Lecture 30 - Newton's Method

Problem: Solve equations like \( \cos x = x \)
- No closed form solution.
- Want numerical approximation.

Goal: Find solution to \( f(x) = 0 \) to high precision.

Idea:
1. Guess near a root.
2. Calculate tangent line at your guess. Find its x-intercept.
3. Use x-intercept as your new guess. Repeat.

Tangent line at \( x_1 \):
\[
\frac{y - f(x_1)}{f'(x_1)} = \frac{x - x_1}{f'(x_1)}
\]

Find x intercept:
\[
0 - f(x_1) = f(x_1) x - f(x_1) x_1
\]
\[
f'(x_1) x_1 - f(x_1) = f'(x_1) x
\]
\[
x = x_1 - \frac{f(x_1)}{f'(x_1)}
\]

So
\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}
\]
Procedure

1. Start with guess $X_1$.

2. Obtain sequence $X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$.

If you are lucky sequence $X_1, X_2, X_3, ...$ converges to root $r$.

Example: Find $\sqrt[3]{2}$ to 4 decimal places. So $f(x) = x^3 - 2$

Guess $X_1 = 1.3$

$X_2 = 1.3 - \frac{1.3^3 - 2}{3 \cdot 1.3^2} = 1.261143$

$X_3 = 1.2599222$

$X_4 = 1.2599210$

$\sqrt[3]{2} \approx 1.2599$

Example: Solve $\cos x = x$ to 3 dec.

$f(x) = \cos x - x$

$g'(x) = -\sin x - 1$

Guess $X_1 = 1$

$X_{n+1} = X_n - \frac{\cos(X_n) - X_n}{-\sin(X_n) - 1}$

$X_1 = 1$

$X_2 = 0.739085$
What can Go Wrong?

1. May overshoot or diverge
   \[ \text{Ex } y = \tan^{-1} x, \quad x_1 = 1, y \]

2. Closed cycle - see attached

3. Sensitivity to initial guess
   - see maple worksheet

4. Fractals
   \[ \text{Ex } z^3 - 1 = 0. \text{ Roots } 1, \quad -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad -\frac{1}{2} - \frac{\sqrt{3}}{2} i \]

   \[ \text{Newton's Method works over } \mathbb{C} \text{ also.} \]
Newton Method Cycle

Figure 2: Newton's method cycling between $x_0$ and $x_1$. 
Antiderivatives

Def. If \( F \) is an antiderivative of \( f \) on \( I \) if

\[ F'(x) = f(x) \text{ on } I. \]

Ex. \( \frac{x^3}{3} \) is an antiderivative of \( x^2 \) on \( (-\infty, \infty) \).

Thm. If \( F(x) \) is an antiderivative of \( f(x) \) then the most general antiderivative is \( F(x) + C \).

Proof. We saw earlier if \( F'(x) = G'(x) \) then \( F(x) - G(x) = C \).

Ex. Find the most general antiderivative of \( f(x) \):

1. \( f(x) = x^3 + 2x + 3 \)
2. \( f(x) = \cos x + \sec^2 x \)
3. \( f(x) = \frac{1}{x} \)