Math 353 Homework #9- SOLUTIONS

1. 12.2.2B
The integers $\mathbb{Z}$ are acting on G by $n \rhd g = g_0^n g$. So $0 \rhd g = g_0^0 g = eg = g$ and the first axiom is satisfied. Let $n_1, n_2 \in \mathbb{Z}$. Then :

$$n_1 \rhd n_2 \rhd g = n_1 \rhd g_0^{n2} g = g_0^{n_1} g_0^{n2} g = g_0^{n_1+n_2} g = (n_1 + n_2) \rhd g$$

so the second axiom is satisfied.

2. Let H and K be two subgroups of a group G. Prove that their intersection $H \cap K$ is also a subgroup. For extra credit prove that the union $H \cup K$ is never a subgroup except in the trivial situation where $H \subseteq K$ or $K \subseteq H$.

Let $x, y \in H \cap K$. Since $H \leq G$ we know x^{-1} and xy are in H. Since $K \leq G$ we know x^{-1} and xy are in K. Thus x^{-1} and xy are in $H \cap K$ and so $H \cap K$ is a subgroup.

For the extra credit suppose H and K are subgroups and neither $H \subseteq K$ nor $K \subseteq H$. We must show $H \cup K$ is *not* a subgroup. By our assumption we can choose $h \in H$ with $h \notin K$. Also choose $k \in K$ with $k \notin H$. So $h, k \in H \cup K$ and we will show $hk \notin H \cup K$. If $hk = h' \in H$ then $k = h^{-1}h' \in H$, a contradiction. Similarly if $hk = k' \in K$ then $h = k^{-1}k' \in K$ a contradiction. Thus hk is in neither h nor K, so not in $H \cup K$. Thus $H \cup K$ is not closed under multiplication, so is not a subgroup.

3. Let G be a group and $g \in G$. Define the *centralizer* of g, denoted $C_G(g)$, as the elements that commute with g,namely:

$$C_G(g) = \{x \in G \mid xg = gx\}.$$

a. Prove that $C_G(g)$ is a subgroup of G.
b. Let $\sigma = (1, 2)(3, 4) \in S_4$ Calculate $C_{S_4}(\sigma)$.
c. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q})$. Calculate the centralizer of A.
d. Describe the center $Z(G)$ in terms of centralizers.

3a. First observe $eg = ge = g$ so $e \in C_G(g)$. Now suppose $x, y \in C_G(g)$ so $xg = gx$ and $yg = gy$ by definition. Then $xyg = xgy = gxy$ so $xy \in C_G(g)$. Take the equation $xg = gx$ and multiply both sides by x^{-1} on the left and on the right we get: $gx^{-1} = x^{-1}g$ so $x^{-1} \in C_G(g)$. Thus $C_G(g)$ is closed under multiplication and taking inverses so $C_G(g) \leq G$.

3b. $C_{S_4}(\sigma) = \{e, (1, 2), (3, 4), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), (1, 3, 2, 4), (1, 4, 2, 3)\}$. Notice this centralizer is isomorphic to D_8.

c. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the centralizer of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if and only if it is invertible and:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Multiplying out we get:

$$\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}.$$

This gives us 4 equations which we solve to show that $c = 0$ and $a = d$. So the centralizer is:

$$\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \neq 0\}$$

where the condition on a ensures the matrix is invertible.

d. The center of G is the intersection of the centralizers of the elements of G.

4. Calculate the conjugacy classes in the dihedral group D_8. Repeat for D_{10}.

For D_8 you should get:

$$\{e\}, \{r, r^3\}, \{r^2\}, \{s, sr^2\}, \{sr, sr^3\}.$$

For D_{10} you should get:

$$\{e\}, \{r, r^4\}, \{r^2, r^3\}, \{s, sr, sr^2, r^3, sr^4\}.$$

Notice all 5 reflections are conjugate for the symmetries of a pentagon whereas for the square there are two conjugacy classes. Can you see why geometrically?

5. 12.3.2A

See back of book.

6. 12.4.1B

i. To get an element of X we can choose anything we like for $(g_1, g_2, g_3, \ldots, g_{p-1})$. Once we do this our choice of g_p is forced on us, since we need $g_1 g_2 \cdots g_p = e$ then we must choose $g_p = g_{p-1}^{-1} g_{p-2}^{-1} \cdots g_1^{-1}$. Thus X has $|G|^{p-1}$ elements.

ii. It is clear that

$$0 \rhd (g_1, g_2, g_3, \ldots, g_p) = (g_1, g_2, g_3, \ldots, g_p) = p \rhd (g_1, g_2, g_3, \ldots, g_p)$$

so the action of Z_p is well-defined. One easily checks that

$$a \rhd b \rhd (g_1, g_2, g_3, \ldots, g_p) = (a + b) \rhd (g_1, g_2, g_3, \ldots, g_p).$$

Rotating by a and then by b is the same as rotating by $a + b$. Finally we need to check that the rotated tuples are still in X. Multiply the equation

$$g_1 g_2 \cdots g_p = e$$

by g_1^{-1} on the left and right to get:

$$g_2 g_3 \cdots g_p g_1 = e.$$

Repeating with g_2 etc... shows us that all the cyclic permutations remain in X.

If any $g_i \neq g_j$ then rotating by $j - i$ will move g_i into the j position and so we will have a different tuple. Thus the only elements fixed by all of Z_p are tuples of the form $(g, g, g, \ldots, g)$.

iii. We know from the orbit stabilizer theorem that all the orbits have order dividing the order of Z_p. We know from part i that X is a multiple of p. Since we have an orbit $(e, e, \ldots, e)$ of size 1, there must be at least $p - 1$ other orbits of size 1. But orbits of size one are tuples $(g, g, \ldots, g)$ with $g^p = e$. Thus G has at least $p - 1$ elements of order p.

7. Let $G = S_4$ be the symmetric group on 4 letters. Let $H = \{e, (12)(34), (13)(24), (14)(23)\}$ and let $K = \{e, (12), (34), (12)(34)\}$. Verify that H and K are both subgroups of S_4 and both are isomorphic to the Klein 4 group. Next compute the left and right cosets of H. Repeat for K. What do you notice?

Left and right cosets of H are the same:

$$\begin{aligned}
eH = He &= \{e, (12)(34), (13)(24), (14)(23)\} \\
(12)H = H(12) &= \{(12), (34), (1324), (1423)\} \\
(13)H = H(13) &= \{(13), (1234), (24), (1432)\} \\
(14)H = H(14) &= \{(14), (1243), (1342), (23)\} \\
(123)H = H(123) &= \{(123), (134), (243), (142)\} \\
(124)H = H(124) &= \{(124), (143), (132), (234)\}
\end{aligned}$$

Left cosets of K are:

$$\begin{aligned}
eK &= \{e, (12), (34), (12)(34)\} \\
(13)K &= \{(13), (123), (134), (1234)\} \\
(14)K &= \{(14), (124), (143), (1243)\} \\
(24)K &= \{(24), (142), (243), (1432)\} \\
(23)K &= \{(23), (132), (234), (1342)\} \\
(13)(24)K &= \{(13)(24), (1423), (1324), (14)(23)\}
\end{aligned}$$

Right cosets of K are:

$$\begin{aligned}
Ke &= \{e, (12), (34), (12)(34)\} \\
K(13) &= \{(13), (132), (143), (1432)\} \\
K(23) &= \{(23), (123), (243), (1243)\} \\
K(24) &= \{(24), (124), (234), (1234)\} \\
K(14) &= \{(14), (142), (134), (1342)\} \\
K(13)(24) &= \{(13)(24), (1324), (1423), (14)(23)\}
\end{aligned}$$