Lecture 29

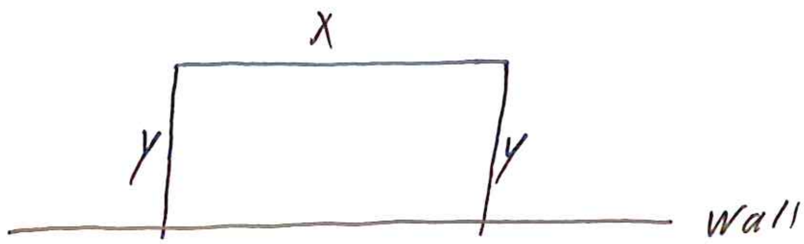
Optimization Problems

Review: Continuous function on a closed interval, how to find max/min?
- critical values
- endpoints

$\rightarrow$ plug in.

"Word problems" Goal: Transform into opt. problem.

Example I have 100 feet of fencing. Want to build a pen, rectangular shape, against a wall. What is max area?



Area $= xy$

- Extra info to eliminate variables:

$$y + y + x = 100$$
$$y = \frac{100 - x}{2}$$

$$A(x) = \frac{x \cdot (100 - x)}{2} = 50x - \frac{1}{2}x^2 \qquad 0 \le x \le 100$$

Reduced problem to maximizing $A(x)$ on a closed interval!

$$A'(x) = 50 - x$$
$$0 = 50 - x$$
$$x = 50$$

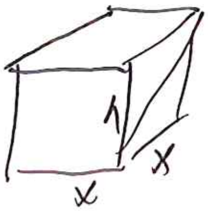
x	$A(x)$
0	0
50	1250
100	0

Maximum area is $1250\,ft^2$ w/ dimensions 25×50.

Ex

We have $1200\ cm^2$ of material! Want to make a box with square base, no lid, of maximum volume

Find dimensions!



$$V = x^2 h$$

Given $\underbrace{x^2 + 4xh}_{\text{surface area}} = 1200$

$$h = \frac{1200 - x^2}{4x}$$

$$V(x) = x^2 \cdot \left(\frac{1200 - x^2}{4x}\right) = \frac{1200x - x^3}{4} = 300x - \frac{1}{4}x^3$$

$$0 < x \leq \sqrt{1200}$$

$V'(x) = 300 - \frac{3}{4}x^2$ set $= 0$

$$\frac{3}{4}x^2 = 300$$

$$x^2 = 400$$

$$x = \pm 20$$

x	V
20	$300 \cdot 20 - \frac{1}{4} \cdot 20^3$

$+$ | $-$

20

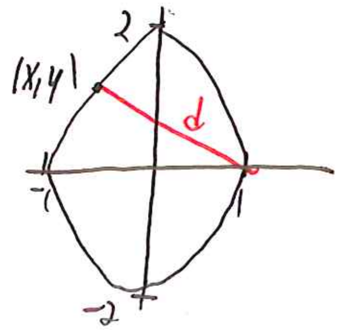
V'

1st Der Test: Suppose c is a critical # of continuous function $f(x)$ defined on I.

- If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ then global max at $x = c$.

- If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ then global min at $x = c$.

Ex Find points on $4x^2+y^2=4$ farthest from $(1,0)$!



$d=\sqrt{(x-1)^2+(y-0)^2}$ maximize d

Trick: Maximize $d^2=(x-1)^2+y^2$

$y^2=4-4x^2$

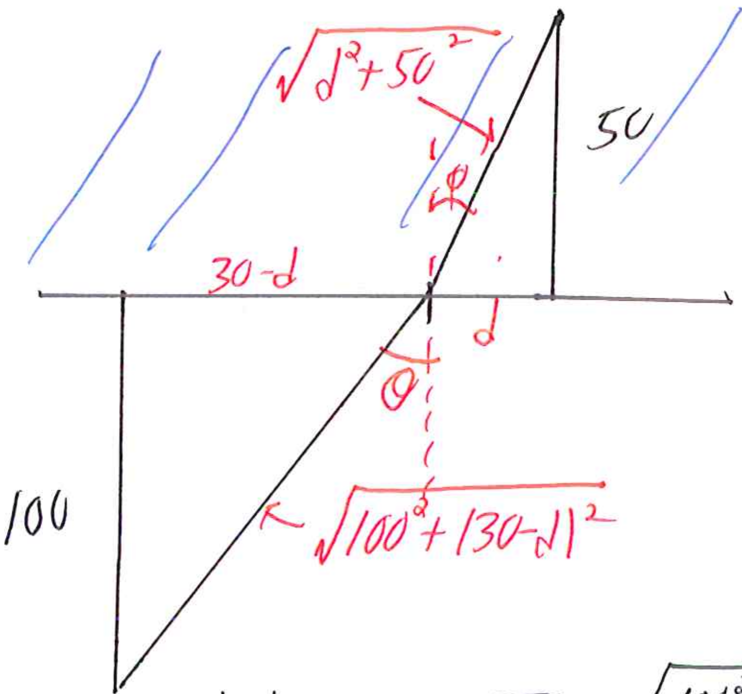
Maximize $(x-1)^2+4-4x^2 \qquad -1\le x\le 1$

$\frac{d}{dx}$ $2(x-1)-8x=0 \to x=-1/3 \quad y=\pm\frac{4}{3}\sqrt{2}$

x	D^2
-1	4
$-1/3$	?
1	0

Ex Lifeguard runs 10 mph, swims 3 mph.
She is 100 ft from water, sees someone drowning 60 feet offshore and 30 feet to her right.

What path should she take to minimize time?



So $0 \le d \le 30$

time = $\frac{\text{distance}}{\text{speed}}$ $\qquad T = \frac{\sqrt{100^2+(30-d)^2}}{10} + \frac{\sqrt{d^2+50^2}}{3}$

$$\frac{dT}{dd} = \frac{1}{10} \cdot \frac{-2(30-d)}{2\sqrt{100^2+(30-d)^2}} + \frac{1}{3}\frac{2d}{2\sqrt{d^2+50^2}} \qquad \text{set} = 0$$

$$\frac{1}{10}\frac{30-d}{\sqrt{100^2+(30-d)^2}} = \frac{1}{3}\frac{d}{2\sqrt{d^2+50^2}}$$

$$\frac{1}{10}\sin\theta = \frac{1}{3}\sin\phi$$

$$\frac{10}{3} = \frac{\sin\theta}{\sin\phi}$$

Snell's Law!

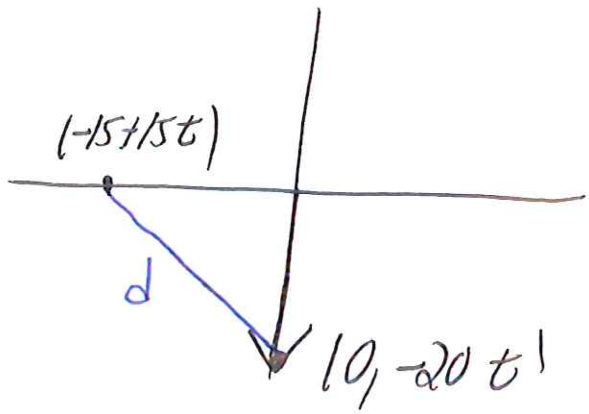
Ex Boat leaves dock at 2pm, heads south at 20 km/h

Another boat heads east at 15 km/h and reaches dock at 3p.m.

When were they closest together.

A Put dock at origin

Let $t=0$ be 2p.m.



$$d(t)=\sqrt{(-15+15t)^2+400t^2}$$

$$d'(t)=\frac{1}{2\sqrt{\quad}}\cdot 2(-15+15t)(15)+800t$$

$$-450+450t+800t=0$$

$$1250t=450$$

$$t=9/25$$