Math 353 Homework #11- Due Monday 12/5/16

1. Use Polya's theorem to compute the number of 5×5 chessboards with 10 red squares, 12 blue squares and 3 green squares, up to symmetry. You will likely need to use a computer algebra system like Maple in the final step.

Solution: G here is D_8. First we calculate $CI(G)$. The elements r and r^3 have cycle type $x_1x_4^6$ on the 25 squares. The element r^2 has type $x_1x_2^{12}$. Since 5 is odd we see all four reflections fix five squares and have type $x_1^5x_2^{10}$. Thus:

$$CI(G) = \frac{x_1^{25} + 2x_1x_4^6 + x_1x_2^{12} + 4x_1^5x_2^{10}}{8}.$$

So to apply Polya we sub in $x_1 = (r + b + g), x_2 = (r^2 + b^2 + g^2), x_4 = (r^4 + b^4 + g^4)$ and take the coefficient of $r^{10}b^{12}g^3$. Using Maple I calculate **185937878**.

2. Taking rotational symmetries into account, how many ways are there to color the vertices of a cube so that four are blue, two are red and two are green?

Solution: We have 24 rotational symmetries, as above we need to calculate the cycle index polynomial on the 8 vertices. You should get the following:

$$CI(G) = \frac{x_1^8 + 6x_4^2 + 9x_2^4 + 8x_1^2x_3^2}{24}$$

Plugging in $x_1 = (b + r + g), x_2 = (b^2 + r^2 + g^2), x_3 = (b^3 + r^3 + g^3), x_4 = (b^4 + r^4 + g^4$ we see the coefficient of $b^4r^2g^2$ is **22**.

3. Recall that the integer lattice $\mathbb{Z}^3$ consists of all points $(a_1, a_2, a_3) \in \mathbb{R}^3$ such that a_1, a_2, a_3 are integers. Suppose we choose 9 distinct points in $\mathbb{Z}^3$. Prove the line segment between some two of the 9 points contains another point in $\mathbb{Z}^3$. Hint: You can actually show the "another point" may be chosen to be the midpoint.

Solution:

Reduce each point module 2 so our points now like in $\mathbb{Z}/2\mathbb{Z}^3$, which has 8 points. By the pigeonhole principle then we have at least two points that reduce to the same, i.e. $(a_1, a_2, a_3) = (b_1, b_2, b_3)$ modulo 2. This means $a_1 + b_1, a_2 + b_2$ and $a_3 + b_3$ are all even. Those the midpoint $(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2})$ has integer coordinates, as desired.

4. Show that given any 9 distinct natural numbers it is possible to chose 5 whose sum is divisible by 5.
Solution: Coming soon!

5. 15.3.4B- Show that if there are nine points inside and equilateral triangle of side length 1 unit then there are 2 of these points within 1/3 unit of each other.

Solution: Divide the triangle into 9 smaller equilateral triangles. Removing the three corner triangles we get a hexagon H made up of the six center triangles. If any of the three corner triangles contain two points then we are clearly done. If not then there are at least 6 points in or on the hexagon H. None of them lies on a vertex (since it would be also on a corner). Thus we have six points strictly inside a circle of radius 1/3 passing through the vertices of the hexagon. Now we have reduced the problem to 15.3.4A with solution in the back.