf{Q}_n(t)\Theta(n) + \mathsf{Q}_{-n+1}(t)\Theta(-n). \tag{3.13}$$

For brevity, we will refer to this as the 'even-shift' extension of the potential. The extended potential now enjoys the shifted symmetry

$$\mathsf{Q}_{-n}^{\mathrm{ext}}(t) = \mathsf{Q}_{n+1}^{\mathrm{ext}}(t), \tag{3.14}$$

for all $n \in \mathbb{Z}$. It is then obvious that the corresponding extension of the AL field, namely, $q_n^{\mathrm{ext}}(t) = q_n(t)\Theta(n) + q_{-n+1}(t)\Theta(-n)$, satisfies the BC (1.2) with $\alpha = 1$. We next show that the symmetries of the eigenfunctions and scattering coefficients are

$$\mu_n^{(1)}(z,t) = C_{-n+2}(t)\,\sigma_3\mu_{-n+1}^{(2)}(1/z,t)\sigma_3, \quad \mathsf{A}(1/z) = C_{-\infty}\,\mathsf{Z}^2\sigma_3\mathsf{A}^{-1}(z)\sigma_3\mathsf{Z}^{-2}. \tag{3.15}$$

Starting from (3.1), let $\psi_{n+1}(z,t) = \sigma_3 w_{-n+1}(1/z,t)\sigma_3$. Using the symmetry of the shifted potential and noting that $\sigma_3 \mathsf{Q}_n^{\text{ext}}\sigma_3 = -\mathsf{Q}_n^{\text{ext}}$, we then find, after re-indexing,

$$\psi_{n+1} - \mathsf{Z}\psi_n\mathsf{Z}^{-1} = \mathsf{Q}_n^{\text{ext}}\psi_n\mathsf{Z}^{-1}. \tag{3.16}$$

Comparing the asymptotic behaviour of $\psi_n(z,t)$ with that of $\mu_n^{(1)}$ as before, we then obtain the symmetry of the eigenfunctions as the first of (3.15). Performing similar steps as before and using the fact that $C_n(t)C_{-n+2}(t) = C_{-\infty}$ thanks to the even-shift extension of the potential. From the scattering relation (2.6) we then obtain the following symmetry of the scattering matrix as the second of (3.15).

As a result of (3.15), we have the following symmetries for the scattering coefficients:

$$a(z) = a^*(z^*), \quad b(1/z) = z^2 b(z). \tag{3.17}$$

In particular, the first of these implies that the symmetry (3.9) of the discrete spectrum also holds for the IBVP with BCs (1.2) with $\alpha = 1$.

4. Normalization constant $C_{-\infty}$ in terms of scattering data

We will see later that the relation between the norming constants associated with symmetric eigenvalues involves the overall normalization constant $C_{-\infty}$ defined in (3.5). It is well-known that $C_{-\infty}$ is a conserved quantity [27]. In order for the method to provide an effective characterization of the discrete spectrum and the soliton solutions of the IBVP, however, one must be able to express $C_{-\infty}$ only in terms of the scattering data. We next show that this indeed is the case. Interestingly, the relation that we will derive holds for all solutions of the AL system on the integers; i.e. it applies also to the solutions of the IVP, not just to those of the IBVP with extended potential.

In the reflectionless case with $\nu = -1$, one can in principle compute explicitly $C_{-\infty}$ in terms of the eigenvalues as follows. In the reflectionless case, (2.17) yields

$$\mathsf{M}_n^{(+,R)}(z,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{J} K_j z_j^{2n} \mathrm{e}^{-2i\omega(z_j)t}\left(\frac{1}{z - z_j}\mathsf{M}_n^{(+,L)}(z_j,t) + \frac{1}{z + z_j}\mathsf{M}_n^{(+,L)}(-z_j,t)\right). \tag{4.1}$$

Comparing this with (2.16), and noting that $\mathsf{M}_{n,21}^{+}$ and $\mathsf{M}_{n,22}^{+}$ are, respectively, an odd function and even function of z[16,27], we then have

$$C_n(t) = 1 - 2\sum_{j=1}^{J} K_j z_j^{2n-1}\mathrm{e}^{-2i\omega(z_j)t}\mathsf{M}_{n,21}^{+}(z_j,t), \tag{4.2}$$

where $\mathsf{M}_{n,21}^{+}(z_j,t)$ can be obtained from the solution of an algebraic system obtained by evaluating Equations (2.17) at $z = z_j$ and $z = \bar{z}_j$ with $\rho(z) \equiv 0$. For example, when $J = 1$, solving this algebraic system, substituting in (4.2) and taking the limit $n \to -\infty$ one obtains $C_{-\infty} = (\bar{z}_1/z_1)^2$. It is also straightforward, although tedious, to show that when $J = 2$, one obtains $C_{-\infty} = [\bar{z}_1\bar{z}_2/(z_1 z_2)]^2$. The calculations with this approach, however, quickly become cumbersome for larger systems, and the method cannot be used for solutions with a nonzero reflection coefficient.

We now present a different method to obtain $C_{-\infty}$, which results in an explicit expression in terms only of the scattering data. Note first that $\det \mathsf{A}(z) = C_{-\infty} = |a(z)|^2 - \nu |b(z)|^2$, where $a(z) = a_{22}(z)$ and $b(z) = a_{12}(z)$ as before. This implies, for all $|z| = 1$,

$$C_{-\infty} = |a(z)|^2(1 + |\rho(z)|^2) \tag{4.3}$$

where as before $\rho(z) = b(z)/a(z)$ and $\nu = -1$. Multiplying (4.3) by $1/z$ and integrating over the unit circle we then have

$$\log C_{-\infty} = \frac{1}{2\pi i}\int_{|z|=1} \log|a(z)|^2 \frac{\mathrm{d}z}{z} + \frac{1}{2\pi i}\int_{|z|=1} \log(1 + |\rho(z)|^2)\frac{\mathrm{d}z}{z}. \tag{4.4}$$

A further simplification can be obtained by recalling that the analytic scattering coefficients obeys trace formulae. In particular, for all $|z| \le 1$, it is [27]

$$\log a(z) = \sum_{j=1}^{J} \log\left(\frac{z^2 - z_j^2}{z^2 - \bar{z}_j^2}\right) + \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\log(1 + |\rho(\zeta)|^2)}{\zeta^2 - z^2}\,\mathrm{d}\zeta, \tag{4.5}$$

where again $\bar{z}_j = 1/z^*$. An identical relation holds for $a_{11}(z)$, except for the interchange of z_j and $\bar{z}_j$ in the sum and a minus sign in front of the integral. Recall that, on $|z| = 1$ it is $a_{11}(z) = a^*(z)$. We can then use (4.5) and the corresponding relation for $a_{11}(z)$ in the first integral in (4.4), obtaining

$$\log C_{-\infty} = 2\sum_{j=1}^{J} \log(\bar{z}_j/z_j) + \frac{1}{2\pi i}\int_{|z|=1} \log(1 + |\rho(z)|^2)\frac{\mathrm{d}z}{z}, \tag{4.6}$$

where the integral is obviously absent for reflectionless solutions. This relation, which holds for all solutions of the IVP for the AL system, was not known in the literature to the best of our knowledge. For reflectionless soliton solutions of the IBVP one can further use the symmetries of the discrete spectrum to obtain simply

$$C_{-\infty} = 1/|z_1 \cdots z_J|^2. \tag{4.7}$$

5. Relations between norming constants and symmetric eigenvalues

We denote the eigenvalue symmetric to z_j as $z_{j'} = z_j^*$. Note that real eigenvalues are self-symmetric; that is, $\operatorname{Im} z_j = 0$, implies $z_{j'} = z_j$. Thus, for these eigenvalues the octet (3.9) degenerates into a quartet. We can therefore divide the discrete eigenvalues in two classes by writing $J = J_0 + J_1$, where J_0 and J_1 are the number of discrete eigenvalues z_j inside the unit circle with $\operatorname{Im} z_j = 0$ and $\operatorname{Im} z_j > 0$, respectively. The corresponding numbers of eigenvalues in the IBVP are $S_0 = J_0$ and $S_1 = J_1/2$, with $S = S_0 + S_1$.

5.1. *Derivative of the analytic scattering coefficients*

Recall from (2.11) that the norming constants contain $a'(z)$. Also recall that $a(z)$ can be obtained via the trace formula (4.5). In particular, in the reflectionless case with $J = 1$, one obtains simply

$$a'(z_1) = 2z_1/[z_1^2 - 1/(z_1^*)^2]. \tag{5.1}$$

In the general reflectionless case with $J > 1$, it is, for all $j = 1, \ldots, J$,

$$a'(z_j) = \frac{2z_j}{z_j^2 - 1/(z_j^*)^2} \prod_{m=1}^{J}{}' \frac{z_j^2 - z_m^2}{z_j^2 - 1/(z_m^*)^2}, \tag{5.2}$$

where the prime in the product indicates that the term $m = j$ is omitted. Using the symmetry of the discrete eigenvalues one then obtains the corresponding expressions in the IBVP as

$$a'(z_s) = \frac{2z_s}{z_s^2 - 1/(z_s^*)^2} \prod_{p=1}^{S_0}{}' \frac{z_s^2 - z_p^2}{z_s^2 - 1/(z_p^*)^2} \prod_{m=1}^{S_1} \frac{(z_s^2 - z_m^2)(z_s^2 - (z_m^*)^2)}{(z_s^2 - (z_m^*)^{-2})(z_s^2 - z_m^{-2})}, \tag{5.3a}$$

for all $s = 1, \ldots, S_0$, and

$$a'(z_s) = \frac{2z_s(z_s^2 - (z_s^*)^2)}{(z_s^2 - (z_s^*)^{-2})(z_s^2 - z_s^{-2})} \prod_{p=1}^{S_0} \frac{z_s^2 - z_p^2}{z_s^2 - 1/(z_p^*)^2} \prod_{m=1}^{S}{}' \frac{(z_s^2 - z_m^2)(z_s^2 - (z_m^*)^2)}{(z_s^2 - (z_m^*)^{-2})(z_s^2 - z_m^{-2})}, \tag{5.3b}$$

for all $s = 1, \ldots, S_1$.

5.2. *Odd extension*

Note first that from (2.9), we have

$$\mu_n^{(1,R)}(z_{s'}) = b_{s'} z_{s'}^{2n} \mathrm{e}^{-2i\omega(z_{s'})t} \mu_n^{(2,L)}(z_{s'}). \tag{5.4}$$

Using the symmetry (3.4) with (2.9), we also obtain

$$\mu_n^{(1,R)}(z_{s'}) = C_{-n+1} \bar{b}_s^{-1} \bar{z}_s^{-2(n-1)} \mathrm{e}^{-2i\omega(\bar{z}_s)t} \mu_{-n+1}^{(1,L)}(\bar{z}_s). \tag{5.5}$$

Finally, we know that $\mu_n^{(2,L)}(z_{s'}) = C_n^{-1} \mu_{-n+1}^{(1,L)}(\bar{z}_s)$. Recalling that $\bar{z}_s = 1/z_s^*$ and $\bar{b}_s = -b_s^*$ and combining these with the above equations we then obtain

$$b_s^* b_{s'} = -C_{-\infty}/(z_s^*)^2, \tag{5.6a}$$

from which we can derive the relations between the norming constants associated with symmetric eigenvalues. Note first, however, that since $C_{-\infty} > 0$ and self-symmetric eigenvalues are real, (5.6a) implies that there can be no self-symmetric eigenvalues in this case.

Now, using the symmetries of the scattering coefficients we have $a'_{11}(\bar{z}_s) = -(z_s^*)^2 a'_{22}(z_s)^*$ and $a'_{22}(z_{s'}) = a'_{22}(z_s)^*$. From the definitions of the norming constants K_j and $\bar{K}_j$, we then obtain

$$K_s^* K_{s'} = -C_{-\infty}/\big(z_s^* a'(z_s)^*\big)^2. \tag{5.6b}$$

By analogy with the one-soliton solution (2.20), let $z_j = \mathrm{e}^{(\alpha_j + i\beta_j)/2}$ and $K_j = \mathrm{e}^{\xi_j + i\eta_j}$, with

$$\xi_j = \log(\sinh \alpha_j) - \alpha_j \delta_j. \quad \eta_j = \varphi_j - \pi. \tag{5.7}$$

Using (5.6b) we have

$$\xi_s + \xi_{s'} = \log|C_{-\infty}| - 2\log|a'(z_s)| - \alpha_s, \tag{5.8a}$$

$$\eta_{s'} - \eta_s = \arg C_{-\infty} + 2\arg\big[a'(z_s)\big] + \beta_s + \pi. \tag{5.8b}$$

For one-soliton solutions of the IBVP (that is, for $S=S_1=1$), (4.7) is simply $C_{-\infty}=\mathrm{e}^{-4\alpha_1}$. Substituting this relation and (5.3b) into (5.8), one can then express the relation of the norming constants associated with symmetric eigenvalues only in terms of z_1:

$$\xi_1+\xi_2=2\log\left|\sinh\alpha_1\right|+2\log\left|\sinh(\alpha_1+i\beta_1)/\sin\beta_1\right|-8\alpha_1, \tag{5.9a}$$

$$\eta_2-\eta_1=2\arg\left[\sin\beta_1\operatorname{csch}(\alpha_1+i\beta_1)\right]+\pi. \tag{5.9b}$$

5.3. *Even extension*

Performing similar calculations as in the case of the odd extension we obtain

$$b_s^*b_{s'}=C_{-\infty}/(z_s^*)^2,\quad K_s^*K_{s'}=C_{-\infty}/\left(z_s^*a'(z_s)^*\right)^2. \tag{5.10}$$

As a result,

$$\xi_s+\xi_{s'}=\log|C_{-\infty}|-2\log|a'(z_s)|-\alpha_s, \tag{5.11a}$$

$$\eta_{s'}-\eta_s=\arg[C_{-\infty}]+2\arg\left[a'(z_s)\right]+\beta_s. \tag{5.11b}$$

Then, when $S=S_1=1$ one has the same relation as (5.9b) between the norming constants, except that the π at the end of (5.9b) is now absent. But (5.10) implies that self-symmetric eigenvalues are now also allowed. When $S=S_0=1$, we then obtain a condition that the self-symmetric soliton must satisfy in order for it to be a solution of the IBVP:

$$\xi_1=\log\left|\sinh\alpha_1\right|-2\alpha_1, \tag{5.12}$$

with η_1 arbitrary.

5.4. *Even-shift extension*

In a similar way as in the previous cases, from Equations (3.15) we obtain the relation of the norming constants as

$$b_s^*b_{s'}=C_{-\infty}/(z_s^*)^4,\quad K_s^*K_{s'}=C_{-\infty}/\left[(z_s^*)^2a'(z_s)^*\right]^2. \tag{5.13}$$

As a result

$$\xi_s+\xi_{s'}=\log|C_{-\infty}|-2\log|a'(z_s)|-2\alpha_s, \tag{5.14a}$$

$$\eta_{s'}-\eta_s=\arg[C_{-\infty}]+2\arg\left[a'(z_s)\right]+2\beta_s. \tag{5.14b}$$

Then in the case $S=S_1=1$ we have

$$\xi_1+\xi_2=2\log\left|\sinh\alpha_1\right|+2\log\left|\sinh(\alpha_1+i\beta_1)/\sin\beta_1\right|-9\alpha_1, \tag{5.15a}$$

$$\eta_2-\eta_1=2\arg\left[\sin\beta_1\operatorname{csch}(\alpha_1+i\beta_1)\right]+\beta_1. \tag{5.15b}$$

As in the even case, however, self-symmetric solitons are now possible. In particular, when $S = S_0 = 1$, we obtain

$$\xi_1 = \log\left|\sinh\alpha_1\right| - \frac{5}{2}\alpha_1, \tag{5.16}$$

with η_1 arbitrary as before.

5.5. *Multi-soliton solutions*

The above results are easily generalized to the case of multi-soliton solutions. Recall that (4.7) and (5.3) give, respectively, the general form of $C_{-\infty}$ and of $a'(z)$ for an arbitrary number of discrete eigenvalues. Inserting these expressions into (5.6), (5.10) and (5.13) one can easily obtain the analogue of (5.9) and (5.15). Since these extensions are straightforward, we omit the relevant formulae for brevity.

6. Soliton behaviour

We now discuss the behaviour of the soliton solutions of the AL system on the half-line. For simplicity we first restrict ourselves to the case in which no self-symmetric eigenvalues are present.

Recall that each discrete eigenvalue of the scattering problem is associated with a zero of $a(z)$ and generates a soliton travelling with constant velocity. Let $z_s = \mathrm{e}^{(\alpha_s + i\beta_s)/2}$ be one such eigenvalue. By the symmetry (3.9), we know that $z_{s'} = z_s^* = \mathrm{e}^{(\alpha_s - i\beta_s)/2}$ is also a discrete eigenvalue. Since the velocity of the soliton is $v_s = v(\alpha_s, \beta_s) = (2/\alpha_s)\sinh\alpha_s \sin\beta_s$, we have immediately $v_{s'} = v(\alpha_s, -\beta_s) = -v_s$. That is, solitons generated by symmetric eigenvalues travel with equal and opposite velocities. We refer to the soliton that appears to the right of the boundary (i.e. $n > 0$), as the *physical* soliton. The symmetry of the discrete spectrum and norming constants implies that there exists a counterpart to each physical soliton, generated by the symmetric eigenvalue, and located to the left of the boundary (i.e. $n < 0$). Since this counterpart can be considered a reflected image of the physical soliton, we refer to it as a *mirror* soliton. It should be obvious that the number of physical solitons equals that of mirror solitons.

Let us now discuss more in detail the behaviour of the soliton solutions. Once the eigenvalues and norming constants of the physical solitons are assigned, the eigenvalues and norming constants of the mirror solitons are determined by (5.9), (5.11) and (5.15) in such a way that the solution satisfies the given BCs. Solving the algebraic system (2.19) one then obtains the corresponding two-soliton solution of the AL system. Consider first the case $S = 1$ (i.e. $J = 2$). Figure 1 (a) shows such a solution for the IBVP with BC $q_0(t) = 0$. The physical soliton appears to be reflected at the boundary of the lattice. Since the soliton velocity is completely determined by the discrete eigenvalue, however, it cannot change with time. The contour plot (Figure 1b), which includes the mirror soliton, shows what is really going on: the two solitons interact at the boundary, and the amplitudes and velocities of both solitons return to their initial values after the interaction, but the roles of the physical and mirror solitons are interchanged. A similar scenario occurs in the case of the even and even-shift extensions, as shown respectively in Figures 2 and 3, the only

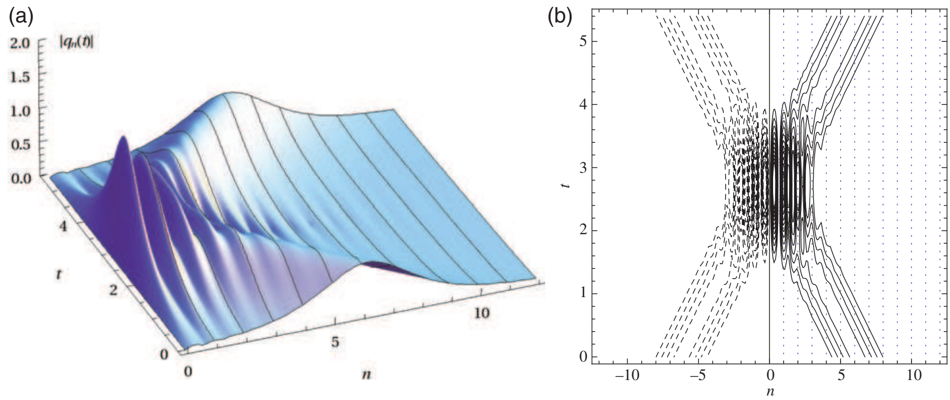

Figure 1. Soliton reflection at the boundary in the case of BCs $q_0(t)=0$, with $z_1=(-1+i)/2$, $\xi_1=4$, and $\arg K_1=\pi/2$. (a) three-dimensional (3D) plot of $|q_n(t)|$. (b) contour plot showing the mirror soliton (dashed) to the left of the boundary. Note that in both plots we allow n to be real-valued in the expression obtained from (2.11); integer values of n are shown as solid lines in the 3D plot and as dotted lines in the contour plot.

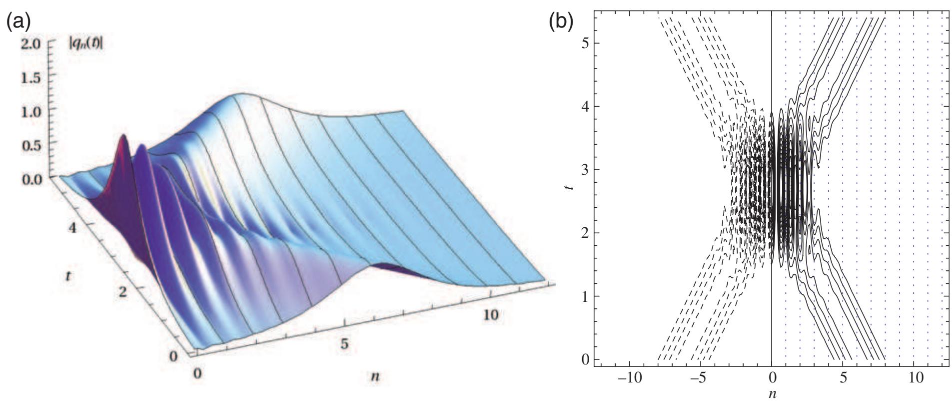

Figure 2. Same as Figure 1 but for the even extension, with $z_1=(-1+i)/2$, $\xi_1=4$, and $\arg K_1=\pi/2$.

difference being that the norming constant of the mirror soliton in each case is such that the appropriate BCs are satisfied.

Note that the symmetry of the discrete spectrum and the relations between eigenvalues and norming constants apply independently of whether the physical soliton has a positive or negative velocity. Of course, if the physical soliton has a positive velocity, no reflection occurs for $t>0$, and the solution is exponentially small at the origin for all $t>0$. Nonetheless, a mirror soliton is still needed to satisfy the BCs at the origin – as in the continuum case [23]. Note also that these results are not limited to reflectionless solutions, nor to solutions with $S=1$. Indeed, Figure 4 displays the reflection of two physical solitons.

It is convenient to label the discrete eigenvalues (including symmetric ones) so that $v_1 \le v_2 \le \cdots \le v_{2S}$. With this convention, as $t \to -\infty$ the physical solitons

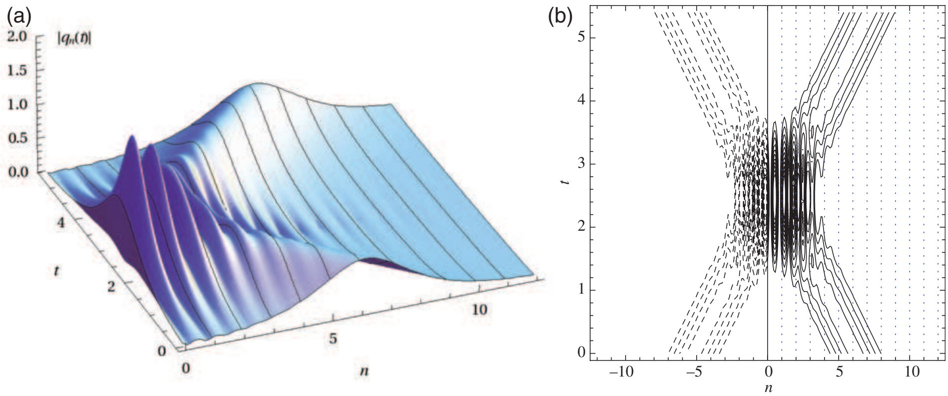

Figure 3. Same as Figure 1 but for the even-shift extension, with $z_1 = (1 - i)/2$, $\xi_1 = 4$, and $\arg K_1 = \pi/2$.

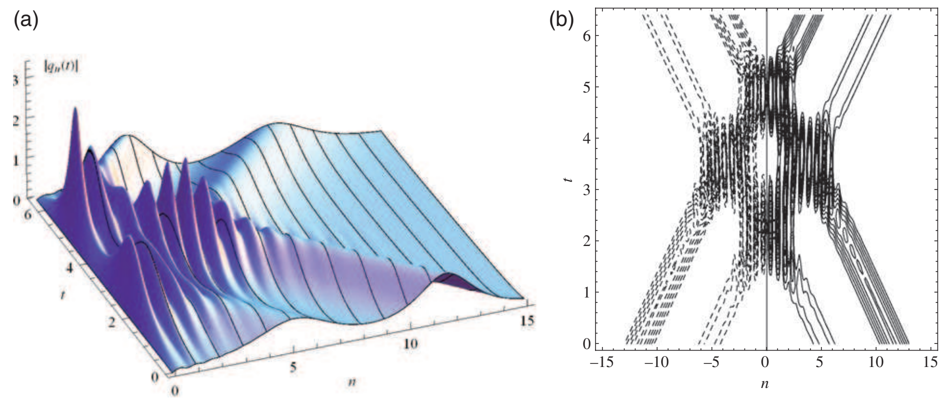

Figure 4. A multi-soliton solution of the IBVP with BC $q_0(t) = 0$, with $z_1 = (-1 + i)/2$, $\xi_1 = 6$, $z_2 = -1/3 + i/2$, $\xi_2 = 12$ and $\arg K_1 = \arg K_2 = \pi/2$. (a) 3D plot of $|q_n(t)|$. (b) Contour plot showing the mirror solitons (dashed).

correspond to eigenvalues in the second quadrant of the complex z-plane (i.e. those with $v_s < 0$). Then, as each soliton is reflected in succession, the corresponding discrete eigenvalue associated with the physical soliton switches role with its symmetric counterpart, until, as $t \to \infty$, all physical solitons correspond to the S discrete eigenvalues in the first quadrant (i.e. with $v_s > 0$). Thus, as $t \to -\infty$ the discrete eigenvalues associated with the physical solitons are $z_1, \ldots, z_S$, and the corresponding mirror solitons are, respectively, $z_{2S}, \ldots, z_{S+1}$. Conversely, as $t \to \infty$ the eigenvalues associated with the physical solitons are $z_{S+1}, \ldots, z_{2S}$, and the corresponding mirror solitons are, respectively, $z_S, \ldots, z_1$.

We now briefly discuss the situation in which self-symmetric eigenvalues are present. Since $z_{s'} = z_s^*$, the condition $z_{s'} = z_s$ implies $\beta_s = 0$, and therefore $v_s = 0$. That is, all solitons associated with self-symmetric eigenvalues are stationary, and no soliton reflection occurs. Multiple self-symmetric eigenvalues give rise to bound states of stationary solitons. Generically, the solution of the IBVP in the even and even-shift cases will contain a mixture of self-symmetric and non-self-symmetric eigenvalues.

7. Conclusions

In summary, we studied the solutions of the IBVP for the AL system with certain linearizable BCs at the origin, using a nonlinear method of images. We have characterized the symmetries of the discrete spectrum and norming constants, and we have discussed the behaviour of the soliton solutions. Our approach provides the discrete analogue of the results obtained in Refs. [5,23] for the NLS equation. We have also identified a new linearizable BC, which does not fall into the kind (1.2) that was studied previously.

We emphasize that the method we used here to solve the IBVP is fundamentally different from the one we used in Ref. [16], which was based on the simultaneous spectral analysis of both parts of the Lax pair. That method is more general, since it can deal with both linearizable and nonlinearizable BCs. For linearizable BCs, however, the present method has the advantage of being considerably simpler, and it easily allows one to characterize the soliton solutions explicitly. At the same time, it should be clear that the present results (symmetries of the discrete spectrum, norming constants and the characterization of the soliton solutions) only apply for linearizable BCs, and do not hold for generic BCs. A simple counterexample is given by the one-soliton solution (2.20), which provides a solution of the IBVP posed with IC $q_n(0) = q_n^{(1s)}(0)$ and BC $q_0(t) = q_0^{(1s)}(t)$ for which the discrete spectrum does not possess any symmetry.

We were not able to identify an extension of the potential suitable for solving the IBVP with generic values of $\alpha \neq 0$, 1 in (1.2). Thus, a proper characterization of the IBVP with the most general linearizable BCs is still at present an open issue. We believe that no simple extension of the potential exists in general. One promising approach instead might be to use an extension provided by certain Bäcklund transformations of the AL system [17], in a similar way to what was done for the NLS equation in Refs [7–10]. This, however, is a nontrivial task that is left for future work.

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