Math 353 Fall 2016 - Homework #3 Solutions

3.1.5B We have an odd number of balls in each cup, so let these numbers be $2x+1, 2y+1, 2z+1$ and $2w+1$ where $x, y, z, w \geq 0$. We need to count solutions to

$$2x+1+2y+1+2z+1+2w+1=20$$

with $x, y, z, w \geq 0$. These are the same as solutions to $x+y+z+w=8$, so there are $\binom{11}{3}=165$ solutions.

3.3.2B Done in class.

3. The $n=0$ case is clear. We have:

$$\begin{aligned}
B_0 &= S(0,0)\\
B_1 &= S(1,0)+S(1,1)\\
B_2 &= S(2,0)+S(2,1)+S(2,2)\\
B_3 &= S(3,0)+S(3,1)+S(3,2)+S(3,3)\\
B_4 &= S(4,0)+S(4,1)+S(4,2)+S(4,3)+S(4,4)\\
\cdots &= \cdots\\
B_{n-1} &= S(n-1,0)+S(n-1,1)+S(n-1,2)+S(n-1,3)+\cdots+S(n-1,n-1)
\end{aligned}$$

Now multiply the equation for B_j by $\binom{n-1}{j}$ and add up to get $\sum_{j=0}^{n-1} B_j$. Rather than computing the sum of the right hand sides row by row we do it column by column. The first column contributes:

$$S(0,0)\binom{n-1}{0}+S(1,0)\binom{n-1}{1}+\cdots S(n-1,0)\binom{n-1}{n-1}$$

which equals $S(n,1)$ by 3.3.2B. The second column contributes:

$$S(1,1)\binom{n-1}{1}+S(2,1)\binom{n-1}{2}+\cdots S(n-1,1)\binom{n-1}{n-1}$$

which equals $S(n,2)$ by 3.3.2B. The penultimate column contributes:

$$S(n-2,n-2)\binom{n-1}{n-2}+S(n-1,n-2)\binom{n-1}{n-1}$$

which equals $S(n,n-1)$ by 3.3.2B

And the final column is just $S(n-1,n-1)\binom{n-1}{n-1}$ which is equal to 1, so also $S(n,n)$.

Thus the total sum is $S(n,1)+S(n,2)+\cdots+S(n,n)=B_n$ as desired.

4. B_n counts all ways to partition an n element set. Fix a in the set. Suppose there are j elements that are not in the class with a so j ranges from 0 to $n-1$. The number of such partitions is $\binom{n-1}{j}B_j$ since we must choose the j elements and then

partition them in B_j ways. So $B_n = \sum_{j=0}^{n-1} \binom{n-1}{j} B_j$, as desired.

5. This is just the number of solutions to $C_1 + C_2 + \cdots + C_8 = 15$, which is $\binom{22}{7}$ by Theorem 3.1.

6. The easiest way to do this problem is imagine the 3 shelves side by side. Instead we can think of one long shelf where we must place all the books together with two dividers, which split the long shelf into the 3 shelves. The total number of ways to do this is just the multinomial coefficient:

$$\binom{n+r+s+t+2}{n, r, s, t, 2}.$$

7. The first problem just the Stirling number $S(12, 4)$ by definition. If the bags are now distinct children, each such arrangement has 4! labels with the childrens' names so the answer is $4!S(12, 14)$.

8. Using the table of Stirling numbers in the book it's easy to add up the rows and get that the Bell numbers B_1 through B_8 are

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147$$

The ratios are 2, 2.5, 3, 3.4, 3.9, 4.3, 4.7, 5.1. So for example it seems that when n gets large, $B_n \geq 5B_{n-1}$ which suggest B_n grows exponentially with n, for example bigger than some constant times 5^n. Much more is known, the Wikipedia article on the Bell numbers is a good place to start.