Lecture 5
Review Limit laws
- Many functions can sub in \( x=a \) in domain
- 1-sided versions, proofs in App

Example: \( \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) \)
- Since \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) DNE. Limit laws do not apply.

Need another tool.

Idea

\[
\begin{align*}
\text{Squeeze Theorem} \\
\text{Suppose } f(x) &\leq g(x) \leq h(x) \text{ for } x \text{ in a neighborhood of } a \ (\text{except } a)
\end{align*}
\]

Suppose \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L. \)

Then \( \lim_{x \to a} g(x) = L. \)

Proof: Suppose \( \epsilon > 0 \) is given. Choose \( \delta_1 \) so \( 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2} \)

Choose \( \delta_2 \) so \( 0 < |x-a| < \delta_2 \Rightarrow |g(x) - L| < \frac{\epsilon}{2} \)

Smaller of \( \delta_1 \) or \( \delta_2 \) will work for \( g(x) \).
Remarks

- Don't need \( g(\alpha) \) squeezed between \( \pm h \) for all \( x \), only in a neighborhood of \( \alpha \).
- Choice of squeezing functions is key.

\[ \lim_{x \to 0} x^2 \sin(1/x) = 0 \]

**Proof**

\[-1 \leq \sin(1/x) \leq 1 \text{ for all } x \neq 0 \]
\[-x^2 \leq x^2 \sin(1/x) \leq x^2 \text{ for all } x \neq 0 \]

But \( \lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0 \) \]

\[ \lim_{x \to 0^+} \sqrt{x} e^{\sin(1/x)} \]

between \( \frac{1}{e} \) and \( e \)

Later \( \cos \theta \leq \frac{\sin \theta}{\theta} \leq 1 \) for \( \theta \) near 0.
Continuity

Informal: Precise: if \( f(x) \) is continuous it can sketch graph without picking up pencil.
- Small changes in \( x \) produce "small" changes in \( f(x) \).

**Example** 
\[ f(x) = x^2 \] 
continuous for all \( -\infty < x < \infty \)

\[ f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \]
Not continuous at \( x=0 \)

**Definition** 
A function \( f(x) \) is **continuous at** \( x=a \) if \( \lim_{x \to a} f(x) = f(a) \).

**Remark** 
- \( f(a) \) defined
- \( \lim_{x \to a} f(x) \) exists
- They are equal.

If not, say \( f(x) \) is **discontinuous at** \( x=a \) or has a discontinuity at \( x=a \).

**Examples**

1. Polynomials are continuous on \( (-\infty, \infty) \).

2. Rational Functions are continuous on their domains.
   \[ \frac{x}{x^2 + 1} \] 
   cont on \( (-\infty, \infty) \) U \( 0, \infty \)

3. \( e^x, \sin x, \cos x \) continuous on \( (-\infty, \infty) \).

4. \( \tan x \) etc., continuous on domain.

5. Physical Quantities usually continuous.
Types of discontinuities

1. \( f(x) = \frac{x^2-2}{x-2} \)
   \[ \lim_{x \to 2} f(x) = \frac{0}{0} \]
   but \( f(2) \) not defined,
   \[ \lim_{x \to 2} |f(x)| \text{ exists but } \not\in \mathbb{R} \]
   \(1 \& 2 \text{ have removable discontinuities at } x = 2 \]
   \( \text{limit exists but is not } = f(2) \)

2. \( f(x) = \frac{|x|}{x} \)
   \[ \text{has a jump discontinuity at } x=0 \]
   \( \text{limits from left & right exist but are } \neq \)

3. \( f(x) = \frac{1}{x} \)
   \[ \text{has an infinite discontinuity at } x=0 \]
Ex \[ y = \sqrt{x} \]

Def: \( f(x) \) is continuous from right at \( x = a \) if

\[
\lim_{x \to a^+} f(x) = f(a)
\]

Def: Continuous on a closed interval.