Name: SOLUTIONS

Math 353 Midterm Exam - October 17, 2014

Instructions: You may not use any notes, books, calculators, etc... It is ok if your final answers include binomial coefficients and if you do not multiply out exponentials.

1. **(55 points)** Short answer, little or partial credit.

a. Define a *derangement.*

A permutation with no fixed points,

i.e. $\sigma(i) \neq i \ \forall i.$

b. A class of 20 students wishes to elect a president, vice president, and three senators. How many ways can this be done?

$$20 \cdot 19 \cdot \binom{18}{3}$$

c. A five-card poker hand is dealt. What is the probability of getting two-pair? (for example AATT3 is two-pair but AAATT is not, it is a full-house.)

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot 44}{\binom{52}{5}}$$

d. Let $\sigma = (123)(45)(678)$ and $\tau = (1568)(234)(7)$ be elements of the symmetric group S_8.. Write the products $\sigma\tau$ and σ^2 in cycle notation.

$$\sigma\tau = (1435782)(6)$$

$$\sigma^2 = (132)(687)(4)(5)$$

e. Let $S(n,k)$ denote the Stirling number of the second kind. Calculate $S(5,2)$.

5 distinct objects, 2 boxes, no empties

4 + 1   5 choices

3 + 2 $\binom{5}{2} = 10$ choices

15

f. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 15$ where the x_i are nonnegative integers?

15 "dots" 3 plus signs

$\binom{18}{3}$

g. How many triangulations are there of a octagon?

n-gon has C_{n-2} triangulations

$C_6 = \binom{12}{6}/7 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 7} = 12 \cdot 11 = 132$

132

h. Calculate $p(8,3)$, the number of partitions of 8 with less than or equal to three parts.

8
71
62
611
53
521
44
431
422
332

10

i. A fair coin is tossed 6 times. What is the probability of getting at least 4 heads?

$$\frac{\binom{6}{4}}{2^6} + \frac{\binom{6}{5}}{2^6} + \frac{\binom{6}{6}}{2^6} = 22/64 = 11/32$$

j. A biased coin comes up heads with probability 0.8. If it is tossed 6 times what is the probability of getting at least 4 heads.

4 heads: $\binom{6}{4}(.8^4)(.2^2)$ 5 heads $\binom{6}{5}(.8^5)(.2)$

6 heads $\binom{6}{6}(.8^6)$

$$\boxed{\binom{6}{4}(.8^4)(.2^2) + \binom{6}{5}(.8^5)(.2) + \binom{6}{6}(.8^6)}$$

k. Find the generating function for the sequence $a_n = n$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$\frac{d}{dx}$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3$$

$\cdot x$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \ldots = \sum n x^n$$

$$\frac{x}{(1-x)^2}$$

2. **(15 points)** A bag contains 3 red, 2 blue and 6 green marbles. A marble is sampled and replaced 7 times. Find the probability that all three colors have already been seen.

Let M_R = # ways to have no reds. Similarly M_B, M_G.

Total of 11^7 draws, need to subtract $|M_R \cup M_B \cup M_G|$.

$$|M_R \cup M_B \cup M_G| = |M_R| + |M_B| + |M_G| - |M_R \cap M_B| - |M_R \cap M_G| - |M_B \cap M_G| + |M_R \cap M_B \cap M_G|$$

$$= 8^7 + 9^7 + 5^7 - 6^7 - 2^7 - 3^7 + 0$$

$$\text{Prob} = \frac{11^7 - 8^7 - 9^7 - 5^7 + 6^7 + 2^7 + 3^7}{11^7}$$

3. **(15 points)** a. Define the Stirling number of the second kind, $S(n,k)$.

b. Give a combinatorial proof of the identity: $S(n+1,k+1) = \sum_{i=k}^{n} \binom{n}{i} S(i,k)$.

a. The # of ways to assign n distinct balls into k identical boxes with no empty boxes

b. Fix a ball b. Divide up the possible assignments based on the # of balls in the remaining k boxes w/out b. There must be at least k since each box has $\geq$ one. There are at most n if b is alone

The # of ways to assign w/ i balls not in "b"'s box is

$$\binom{n}{i} \cdot S(i,k)$$

choose balls not with b

assign to k boxes no empties

all assignments of $n+1$ balls to $k+1$ boxes are counted once

Thus

$$S(n+1,k+1) = \binom{n}{k} S(k,k) + \binom{n}{k+1} S(k+1,k) + \dots + \binom{n}{n} S(n,k)$$

as desired //

4. **(15 points)** Consider the sequence given by $a_1 = 8, a_2 = 46$ and $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 3$. Solve the recursion to find a formula for a_n.

$a_n - 7a_{n-1} + 10a_{n-2} = 0$

Polynomial is $x^2 - 7x + 10 = (x-5)(x-2)$

General solution $A \cdot 5^n + B \cdot 2^n$

$8 = A \cdot 5 + 2 \cdot B$

$46 = A \cdot 25 + 4 \cdot B$

$\Rightarrow 30 = 15A \Rightarrow A = 2$

$\Rightarrow B = -1$

$a_n = 2 \cdot 5^n - 1 \cdot 2^n$