Math 141A - Final Exam - December 8, 2014

Instructions: The exam is worth 150 points. You should not use any aids, electronic or paper, other than a writing utensil.

1. (15 points) Evaluate the following integrals:

   a. $\int_{0}^{\pi} \sin(3t) \, dt = -\frac{\cos(3t)}{3} \bigg|_{0}^{\pi} = -\frac{1}{3} (\cos(3\pi) - \cos(0)) = -\frac{1}{3}(-1-1) = \frac{2}{3}$

   b. $\int \frac{4x+2}{x^2+x-1} \, dx$

      $u = x^2 + x - 1 \quad du = 2x \, dx$

      $= \int \frac{2u}{u-1} \, du = 2 \ln |u-1| + C = 2 \ln |x^2 + x - 1| + C$

   c. $\int_{0}^{2} y^2 \sqrt{1 + y^2} \, dy$

      $u = 1 + y^2 \quad du = 2y \, dy \quad y = 0 \rightarrow u = 1$

      $y = 2 \rightarrow u = 5$

      $= \int_{1}^{5} \frac{u^{1/2}}{2} \, du = \frac{2}{3} u^{3/2} \bigg|_{1}^{5} = \frac{2}{3} \cdot 27 = 54/9 = 6$

   d. $\int x^3 + 2x^2 + \frac{1}{x} \, dx$

      $= \left[ \frac{x^4}{4} + \frac{2x^3}{3} + \ln x \right] + C$

   e. $\int_{e}^{e^y} \frac{dx}{x \sqrt{\ln x}}$

      $u = \ln x \quad du = \frac{1}{x} \, dx \quad x = e \rightarrow u = 1$

      $x = e^y \rightarrow u = y$

      $= \int_{1}^{y} \frac{1}{u^{1/2}} \, du = 2\sqrt{u} \bigg|_{1}^{y} = \sqrt{y} - \sqrt{1} = \sqrt{y} - \sqrt{1} = \sqrt{y} - 1$
2. (15 points) a. Estimate the area under the curve \( y = 1 + x^2 \) for \( 0 \leq x \leq 2 \) using 4 intervals and right hand endpoints (i.e. compute \( R_4 \)). Is your answer an overestimate or underestimate? Explain.

\[
R_4 = \frac{1}{4} \left( f(1) + f(1) + f(\frac{3}{2}) + f(2) \right) = \frac{1}{4} \left( \frac{5}{4} + 2 + \frac{9}{4} + 5 \right) = \frac{1}{4} \left( \frac{24}{4} \right) = \frac{6}{4} = \frac{3}{2}
\]

\( \approx 1.5 \)  

overestimate because \( 1 + x^2 \) is increasing on \( [0, 2] \)

b. Now find the exact area by computing the following definite integral using the definition as a limit of Riemann sums:

\[
\int_0^2 1 + x^2 \, dx.
\]

\( \Delta x = \frac{2}{n} \)

\( x_i = \frac{2i}{n} \)

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} f(x_i) \Delta x}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( 1 + \left( \frac{2i}{n} \right)^2 \right) \frac{2}{n}
\]

\[
= \lim_{n \to \infty} \left( \frac{2}{n} \sum_{i=1}^{n} + \frac{8}{n^3} \sum_{i=1}^{n} \right) = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} i^2 + \lim_{n \to \infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2
\]

\[
= \frac{2}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}
\]

\[
= 2 + \lim_{n \to \infty} \frac{16n^3 + \ldots}{6n^3} = 2 + \frac{8}{3} = \frac{14}{3}
\]

Check: \[
\int_0^2 1 + x^2 \, dx = \left[ x + \frac{x^3}{3} \right]_0^2 = 2 + \frac{8}{3} = \frac{14}{3}
\]
3. (10 points) State both parts of the Fundamental Theorem of Calculus.

Let \( f(x) \) be continuous on \([a,b]\)

1. \[ \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \]

2. Suppose \( F'(x) = f(x) \)

   Then \[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

4. (10 points) Use the Fundamental Theorem of Calculus to find the derivative \( g'(x) \):

   a. \( g(x) = \int_{0}^{x} e^{t^2} \, dt \).

      \[ g'(x) = e^{x^2} \]

   b. \( g(x) = \int_{0}^{x} \cos(t) \, dt \).

      \[ g'(x) = 2x \cdot \cos(x^2) \]
5. (10 points) Consider the following limit:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{1 + (i/n)^2} \cdot \frac{1}{n}.$$ 

Recognize this limit as a definite integral and then evaluate it by evaluating the integral.

This is:

$$\int_{0}^{1} \frac{1}{1 + x^2} \, dx = \tan^{-1} x \bigg|_{0}^{1}$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

6. (10 points) Let $f(t) = t\sqrt{4 - t^2}$. Find the absolute maximum and absolute minimum of $f(t)$ on the interval $[-1, 2]$.

$$f'(t) = \sqrt{4 - t^2} + t \cdot \frac{-2t}{2\sqrt{4 - t^2}} = \frac{4 - t^2 - t^2}{\sqrt{4 - t^2}}$$

Set $f'(t) = 0$:

$$\Rightarrow \frac{4 - t^2}{\sqrt{4 - t^2}} = 0$$

$$\Rightarrow 4 - t^2 = 0$$

$$t^2 = 4$$

$$t = \pm 2$$

Crit. points in interval $t = \pm 2$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Absolute max value $= 2$

Absolute min value $= 0$
7. (5 points) Using the Fundamental Theorem of Calculus we conclude:

\[
\int_{-1}^{2} 3x^{-4} \, dx = -x^{-3} \bigg|_{-1}^{2} = -\frac{1}{x^3} \bigg|_{-1}^{2} = -(1/8 + 1) = -9/8.
\]

Why is this mistaken?

\[
\frac{3}{x^4} \text{ is not continuous on } [-1, 2]
\]

so FTOC does not apply,

8. (5 points) If \( f(x) \) represents the slope of a trail at a distance \( x \) miles from the start, what does \( \int_{0}^{6} f(x) \, dx \) represent?

The net change in elevation between mile marker 2 and 6.
9. (10 points) The formula for the derivative of a function \( f(x) \) at the value \( x = a \) is given by:

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

if it exists. Explain in words, and using diagrams, where this formula comes from.

\[
\frac{f(a+h) - f(a)}{h}
\]

is the slope of the secant line connecting \( (a, f(a)) \) and \( (a+h, f(a+h)) \).

As \( h \) gets smaller this slope approaches the slope of the tangent line.

Equivalently, \( \frac{f(a+h) - f(a)}{h} \) is an average rate of change of \( f(x) \) and letting \( h \to 0 \) we get an instantaneous rate of change.
10. (10 points) Let \( f(x) = x^3 + 2x + 1. \)

a. Find the equation of the tangent line at \( x = 2. \)

\[
\begin{align*}
f'(x) &= 3x^2 + 2 \\
f'(2) &= 14 \\
\text{point} \ (2, 13) \\
\text{slope} \\
y - 13 &= 14(x - 2)
\end{align*}
\]

b. Prove \( f(x) \) has one real root.

\( f(-1) = -4, \ f(0) = 1 \) so, since \( f \) is continuous on \([-1, 0]\), the Intermediate Value Theorem guarantees a root in \([-1, 0]\).

But \( f'(x) = 3x^2 + 2 \) is always \( > 0 \). If \( f \) had 2 roots, Rolle's Theorem would imply \( f' = 0 \) at some point between them.

11. (10 points) Complete the definitions:

a. \( \lim_{x \to a} f(x) = L \) if ...

For any \( \epsilon > 0 \) there is a \( \delta > 0 \) so that if \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \epsilon. \)

b. A function \( f(x) \) is continuous at \( x = a \) if ...

\[
\lim_{x \to a} f(x) = f(a)
\]
12. (10 points) Find \( \frac{dy}{dx} \).

a. \( y = xe^x \)

\[ e^x + xe^x \]

b. \( y = 6x^2 + 3x + 5 \)

\[ 12x + 3 \]

c. \( y = (1 + \cos(x))^{10} \)

\[ 10(1+\cos x)^9 (-\sin x) \]

d. \( y = \frac{1+x}{1-\ln(x)} \)

\[ \frac{1-\ln x - (1+\ln x) \left( \frac{-1}{x^2} \right)}{(1-\ln x)^2} \]

e. \( xy^2 + \cos(y) = 2x \)

\[ y^2 + 2xxy' - \sin y \cdot y' = 2 \]

\[ y'(2xy - \sin y) = 2 - y^2 \]

\[ y' = \frac{2 - y^2}{2xy - \sin y} \]
13. (15 points)

a. Find the equation of the line passing through \((-2, 1)\) and parallel to the line \(2x + 3y = \frac{2}{5}\). 

\[ y - 1 = -\frac{2}{3}(x + 2) \]

b. Solve the inequality \(x^2 - x - 6 \geq 0\).

\((x-3)(x+2) = 0\)

\(\frac{1}{2} \quad -1 \quad 3 \quad \frac{1}{2} \)

\((-\infty, -2) \cup (3, \infty)\)

c. Find \(\log_3(27)\).

\(3\)

d. Write \(\frac{\cos x}{1+x} + \frac{x}{2+2x}\) as a single fraction.

\[
\frac{\cos x (2tx + x(1+tx))}{(1+x)(2x+1)}
\]

\(3tx = -3\)

\(x = e^{-3}\)

e. Neatly sketch and label the graph of \(y = 2 \sin x\) and \(y = 3 + \ln x\) on the same axes.
14. (5 points) Let $f(x) = 2x^2 + 3x + 1$. Find an $x$ value $c$ such that $c$ satisfies the conclusion of the Mean Value Theorem for $f(x)$ on the interval $[1, 4]$.

$$\frac{f(4) - f(1)}{4 - 1} = \frac{y_5}{3}$$

$$f' = 4x + 3$$

Set $4x + 3 = \frac{y_5}{3}$

$$4x = \frac{30}{3} - 12$$

$$x = 3$$

15. (10 points) Below is the graph of $y = f(t)$ for $0 \leq t \leq 27$. (Please note the scale on the axis, each unit is 1.5)
a. For what intervals is \( f(t) \) increasing or decreasing? Concave up or concave down?

\[
\text{Inc: } (0, 15) \cup (16, 21) \cup (24, 27) \\
\text{Dec: } (15, 16) \cup (18, 24) \cup (27, 24)
\]

b. Estimate \( f'(15) \).

\[-3\]

c. Estimate \( \int_0^6 f(t) dt \).

\[
\int_0^6 f(t) dt \approx -6
\]

d. Now define \( g(x) = \int_0^x f(t) dt \). For what intervals is \( g(x) \) increasing or decreasing?

\[
\text{Increasing: } (9, 13) \cup (19, 15) \cup (21, 27) \\
\text{Decreasing: } (3, 9) \cup (15, 21)
\]

e. What is \( g'(12) \)?

\[
g'(12) = f(12) = 2
\]

f. At what values does \( g(x) \) obtain local maximums? Global maximum?

\[
\text{Local max: } x = 3, 15, 27 \\
\text{Global max: } x = 27
\]