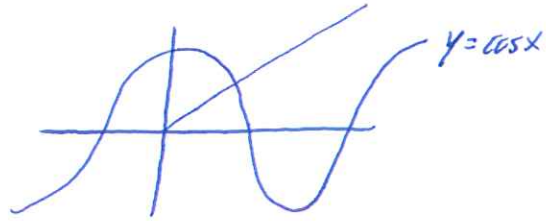
Lecture 30 - Newton's Method



Problem Solve equations like $\cos x = x$

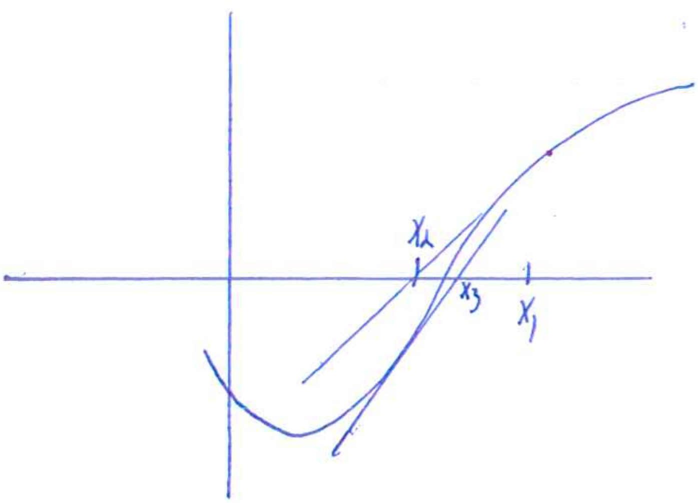
- No closed form solution.
- Want numerical approximation.

Goal Find solution to $f(x) = 0$ to high precision.

Idea 1. Guess near a root.

2. Calculate tangent line at your guess. Find its x-intercept.

3. Use x intercept as your new guess. Repeat!



$x_1 = 1^{st}$ guess

Tangent line at x_1: $y - f(x_1) = f'(x_1)(x - x_1)$

Find x intercept: $0 - f(x_1) = f'(x_1)x - f'(x_1)x_1$

$$f'(x_1)x_1 - f(x_1) = f'(x_1)x$$

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

if 0 method fails, tang line is horizontal

So $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Procedure 1. Start w/ guess x_1.

2. Obtain Sequence $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

If you are lucky sequence $x_1, x_2, x_3, \ldots$ converges to root r.

Example Find $\sqrt[3]{2}$ to 4 decimal places. So $f(x) = x^3 - 2$

Guess $x_1 = 1.3$ $\qquad x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2}$

$x_2 = 1.3 - \frac{1.3^3 - 2}{3 \cdot 1.3^2} = 1.261143$

$x_3 = 1.2599222$ $\qquad \sqrt[3]{2} \approx 1.2599$

$x_4 = 1.2599210$

Example Solve $\cos x = x$ to 3 dec.

$f(x) = \cos x - x$ $\qquad$ Guess $x_1 = 1$

$f'(x) = -\sin x - 1$ $\qquad x_{n+1} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1}$

$\boxed{.739085}$

What can Go Wrong?

1. May overshoot or diverge

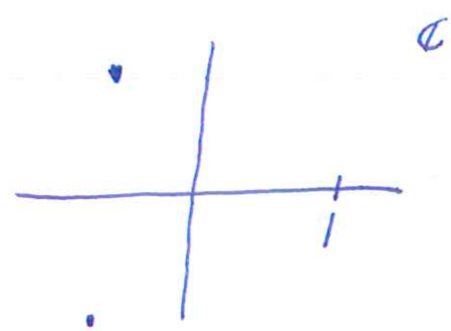
   Ex $y = \tan^{-1} x$, $x_1 = 1.4$

2. Closed cycle - see attached

3. Sensitivity to initial guess
   - see maple worksheet

4. Fractals

   Ex $z^3 - 1 = 0$. Roots 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$



Newton's Method works over ℂ also.

Newton Method Cycle

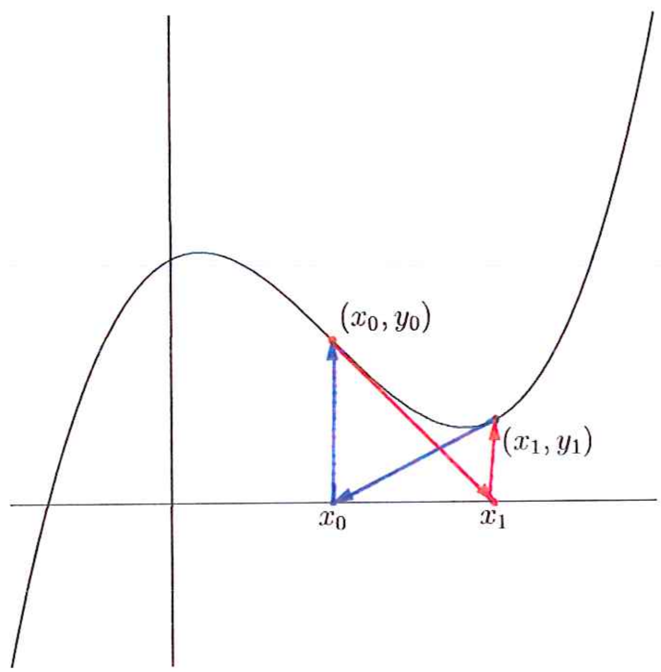


Figure 2: Newton's method cycling between x_0 and x_1.

Antiderivatives

Def F is an *antiderivative* of f on I if

$$F'(x) = f(x) \text{ on } I.$$

Ex $\frac{x^3}{3}$ is an antiderivative of x^2 on $(-\infty,\infty)$

Thm If $F(x)$ is an antiderivative of $f(x)$ then the most general antiderivative is $F(x)+C$.

Proof We saw earlier if $F'(x) = G'(x)$ then $F(x) - G(x) = C$

Ex Find most general antiderivative of $f(x)$:

1. $f(x) = x^3 + 2x + 3$

2. $f(x) = \cos x + \sec^2 x$

3. $f(x) = \frac{1}{x}$