Lecture 27

Indeterminate Forms

Problem How to calculate limits where "plugging in" gives $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$ etc..

So far

1. $\lim_{h\to 0} \frac{\sin h}{h} = 1$, type "$\frac{0}{0}$", proved using sqz thm + geometric argument.

2. $\lim_{x\to\infty} \frac{3x^2+x+1}{5x^2-7} = \frac{3}{5}$, used trick of dividing by $\frac{x^2}{x^2}$

3. What about $\lim_{x\to\infty} \frac{\ln x}{x+2}$. Type $\frac{\infty}{\infty}$, no obvious trick

   or $\lim_{x\to 1} \frac{\ln x}{x-1}$ Type $\frac{0}{0}$.

Idea Suppose $f(a)=g(a)=0$. Recall Linear approx formula

$$L(x) = f(a) + f'(a)(x-a)$$

For x near a, linear approx is very good!

$$\text{So} \quad \frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

* Limit is ratio of derivatives *

L'Hospital's Rule Suppose f, g are differentiable and $g'(x) \neq 0$ on a neighborhood of a (except perhaps at a)!

Suppose $\lim_{x\to a} f(x) = \lim_{x \to a} g(x) = 0$ (Type "$\frac{0}{0}$") **or**

$\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$ (Type "$\frac{\infty}{\infty}$")

Then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ if it exists or is $\pm\infty$

Remark: Also works for one-sided limits and limits at ∞, although our intuitive "proof" w/ $L(x)$ needs modifying.

Examples

1. $$\lim_{x\to\infty}\frac{\ln x}{x+2} \overset{LHR}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0$$

2. $$\lim_{x\to\infty}\frac{\ln x}{\cancel{\text{polynomial, deg}>0}\ \ x^a \ \ a>0} = 0.$$ $\ln x$ grows more slowly than any positive power of x

3. $$\lim_{x\to\infty}\frac{e^x}{x^2} \overset{LHR}{=} \lim_{x\to\infty}\frac{e^x}{2x} \overset{LHR}{=} \lim_{x\to\infty}\frac{e^x}{2} = \infty$$

$\to$ e^x grows faster than any polynomial!

Ex $\lim_{x\to 0} \frac{e^{2x}-1}{\sin x}$ type $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{2e^{2x}}{\cos x} = \frac{2}{\cos 0} = 2$$

do not apply LHR again !

Ex $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$ type $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1+2x}} + \frac{2}{\sqrt{1-4x}}}{1} = 1+2 = 3$$

Indeterminate Products

Ex $\lim_{x\to 0^+} x^2 \ln x$. Type "$0\cdot\infty$". Use $f\cdot g = \frac{f}{1/g} = \frac{g}{1/f}$ to convert first.

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{LHR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0$$

Ex $\lim_{x\to 0} \sin(5x)\csc(3x) = \lim_{x\to 0} \frac{\sin(5x)}{\sin(3x)} = \lim_{x\to 0} \frac{5\cos 5x}{3\cos 3x}$

type $0\cdot\infty$

$= 5/3$

Indeterminate Powers Type 0^0; 1^∞; ∞^0; Not 0^∞!

1. Take ln of both sides

2. Proceed as before to get $\lim \ln y$

3. Exponentiate to get $\lim y$

Ex $\lim_{x\to 0^+} (\tan 2x)^x$ type 0^∞

$\ln y = x \ln(\tan 2x)$ type $0\cdot\infty$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(\tan 2x)}{1/x} \quad \text{type } \frac{\infty}{\infty}$$

$$= \lim_{x\to 0} \frac{\frac{2\sec^2(2x)}{\tan 2x}}{-1/x^2} = \lim_{x\to 0} \frac{-2x^2}{\cos^2(2x)\cdot \frac{\sin(2x)}{\cos(2x)}}$$

$$= \lim_{x\to 0} \frac{-2x\cdot x}{\sin(2x)\cos 2x}$$

$$= \lim_{x\to 0} \frac{-2x}{\sin 2x} \cdot \lim_{x\to 0} \frac{x}{\cos 2x} = 1\cdot 0 = 0$$

So $\lim_{x\to 0^+} \ln y = 0$

$$\lim_{x\to 0^+} y = e^0 = 1$$