Math 141- Midterm Exam #2 - October 22, 2014

1. (50 points) Find $\frac{dy}{dx}$. You do not need to simplify your answers.

a. $y = xe^{\cos x}$

$$y' = e^{\cos x} - x\sin x e^{\cos x}$$

b. $y = \log_3(x)$

$$y' = \frac{1}{x \ln 3}$$

c. $y = \ln(x^2 + 2x + 1)$

$$y' = \frac{2x + 2}{x^2 + 2x + 1}$$

d. $y = \frac{\sin x}{e^x}$

$$y' = \frac{e^x \cos x - e^x \sin x}{e^{2x}}$$

$$= \frac{\cos x - \sin x}{e^x}$$
e. $y = x^{\sec x}$

\[
\begin{align*}
\ln y &= \ln (x^{\sec x}) = \sec x \ln x \\
\frac{1}{y} y' &= \sec x \tan x \ln x + \frac{\sec x}{x} \\
y' &= \left(\sec x \tan x \ln x + \frac{\sec x}{x}\right)
\end{align*}
\]

f. $y = x^2 \sin x \cos x$

\[
y' = 2x \sin x \cos x + x^2 \cos^2 x - x^2 \sin^2 x
\]

g. $y = \sqrt{2 + \tan(1 + x^3)}$

\[
y' = \frac{1}{2} \left(2 + \tan(1 + x^3)\right)^{-\frac{1}{2}} \left(\sec^2(1 + x^3)\right) \frac{1}{3} \cdot 3x^2
\]
h. \( xy^2 + 5x^2 - 2y = 10 \)

\[
\frac{y^2 + 2xyy'}{10x - 2y'} = 0
\]

\[
y'(2xy - 2) = -10x - y^2
\]

\[
y' = \frac{y^2 + 10x}{2 - 2xy}
\]

i. \( y = \sqrt{(x^2 + 1)e^x} \)

\[
\ln y = \frac{1}{2} \left( 5 \ln(x^2 + 1) + x - \ln(x^2 + 2) \right)
\]

\[
\frac{1}{y} y' = \frac{1}{2} \left( \frac{10x}{x^2 + 1} + 1 - \frac{2x}{x^2 + 2} \right)
\]

\[
y' = \left( \sqrt{\frac{(x^2 + 1)e^x}{x^2 + 2}} \right) \cdot \frac{1}{2} \cdot \left( \frac{10x}{x^2 + 1} + 1 - \frac{2x}{x^2 + 2} \right)
\]

j. \( y = \sin^{-1}(4x) \)

\[
y' = \frac{4}{\sqrt{1 - 16x^2}}
\]
2. (10 points) Suppose I deposit $1000 in a bank account with continuously compounding interest. After three years time I now have $1300. What is the annual rate in percent? (it is ok to leave your answer in terms of $\ln$.)

\[ P(t) = P_0 e^{\lambda t} = 1000 e^{\lambda t} \]
\[ 1300 = P(3) = 1000 e^{3\lambda} \]
\[ 1.3 = e^{3\lambda} \]
\[ \ln(1.3) = 3\lambda \]
\[ \lambda = \frac{\ln(1.3)}{3} \]

\[ \text{in percent need} \cdot 100 \]
\[ \frac{100 \ln(1.3)}{3} \% \]

3. (10 points) Estimate $\ln(0.99)$ using a linear approximation to an appropriate function.

\[ f(x) = \ln x \quad \text{use lin approx } a + a = 1 \]
\[ f'(x) = \frac{1}{x} \quad f(a) = \ln 1 = 0 \]
\[ f'(a) = 1 \]

\[ L(x) = f(a) + f'(a)(x-a) \]
\[ = 0 + 1 (x-1) = x-1 \]

\[ L(.99) = .99 - 1 = -.01 \]
4. (10 points) Suppose \( xy + e^y = e \).

a. Find the equation of the tangent line at the point on the curve where \( x = 0 \).

b. Find \( y'' \) at that same point.

\[
y + xy' + e^y y' = 0
\]

\[
y' = \frac{-y}{x + e^y}
\]

\[
y'' = \frac{(x + e^y)l - y' + y(1 + e^y y')}{(x + e^y)^2}
\]

a. At \( x = 0 \), \( y = 1 \), \( y' = \frac{-1}{\alpha + e} = -\frac{1}{e} \).

\[
Y - 1 = -\frac{1}{e} (x - 0)
\]

b. Plug in \( x = 0 \), \( y = 1 \), \( y' = -\frac{1}{e} \) into \( y'' \):

\[
y'' = \frac{e^{\frac{1}{e}} + (1 + e^{\frac{-1}{e}})}{e^2} = \frac{1 + 1}{e^2} = \frac{1}{e^2}
\]
5. (10 points) A spotlight on the ground shines on a wall 12m away. If a man 2m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4m from the building? (Hint: use similar triangles)

\[ \frac{dx}{dt} = 1.6 \text{ m/s} \]

Find \( \frac{ds}{dt} \) when \( x = 8 \)

Similar triangles gives \( \frac{s}{12} = \frac{2}{x} \)

\( 5x = 2y \)

\( 5 \cdot \frac{dx}{dt} + \frac{ds}{dt} \cdot x = 0 \)

When \( x = 8 \) \( \Rightarrow \) \( \frac{s}{12} = \frac{2}{8} \) \( \Rightarrow \) \( s = 3 \)

3 \cdot (1.6) + \frac{ds}{dt} (8) = 0

\[ \frac{ds}{dt} = \frac{-4 \cdot 8}{8} = -0.6 \text{ m/s} \]

Shadow is decreasing at \(-0.6 \text{ m/s}\)
6. (10 points) The equation $x^2 - xy + y^2 = 3$ represents a “rotated ellipse”, that is an ellipse whose axes are not parallel to the coordinate axes. Find the points where this ellipse intersects the $x$ axis (you should get two points). Show that the tangent lines at these points are parallel.

Intersects $x$ axis means $y = 0$ so $x^2 = 3 \quad x = \pm \sqrt{3}$

Points $(\sqrt{3}, 0), (-\sqrt{3}, 0)$

Find $y'$:

$$2x - xy' - y + 2yy' = 0$$

$$y' = \frac{y - 2x}{2y - x}$$

At $(\sqrt{3}, 0)$, $y' = \frac{0 - 2\sqrt{3}}{-\sqrt{3}} = 2$

At $(-\sqrt{3}, 0)$, $y' = \frac{0 + 2\sqrt{3}}{0 + \sqrt{3}} = 2$

Tangent lines have same slope so parallel.