Math 353 Homework #10- Due Tuesday 11/22/16

1. 13.2.1B

The regular octagon has 16 symmetries. Let r be the clockwise rotation by $\pi/4$, so there are 8 rotations, namely $\{e, r^2, r^3, \dots, r^7.\}$ There are 8 reflections, 4 of them have axes connecting midpoints of opposite sides, and 4 have axes connecting opposite corners. For each of the 16 elements we need to calculate the cycle structure on the 8 vertices. For example r^2 has cycle structure $(4,4)$, i.e. two 4 cycles. The first reflections have cycle structure $(2,2,2,2)$ and the second type have structure $(2,2,2,1,1)$, since two vertices are fixed. We have:

symmetry	cycle structure on vertices
e	(1,1,1,1,1,1,1,1)
r, r^3, r^5, r^7	(8)
r^2, r^6	(4,4)
r^4	2,2,2,2
midpoint reflections (x4)	(2,2,2,2)
vertex reflections(x4)	(2,2,2,1,1)

Using Frobenius Theorem we get that the number of patterns is:

$$\frac{1}{16}(c^8 + 4c + 2c^2 + c^4 + 4c^4 + 4c^5) = \frac{1}{16}(c^8 + 4c + 2c^2 + 5c^4 + 4c^5).$$

2. 13.2.2B (Hint: The symmetries of the object in Figure 13.5 are the same as the symmetries of a square)

This is similar to the previous problem, we have 8 symmetries and for each we need to figure out how many orbits there are on the 45 tiny squares in the picture. The identity has 45 orbits, contributing c^{45} The rotations r and r^3 fix the center square and the rest are in 11 4-cycles, so 12 orbits, contributing c^{12}. The rotation r^2 fixes the center square and the rest are in 22 2-cycles, so 23 orbits and a contribution of c^{23}. The horizontal and vertical reflections fix 9 squares and the rest are in 18 2-cycles, so contributes c^{27}. Finally the two diagonal reflections fix 3 squares and have 21 2-cycles, for a contribution of c^{24}. Thus the total for c colors:

$$\frac{1}{8}(c^{45} + 2c^{12} + c^{23} + 2c^{27} + 2c^{24}).$$

Plug in $c = 3$ to get the answer.

3. 13.2.4B
Recall the rotational symmetries of a tetrahedron. Of course there is the identity. Each corner can be fixed and the opposite side rotated 120 degrees either way, so this gives 8 symmetries of order 3. Finally for each pair of opposite sides there is a 180 degree rotation with axis connecting the midpoints.

So for each type we need to figure out the action on the 16 small triangles. The identity fixes them all, so contributes c^{16}. The 8 rotations through 1/3 turn have 1 triangle fixed (center of the opposite face) and the other 15 are in 5 3-cycles, so we get a contribution of $8c^6$. Finally the 3 180 degree rotations interchange pairs (none is fixed), so we get 8 2-cycles and a contribution of $3c^8$. The final answer then is:

$$\frac{1}{12}(c^{16} + 8c^6 + 3c^8).$$

4. Suppose we want to place 4 red, two yellow and two green keys on a circular key ring. Use Burnside's Theorem to count the number of ways to do this.

Since we can put our keys at the vertices of a regular octagon, the symmetry group here is D_{16}. To use Burnside's theorem we need to calculate $|Fix(g)|$ for each of the 8 elements acting on all possible colorings with 4 red, two yellow and two green keys. So we have the same elements as in 13.2.1B but now with a more difficult computation of fixed points.

g	*cycletype*	$\lvert Fix(g)\rvert$
e	x_1^8	$\binom{8}{4}\binom{4}{2}\binom{2}{2} = 420$
r, r^3, r^5, r^7	x_8	0
r^2, r^6	x_4^2	0
r^4	x_2^4	$\binom{4}{2}\binom{2}{1} = 12$
s, sr^2, sr^4, sr^6	x_2^4	$\binom{4}{2}\binom{2}{1} = 12$
sr, sr^3, sr^5, sr^7	$x_1^2x_2^3$	$6 + 6 = 12$

The last line 6+6 comes as follow. If the two one-cycles are not red there are two choices (yellow or green) and then $\binom{3}{2}$ choices for which two of the 3 two-cycles are red. If the two one-cycles are red then there are 3 choices for the red 2-cycle, then 2 choices for the yellow 2-cycle.

So the Frobenius Theorem says the number of orbits is:

$$\frac{1}{8}(420 + 12 + 4 * 12 + 4 * 12) = 66.$$

5. Find the number of different colorings of a cube with two white, one black and three red faces.

This problem was done in class.

6. How many different chemical compounds can be made by attaching H, CH_3, C_2H_5 or Cl radicals to the four bonds of a carbon atom. (The radicals lie at the vertices of a regular tetrahedron with the carbon atom in the center).

This problem is equivalent to coloring the vertices of a tetrahedron with 4 colors. The rotational symmetry group has 8 elements, each of them with cycle type $(3, 1)$ on the vertices. There are 3 180 degree rotations, with cycle type $(2, 2)$. So applying Burside's theorem we have 4^2 fixed point for each, and 4^4 for the identity. Thus the answer is:

$$\frac{4^4 + 11 \cdot 4^2}{12} = 36.$$

7. Give a simple proof of Cauchy's theorem for $p = 2$. (Hint: pair up)

We are looking to show there is a non-identity element of order two, i.e. $g^2 = e$. This is the same as finding an element $g \neq e$ with $g = g^{-1}$. Suppose there is not. Then the group consists of the identity together with pairs $g \neq g^{-1}$ which would result in an odd number of elements. So if $|G|$ is even there must be another element not equal to e which is its own inverse.

8. Suppose H is a subgroup of G and $g \in G$. Let:

$$gHg^{-1} = \{ghg^{-1} \mid h \in H.\}$$

a. Prove that gHg^{-1} is also a subgroup.
Choose two arbitrary elements of gHg^{-1}, call them gh_1g^{-1} and gh_2g^{-1}. Multiply them we get:

$$gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} \in gHg^{-1}$$

since $h_1h_2 \in H$ because H is a subgroup. Also:

$$(gh_1g^{-1})^{-1} = gh_1^{-1}g^{-1} \in gHg^{-1}$$

since $h_1^{-1} \in H$ because H is a subgroup. Finally note that $e = geg^{-1}$.

b. Let X be the set of all subgroups of G. Prove that G acts on X by conjugation, as in part a.

So define $g \triangleright H = gHg^{-1}$, which is another subgroup of G by part a. Since $ehe = h$ it is clear that $e \triangleright H = H$ as desired. For the second axiom observe that:

$$g_1 \triangleright (g_2 \triangleright H) = g_1 \triangleright (g_2Hg_2^{-1}) = g_1g_2Hg_2^{-1}g_1^{-1} = (g_1g_2)H(g_1g_2)^{-1} = g_1g_2 \triangleright H.$$

c. The stabilizer of a subgroup H under this action is called the *n*ormalizer:

$$N_G(H) = \{g \in G \mid gHg^{-1} = H.\}$$

Let $G = S_4$ and $H = \langle(1,2,3,4)\rangle$ be a cyclic subgroup of order 4. Determine the normalizer of H.

$$N_{S_4}(H) = \{e, (1,2,3,4), (1,3)(2,4), (1,4,3,2), (1,4)(2,3), (1,2)(3,4), (2,4), (1,3)\}.$$

As a way to see this notice that H is always in its normalizer. Also:

$$(1,3)(2,4)(1,2,3,4)(1,4)(2,3) = (4,3,2,1) = (1,2,3,4)^{-1}$$

which proves that $(1,4)(2,3)$ is in the normalizer. The other 3 elements we get by multiplication.