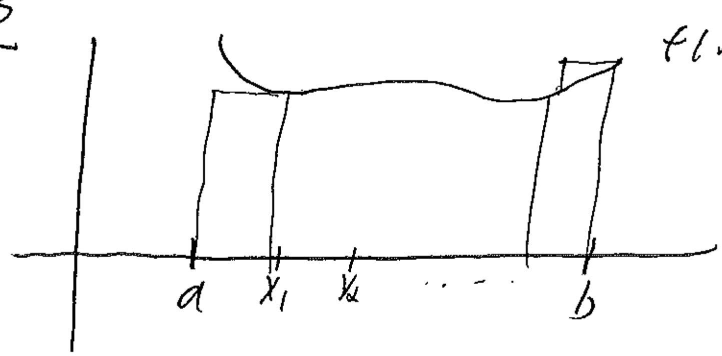
Lecture 33

Review



n rectangles, $\Delta x = \frac{b-a}{n}$ $\qquad x_1 = a + \Delta x,\ x_2 = a + 2\Delta x, \ldots$

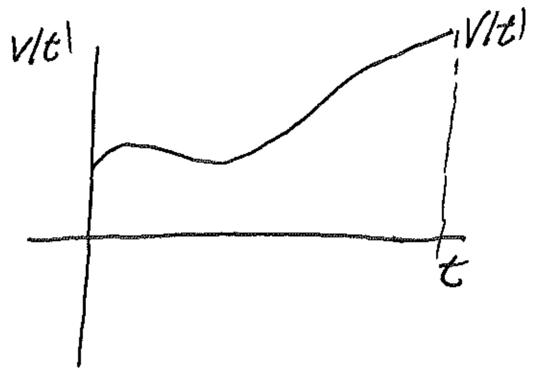
Area of i^{th} rectangle is $f(x_i)\Delta x$, $\quad R_n = \sum_{i=1}^{n} f(x_i)\Delta x$.

Def Area of region under $y = f(x)$ and above $[a,b]$ is

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Fact Suppose $f(x)$ is continuous. Get same area using R_n, L_n, M_n or indeed any $x_i^* \in [x_{i-1}, x_i]$

Example



distance = speed × time

area is distance!

EX

time(s)	0	0.5	1.0	1.5	2	2.5	3.0
v(ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

runner speeding up. Find lower & upper estimates for distance.

Definite Integrals

Def Let $f(x)$ be defined on $[a,b]$. Divide into n subintervals, each of width $\Delta x = \frac{b-a}{n}$.

Let $x_0 = a$, $x_1 = a+\Delta x$, $x_2 = a+2\Delta x, \ldots,$ $x_n = a + n\Delta x = b$, be endpoints

Choose sample points $x_i^* \in [x_{i-1}, x_i]$.

The definite integral of $f(x)$ from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

if it exists. If so say $f(x)$ is integrable on $[a,b]$

$$\int_{a \leftarrow \text{lower limit}}^{b \leftarrow \text{upper limit}} \underbrace{f(x)}_{\text{integrand}}\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Rieman Sum}}$$

Rmk 1. Can sometimes work w/ definition but, as with derivatives we will look for a better way.

2. Interpretation as signed area

Thm $f(x)$ continuous or finitely many jump disc. then $f(x)$ is integrable.

Examples

1. Evaluate $\int_2^5 4-2x\,dx$ using areas

2. Evaluate $\int_{-2}^{0} x^2+x\,dx$ from def.

$$\Delta x = \frac{2}{n} \qquad x_i = -2 + 2i/n$$

$$\int_{-2}^{0} x^2+x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(-2+\frac{2i}{n}\right)^2 + \left(-2+\frac{2i}{n}\right)\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{8i}{n} + 4 - 2 + \frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^2}{n^3} - \frac{12i}{n^2} + \frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{12}{n^2}\sum_{i=1}^{n} i + \frac{4}{n}\sum_{i=1}^{n} 1$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{12}{n^2}\left(\frac{n^2+n}{2}\right) + \frac{4}{n}\cdot n$$

$$= \frac{16}{6} - 6 + 4 = \frac{2}{3}$$

We need a better way!

Properties

1. $\int_a^b c\,dx = c(b-a)$

2. $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b f(x) \pm g(x)\,dx$

3. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

all follow from same properties of sum.

More

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Comparisons

1. If $f(x) \geq g(x)$ on $[a,b]$ then

$$\int_a^b f(x) \geq \int_a^b g(x)$$

Cor $m \leq f(x) \leq M$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$