Question: What is area? Answer/Defn: Area of rectangle = \( L \times W \).

Property: If a set is a union of 2 disjoint pieces, area should add

\[
\text{Area}(A \cup B) = \text{area } A + \text{area } B
\]

What if they have 1-dimensional intersection?

\[
\text{Area triangle} = \frac{1}{2} b h
\]

What if boundary is a fractal?

Problem: Find area under \( y = x^2 \) and above \([0, 1]\)

Idea: Similar to how we got slope of tangent line by approximation w/ secant lines.

Approximate w/ rectangles, take a limit.
\[ f(x) = x^2 \]

- Divide \([0, 1]\) into intervals
- Each has base \(1/4\)
- Heights: \(f(1/4) = 1/16\)
  \(f(1/2) = 1/4\)
  \(f(3/4) = 9/16\)
  \(f(1) = 1\)

Area \(\approx \frac{1}{4} \left( f(1/4) + f(1/2) + f(3/4) + f(1) \right) \)

\[ = \frac{1}{4} \left( \frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1 \right) = \frac{1}{4} \left( \frac{30}{16} \right) = \frac{30}{64} = \frac{15}{32} \]

15/32 is an overestimate.

Could use left endpoints.

\[ A \approx \frac{1}{4} \left( f(0) + f(1/4) + f(1/2) + f(3/4) \right) \]

\[ = \frac{1}{4} \left( 0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16} \right) \]

\[ L_4 = \frac{1}{4} \left( \frac{1}{16} \right) = \frac{1}{32} \]


\[ R_8 = \frac{1}{8} \left( f(1/8) + f(3/8) + \ldots + f(7/8) \right) \approx \]
\[ R_n = \frac{1}{n} \left( \frac{1}{n} \right)^2 + \left( \frac{2}{n} \right)^2 + \ldots + \left( \frac{n}{n} \right)^2 \]

\[ = \frac{1}{n^3} \left( 1 + 2^2 + 3^2 + \ldots + n^2 \right) \]

**Facts**

\[ \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2} = \frac{n^2 + n}{2} \]

\[ \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \]

\[ = \frac{1}{n^3} \left( \frac{n(n + 1)(2n + 1)}{6} \right) \]

\[ \lim_{n \to \infty} R_n = \frac{2}{6} = \frac{1}{3} \]

**Other Options**

- midpoint
- lower & upper sums

**More generally**

\[ \text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x + \ldots + f(x_i) \Delta x \]

\[ x_i \in i^{\text{th}} \text{ interval} \]