When it comes to reading mathematics, however, this is not an appropriate beginning. A mathematics book cannot be read like a novel, sitting in a comfortable chair, with a glass at your side. Reading mathematics requires you to be active. You need to be sitting at a table or a desk, with pencil and paper, both to work through the theory and to tackle the problems. A good guide is the amount of time it takes you to read the book. A novel can be read at a rate of about 60 pages an hour, whereas with most mathematics books you are doing well if you can read 5 pages an hour. (It follows that, even at 12 times the price, a mathematics book is good value for the money!)

Since the approach the book takes is to begin with problems and usually to use them to lead into the theory, we have posed a good number of *problems* in the text. The bth problem in Chapter a is labeled **Problem a.b.** These problems are immediately followed by their solutions, but you are strongly encouraged to try the problems for yourself before reading our solution.

At the end of most of the chapter sections, there are *exercises*. Some of these are routine problems to help consolidate your understanding, and some take the theory a bit further or are designed to challenge you. In most cases these problems occur in pairs, labeled A and B. Usually the B question is quite similar to the A question. The difference is that we have included solutions for the A questions at the back of the book but not for the B questions. The solutions to the A questions are usually written out in detail. This is intended to be helpful, but it will not achieve their purpose of helping you to learn the subject, if you give in to the temptation to read the solutions before making your own attempt at the exercises. The B questions, with no solutions, are there for those who cannot resist this temptation!

CHAPTER 2

Permutations and Combinations

2.1 THE COMBINATORIAL APPROACH

In Chapter 1 we gave examples of counting problems that we hope convinced you of their interest and importance. In this chapter we introduce two of the most basic ideas, counting *permutations* and counting *combinations*. These occur over and over again throughout this book. You may have already met these ideas in algebra in connection with the binomial theorem, but the combinatorial approach may be new to you. It can be hard to relearn a topic you are already familiar with but using a different approach. However, we encourage you to adopt the combinatorial approach, which gives more importance to counting methods than to algebraic manipulation, as this is the key to much of the rest of this book.

2.2 PERMUTATIONS

We begin with some problems that are very simple, but the ideas behind their solutions are of fundamental importance in many counting problems.

PROBLEM 2.1

Cayley's Café has the following menu:

Cayley's Café
Starters
Tomato Soup
Fruit Juice
Mains
Lamb Chops
Battered Cod
Nut Bake
Desserts
Apple Pie
Strawberry Ice

How many different three-course meals could you have?

Solution

You have two choices for your starter, and, whichever choice you make, you have three choices for your main course. This makes $2 \times 3 = 6$ choices for the first two courses.

Soup	Soup	Soup	Juice	Juice	Juice
|	|	|	|	|	|
Chops	Cod	Bake	Chops	Cod	Bake

In each of these six cases you have two choices for your dessert, making $6 \times 2 = 12$ possibilities altogether. We can set them out in Figure 2.1, which makes it clear why the number of cases multiplies at each stage and why the final answer is the product of the number of choices at each stage.

So we obtain $2 \times 3 \times 2 = 12$ as the total number of possible meals.

PROBLEM 2.2

In a race with 20 horses, in how many ways can the first three places be filled? (For simplicity, assume that there cannot be a dead heat.)

Solution

There are 20 horses, each of which could come first. Whichever horse comes first, there are 19 other horses that can come second. So there are $20 \times 19 = 380$ ways in which the first two places can be filled. In each of these 380 cases, there are 18 remaining horses that can come third. So there are $380 \times 18 = 20 \times 19 \times 18 = 6840$ ways in which the first three places can be filled.

We now consider the way in which these two problems are different and the way in which they are similar. In Problem 2.1 your choice of a starter did not affect the choice of the main course. Whether you chose the tomato soup or the fruit juice, you still have the choice of lamb chops, battered cod, or nut bake for your main course. And whatever your choices of starter and main course, you still have the same choices, apple pie or strawberry ice, for your dessert.

In Problem 2.2, the horse that wins the race cannot also come in second. So the possibilities for which horse comes in second vary according to which horse wins the race. However, the *number* of possibilities remains the same. Whichever horse wins the race, there are 19 horses that can come second, though which 19 horses these are varies according to which the winner is. Likewise, the possibilities for the third horse vary according to which two horses come in first and second, but, whichever these horses are, there always remain 18 horses each of which can come in third. It is

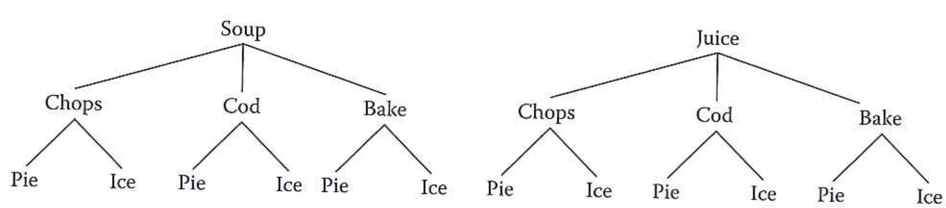

FIGURE 2.1

because the *number of choices* at each stage does not depend on the particular choices made earlier that we could use the same multiplication method to solve Problem 2.2 as we used to solve Problem 2.1, and thus $20 \times 19 \times 18$ does indeed give the number of ways in which the first three positions in the race can be filled.

The multiplication principle we have used in these two problems is sufficiently important to be worth stating explicitly.

THE PRINCIPLE OF MULTIPLICATION OF CHOICES

If there are r successive choices to be made, and for $1 \leq i \leq r$, the ith choice can be made in n_i ways, then the total number of ways of making these choices is $n_1 \times n_2 \times \ldots \times n_r$.

Note that we can use the "pi" notation to write the product in the box as $\prod_{i=1}^{r} n_i$.

Although the principle of multiplication of choices applies equally to Problems 2.1 and 2.2, Problem 2.2 has an additional feature that frequently occurs in problems of this type. The successive choices were all being made from the set of 20 horses taking part in the race. So the number of horses left to choose from goes down by one at each successive stage. That is, in the notation we are using,

$$n_{i+1} = n_i - 1, \quad \text{for } 1 \leq i < r.$$

In such a case, if $n_1 = n$, then for $2 \leq i \leq r$, $n_i = n - i + 1$, so that the product $\prod_{i=1}^{r} n_i$ is $n(n-1)(n-2)\ldots(n-r+1)$. We can express this product more succinctly by making use of factorial notation. We have that

$$n(n-1)(n-2)\ldots(n-r+1) = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots\ldots\times 2\times 1}{(n-r)(n-r-1)\ldots\times 2\times 1} = \frac{n!}{(n-r)!}.$$

Since this situation occurs very frequently, we introduce some special terminology and notation to describe it. We call a choice of r objects from a set of n objects in which the *order* of choice is to be taken into account, a *permutation* of r objects from n. We let $P(n,r)$ be the number of different permutations of r objects from n. Of course, this makes sense only in the case where r and n are nonnegative integers with $r \leq n$. The above remarks yield the general formula for $P(n,r)$.

THEOREM 2.1

For all nonnegative integers r, n with $r \leq n$, $P(n,r) = n!/(n-r)!$

It is important to remember that $P(n,r)$ counts the number of ways of choosing r objects *in order* from a set of n objects. If the order does not matter, the number of choices is smaller, as we shall see in the next section.

In Problem 2.2 we considered only the number of different ways in which the first three positions could be filled. Suppose now we are interested in the number of different ways all 20 horses can finish in order (again, assuming no dead heats). We can see that this number is $20 \times 19 \times 18 \times \ldots \times 2 \times 1$, that is, 20! Note that this is the same as the number of ways of choosing 20 horses in order from a set of 20 horses, and so we could have obtained this by using Theorem 2.1, which tells us that this number is $P(20,20) = 20!/0! = 20!$ (Recall the standard convention that $0! = 1$.) Thus we have:

THEOREM 2.2

The number of different permutations of n objects is $n!$

The values of $n!$ grow very rapidly. Even for quite small values of n, the factorial $n!$ is very large. For example, $10! = 3{,}628{,}800$ and $100!$ is larger than 10^{157}.

PROBLEM 2.3

In how many ways can eight counters be placed on a square 8×8 chessboard in such a way that no two counters lie either in the same row or in the same column? Note that we can reword this problem as: In how many ways can eight rooks be placed on a chessboard so that no two rooks are "attacking" each other? This latter problem is generalized in Chapter 17.

Solution

Let us place one counter in each row in turn. For the first row there are eight columns in which the counter may be put. Having placed this counter, when it comes to the second row, there are just seven columns where we may place a counter, as it must not be in the same column as the counter in the first row. As we place counters in successive rows, the number of possible columns where the next counter may be placed goes down by one at each stage. So the total number of permissible arrangements of the counters is $8 \times 7 \times \ldots \times 2 \times 1$, that is, $8!(= 40{,}320)$. One of these arrangements is shown in Figure 2.2.

Clearly there are, more generally, $n!$ different ways to place n counters on the squares of an $n \times n$ chessboard so that there are neither two counters in the same row nor in the same column. In Exercise 2.2.4A you are asked to generalize this to the case of placing any number of counters on rectangular boards of any size.

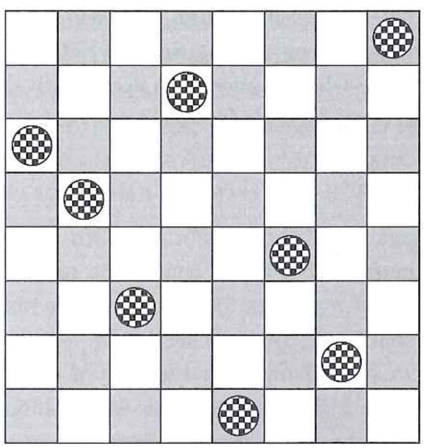

FIGURE 2.2

Exercises

2.2.1A Currently a €10 note has a "serial number" of the form X19298164502, that is, a letter followed by 11 digits. How many different serial numbers of this form are there? A Bank of England £10 note has a serial number of the form CD49000372, that is, two letters followed by eight digits. Are there more of these serial numbers than there are for a €10 note?

2.2.1B A personal identification number (PIN) consists of a sequence of four digits, each drawn from the set {0,1,2,3,4,5,6,7,8,9}, except that the first digit of a PIN cannot be 0. How many different PINs are there? How many different PINs are there in which no digit is repeated?

2.2.2A How many different sequences of length 10 are there in which each of the digits 0,1,2,3,4,5,6,7,8,9 is used once? How long would it take you to list them all if each sequence took one second to write down?

2.2.2B i. How many sequences are there of n digits in which all the digits are different?

ii. How many sequences are there of n digits in which no two consecutive digits are the same?

2.2.3A In three races there are 10, 8, and 6 horses running, respectively. You win a jackpot prize if you correctly predict the first 3 horses, in the right order (assuming no dead heats), in each race. How many different predictions can be made?

2.2.3B A password is a sequence of six characters, the first three being either an upper or a lowercase letter, the next being a digit, and the final two coming from the set {!,£,$,%,^,&,*,(,),_, + , = ,{,},[,], @,#,?} of 19 other symbols occurring on a standard keyboard. How many different passwords are there? How many are there if consecutive characters must be different? How many are there if all the characters must be different?

2.2.4A Let k, m, and n be positive integers with $k \leq m$, $k \leq n$, and $m \leq n$. In how many different ways may k counters be placed on the squares of an $m \times n$ grid so that no two counters are in the same row or in the same column?

2.2.4B In how many different ways may eight red and eight green counters be placed on the squares of an 8×8 chessboard so that there are not two counters on any one square and there is one red counter and one green counter in each row and column?

2.3 COMBINATIONS

Let us now count the number of ways of choosing a specified number of objects from a set when the order of selection does not matter. We tackle this problem by relating it to the problem of counting permutations, which we have already solved. (Reducing a new problem to a case that has already been solved is a common mathematical technique. It is said that many a mathematician who has learned how to make a cup of tea starting with an empty kettle will, when given a full kettle and asked to make tea, first empty the kettle to reduce the problem to one that they already know how to solve.) A couple of examples will make the line of approach clear.

PROBLEM 2.4

A team of three bowls players is to be selected from a squad of six players. How many different teams can be selected?

Solution

We have seen that we can choose three players, in order, from a squad of six players in $P(6,3) = 6 \times 5 \times 4 = 120$ ways. But, *and this is the key point*, there are not 120 different teams of three players. This is because the order in which we pick the members of the team does not matter. For example, choosing first Pat, then Chris, and then Sam leads to the same team as first choosing Chris, then Sam, and then Pat. Thus, each team can be chosen in more than one way. The number of ways in which three given players can be chosen in order is 3!, that is, 6. Since we get 120 when we count each team six times, the number of different teams is $120/6 = 20$. Put another way, the number of different ways to pick three bowls players from six is $P(6,3)/3!$

The technique that we have used in this problem is used again, not only in the next problem, but in many other counting problems. We count the number of arrangements of a particular kind by counting them in such a way that each arrangement is counted more than once. We then adjust our answer to allow for the duplicate counting.

PROBLEM 2.5

How many different hands of 5 cards can be chosen from a pack of 52 cards?

Solution

We can choose 5 cards, in order, from a pack of 52 cards, in $P(52,5)$ different ways. But the order in which the cards are chosen does not affect the hand we end up with. The same hand of 5 cards can be arranged in order in 5! ways and so can be chosen, in order, in 5! ways. Thus $P(52,5)$ gives the number of 5-card hands when each hand is counted 5! times. Hence the number of different 5-card hands is

$$\frac{P(52,5)}{5!} = \frac{(52!/47!)}{5!} = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2{,}598{,}960.$$

We can now generalize the method used in these last two problems. We call a selection of r objects chosen from n objects, *when the order in which they are chosen does not matter*, a *combination* of r objects from n. We use the notation $C(n,r)$ for the number of different combinations of r objects from n. (Notice that the mathematical usage of *permutation*, where the order matters, and *combination*, where it does not, does not correspond to all the uses of these words in everyday life. In football pools permutations or "perms" are selections of football (otherwise known as "soccer") matches where the order does not matter. In a combination lock, the order of the numbers is important.)

The method that we used to solve Problems 2.4 and 2.5 leads us to the general formula for $C(n,r)$.

THEOREM 2.3

For all nonnegative integers r, n with $r \le n$,

$$C(n,r) = \frac{n!}{r!(n-r)!}.$$

Proof

By Theorem 2.2, we know that a set of r objects can be ordered in $r!$ ways. Thus $P(n,r)$, the number of ways in which r objects can be chosen in order from a set of n objects, counts each set of r objects chosen from the given set of n objects $r!$ times. Hence, $C(n,r) = P(n,r)/r!$, and therefore, by Theorem 2.1, $C(n,r) = n!/[r!(n-r)!]$.

The numbers $C(n,r)$ are the well-known *binomial coefficients* that occur in the binomial theorem that we give as Theorem 2.6. There are several common alternative notations for these binomial coefficients. The number for which we have used the notation $C(n,r)$ is often written as $\binom{n}{r}$, or ${}_nC_r$ or C^n_r. We have chosen $C(n,r)$ because it is less cumbersome to print than $\binom{n}{r}$ and, unlike ${}_nC_r$ and C^n_r, it makes the numbers n and r easier to read. It also fits in with the standard mathematical notation, $f(x,y)$, for a function of two variables and also with the notation for two-dimensional arrays in many programming languages. Its disadvantage is that it ties the letter C to a particular meaning. To avoid this, the alternative notation $(n!r)$ was once suggested.*

It is worth noting that since $C(n,r)$ is the number of ways of choosing r objects from n, we must have $0 \le r \le n$. We allow the case $r = n$. In this case the formula gives $C(n,n) = n!/(0!n!) = 1$. This corresponds to the fact that an n-element set A has just one n-element subset, namely, the set A itself. We also have $C(n,0) = n!/(0!n!) = 1$, corresponding to the fact that there is just one subset of A that has zero elements, namely, the empty set $\varnothing$.

The formula for $C(n,r)$ given by Theorem 2.3 can be used to give algebraic proofs of many properties of the binomial coefficients. We prefer, however, to emphasize the combinatorial meaning of these numbers and to give combinatorial proofs whenever this is convenient. In line with this approach, we have given a combinatorial definition of the number $C(n,r)$. The alternative would have been to define $C(n,r)$ by the formula of Theorem 2.3. It would then have been necessary to prove that the number of r-element subsets of an n-element set is indeed $C(n,r)$. Our combinatorial approach is illustrated by our proofs of the next four theorems.

THEOREM 2.4

For all positive integers r, n with $r \le n$, $rC(n,r) = nC(n-1,r-1)$.

Proof

Let X be an n-element set. We evaluate the sum of the numbers of elements in all the r-element subsets of X in two different ways.

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* In *The Printing of Mathematics*, by T. W. Chaundy, P. R. Barrett, and Charles Batey, Oxford University Press, London, 1954. This book is out of date technologically as it was written in the days of hot-metal typesetting, but its advice to mathematical authors is still valuable.

First, as there are $C(n,r)$ subsets of X, each containing r elements, this sum is $C(n,r) \times r$, that is, $rC(n,r)$. Second, consider one particular object, say a, from the n-element set X. To obtain an r-element subset of X containing a we need to choose a further $r-1$ elements from the remaining $n-1$ elements of X. This can be done in $C(n-1,r-1)$ ways. Therefore, each of the n elements of X occurs in $C(n-1,r-1)$ different r-element subsets of X. Consequently, the sum of the numbers of elements in these sets is $n \times C(n-1,r-1)$. As these two different ways of obtaining this sum must lead to the same answer, it follows that $rC(n,r) = nC(n-1,r-1)$.

Note that we can deduce from Theorem 2.4 that $C(n,r) = (n/r)[C(n-1,r-1)]$. This enables us to give a direct, combinatorial proof that $C(n, r) = n!/[r!(n-r)!]$ without the need to consider permutations.

THEOREM 2.5

For all nonnegative integers r, n with $r \le n$, $C(n,r) = C(n,n-r)$.

Proof

Deciding which r objects to select from a set of n objects amounts to exactly the same thing as deciding which $n-r$ objects *not* to select. Hence the number of ways of choosing r objects from n is the same as the number of ways of choosing $n-r$ objects from n.

The next theorem explains how the binomial coefficients get their name.

THEOREM 2.6
The Binomial Theorem

For all variables a, b, and each positive integer n,

$$(a + b)^n = a^n + C(n,1)a^{n-1}b + C(n,2)a^{n-2}b^2 + \dots + b^n,$$

that is,

$$(a+b)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}b^r, \text{ as } C(n,0)=C(n,n)=1.$$

Proof

Consider the product

$$(a + b)(a + b)\dots(a + b)$$

with n pairs of brackets. When we multiply out this product, each separate term that arises comes from choosing either a or b from each pair of brackets and then multiplying these a's and b's together. We obtain the term $a^{n-r}b^r$ each time we choose b from r of these pairs of brackets and a from the remaining $n-r$ pairs. Thus the number of terms of the form $a^{n-r}b^r$ that we obtain equals the number of ways of choosing r pairs of brackets from which to pick b, and this number is $C(n,r)$. Hence when we gather similar terms together, the coefficient of $a^{n-r}b^r$ is $C(n,r)$.

Of course, selecting r b's forces us to select $n-r$ a's. Repeating the argument with a and b interchanged shows that the coefficient of $a^{n-r}b^r$ is also $C(n,n-r)$, as Theorem 2.5 tells us it should be.

The idea that we have used in this combinatorial proof of the binomial theorem will play an important role later in this book (in Chapter 7). The algebraic expression $(a + b)^n$ is called a *binomial* (from the Latin *binomius* meaning "having two names"), and this is why the binomial coefficients were given their name. The binomial theorem is sometimes attributed to Isaac Newton, though the binomial coefficients were known and tabulated long before Newton's time. His main contribution in this area was to prove the form of this theorem that applies when the exponent n is not a positive integer.

Our next theorem about binomial coefficients leads to a very well-known method for calculating their values. We again emphasize that we give a combinatorial proof of this theorem. An algebraic proof, using the formula for $C(n,r)$, is very straightforward but hides the combinatorial meaning of the result.

THEOREM 2.7

For all positive integers r, n with $r \leq n$,

$$C(n+1,r) = C(n,r-1) + C(n,r).$$

Proof

Let X be a set containing $n + 1$ objects, and let a be one of the objects in the set X. We count the number of subsets of X containing r elements by separating them into the set, say $\boldsymbol{Y}$, of those r-element subsets that include a and the set, say $\boldsymbol{Z}$, of those r-element subsets that do *not* include a.

A subset of X in $\boldsymbol{Y}$ contains r elements one of which is a and a further r–1 elements chosen from the n-element set $X \setminus \{a\}$. Thus, there are $C(n,r-1)$ subsets in $\boldsymbol{Y}$.

A subset of X in $\boldsymbol{Z}$ contains r elements none of which is a, and hence consists of r elements chosen from $X \setminus \{a\}$ and hence there are $C(n,r)$ of these.

Each r-element subset of X is either in $\boldsymbol{Y}$ or in $\boldsymbol{Z}$, and none of them is in both. Hence the number of r-element subsets of X is the sum of the number of subsets in $\boldsymbol{Y}$ and the number in $\boldsymbol{Z}$, that is, $C(n+1,r) = C(n,r-1) + C(n,r)$.

The numbers $C(n,r)$, for $0 \leq r \leq n$, are often displayed in a triangle formation, as in Figure 2.3. It then follows from Theorem 2.7 that each number in the $(n + 1)$th row (apart from those at the ends) is the sum of the two adjacent numbers in the row above. For example, the number 21 in the eighth row is the sum of 6 and 15 from the row above. This triangle is usually called *Pascal's triangle*, after the seventeenth-century French mathematician Blaise Pascal, although it was not originated by him.* The first 11 rows of Pascal's triangle are shown in Figure 2.3.

* We quote the following account of the matter from *The Backbone of Pascal's Triangle* by Martin Griffiths, United Kingdom Mathematics Trust (UKMT), Leeds, 2008 p. 10: "Pascal himself called it 'the arithmetical triangle', but after the mathematicians Pierre Rémond de Montmort and Abraham de Moivre referred to it in writing as 'the combinatorial triangle of Mr. Pascal' (in 1708) and 'Pascal's arithmetical triangle' (in 1730) respectively, the name stuck. However the Italian mathematician Nicolo Tartaglia actually published these numbers in 1556, and there is evidence that the Chinese mathematician Yang Hui was working with these numbers in the thirteenth-century (the Chinese do indeed use the term 'Yang Hui's triangle')."

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

FIGURE 2.3

There are innumerable relationships between the binomial coefficients that correspond to patterns that can be found within Pascal's triangle. For example, it follows from Theorem 2.5 that Pascal's triangle is symmetrical about its central vertical axis. Some other relationships are given in the next theorem and in the exercises at the end of this section.

THEOREM 2.8

For all positive integers k, n with $k \leq n$,

$$C(n+1,k+1) = C(n,k) + C(n-1,k) + \ldots + C(k,k).$$

Proof

Let $X = \{x_1, x_2, \ldots, x_n, x_{n+1}\}$. $C(n+1,k+1)$ is the number of subsets of X that contain $k+1$ of the elements of X. We can also count the number of these subsets in the following way. For $k+1 \leq r \leq n+1$ we let X_r be the set of all those $(k+1)$-element subsets, Y, of X such that r is the largest integer for which $x_r \in Y$. Thus, if $Y \in X_r$, then $x_r \in Y$ but $x_{r+1}, \ldots, x_{n+1} \notin Y$.

Clearly, the sets $X_{k+1}, \ldots, X_n, X_{n+1}$ are pairwise disjoint. Also, between them they include all the $(k+1)$-element subsets of X, as each subset of X containing $k+1$ elements must include at least one of the elements $\{x_{k+1}, \ldots, x_{n+1}\}$. Hence

$$\#(X) = \sum_{r=k+1}^{n+1} \#(X_r). \tag{2.1}$$

The sets in X_r contain x_r and k elements chosen from the set $\{x_1, \ldots, x_{r-1}\}$. Therefore, $\#(X_r) = C(r-1,k)$. We can therefore deduce from Equation 2.1 that $C(n+1,k+1) = \sum_{r=k+1}^{n+1} C(r-1,k)$, which, when we rewrite the terms on the right-hand side in reverse order, gives $C(n+1,k+1) = C(n,k) + C(n-1,k) + \ldots + C(k,k)$.

We conclude this section with a simple but intriguing application of Theorem 2.3 to number theory.

THEOREM 2.9

For each positive integer r, the product of any r consecutive positive integers is divisible by $r!$

Proof

We need to prove that for all positive integers k, r the product of the r consecutive integers $k, k+1, k+2, \ldots, k+r-1$ is divisible by r. Now,

$$\frac{k(k+1)(k+2)\ldots(k+r-1)}{r!} = C(k+r-1,r),$$

by Theorem 2.3. Since this binomial coefficient gives the number of r-element subsets of a set of $k+r-1$ elements, it must be an integer. So $k(k+1)(k+2)\ldots(k+r-1)$ is divisible by $r!$

Exercises

2.3.1A A mathematics course offers students the choice of three options from 12 courses in pure mathematics, two options from 10 courses in applied mathematics, two options from 6 courses in statistics, and one option from 4 courses in computing. In how many different ways can the students choose their eight options?

2.3.1B A cricket squad consists of six batsmen, eight bowlers, three wicketkeepers, and four all-rounders. The selectors wish to pick a team made up of four batsmen, four bowlers, one wicketkeeper, and two all-rounders. How many different teams can they pick?

The next three pairs of questions can all be answered by using the binomial theorem. However, you are encouraged to give combinatorial proofs in the style of those we have given for the theorems in this section.

2.3.2A Prove that a set of n elements has 2^n different subsets, and deduce that for each positive integer n, $\sum_{r=0}^{n} C(n,r) = 2^n$.

2.3.2B Prove that, for each positive integer n, $\sum_{r=0}^{n} C(n,r)^2 = C(2n,n)$.

2.3.3A Prove that, for all positive integers n, k, s with $s \leq k \leq n$, $C(n,k)\,C(k,s) = C(n,s)\,C(n-s,k-s)$.

2.3.3B Let X be a finite set. Prove that the number of subsets of X that contain an even number of elements is equal to the number of subsets of X that contain an odd number of elements. Deduce that for each positive integer n, $\sum_{r=0}^{n} (-1)^r C(n,r) = 0$.

2.3.4A Prove that, for each positive integer n, $\sum_{r=0}^{n} rC(n,r) = n2^{n-1}$.
[*Hint:* Let X be an n-element set. Note that as X has $C(n, r)$ subsets containing r elements, $\sum_{r=0}^{n} rC(n,r) = \sum_{A\subseteq X} \#(A)$. Also, we can calculate $\sum_{A\subseteq X} \#(A)$ by pairing off each subset A of X with its complement $X\backslash A$. How many pairs are there, and what is $\#(A) + \#(X\backslash A)$?]

2.3.4B Prove that for each positive integer n, $\sum_{r=0}^{n} r^2 C(n,r) = n(n+1)2^{n-2}$.
(*Hint:* First find $\sum_{r=0}^{n} r(r-1)C(n,r)$ by counting the ordered pairs (a, b), where a and b are chosen from an n-element set, in two ways. Then use the result of Exercise 2.3.4A.)

2.4 APPLICATIONS TO PROBABILITY PROBLEMS

We begin with a very simple problem.

PROBLEM 2.6

A fair coin is tossed 10 times. What is the probability of getting four heads?

Solution

Each toss can have one of two results, either a head or a tail. Thus there are altogether $2^{10} = 1024$ equally likely outcomes for a sequence of ten tosses. The number of these sequences that consist of four heads and six tails is the number of ways in which four of the ten tosses can be heads, that is, $C(10,4) = 210$. Hence the probability of getting four heads is 210/1024, which is 0.205 to three decimal places.

This calculation is straightforward enough, but what does it mean? And how do we know that we have solved the problem correctly? Fortunately, from the point of view of combinatorics we do not have to answer the difficult philosophical question as to what is meant by *probability*. Our work on this and similar probability problems begins after any philosophical work has been done, and the precise mathematical problem has been formulated. These problems will involve a set of events E, taken to be equally likely, together with a subset E_1. The probability of an event falling into the subset E_1 is defined to be ratio of the number of events in E_1 to the number of events in E, that is,

$$\frac{\#(E_1)}{\#(E)}.$$

Thus our task will be to calculate the numbers in this ratio, and we will not need to concern ourselves with its philosophical significance.

However, there remains the question of how we knew, from the formulation of the problem, which the sets E and E_1 of events were in this particular case. You may have noticed that the problem refers to a *fair coin*. This is intended to indicate that on any one throw of the coin the probability (whatever this means) of getting a head is exactly the same as the probability of getting a tail and that the outcome of any one toss is independent of what happens in earlier or later tosses. This means that any one sequence of outcomes is as likely as any other sequence. This is reflected in our solution, where we took the set of events, E, to be the set of all 1024 sequences of 10 results of a toss, each toss resulting in either a head or a tail.

For us, doing combinatorics, that is the end of the matter. If you wish to apply the answer to Problem 2.6 and similar calculations to practical situations, you need to know how realistic the assumption of a *fair coin* is and how statements of probability are to be interpreted. These are not easy questions, and so it is fortunate that, in this book, we can largely avoid answering them.

In general, the statement of a probability problem should indicate which set is to be taken as the set, E, of events. Thus the events in E are events that are to be regarded as equally likely. This indication is often done in a coded way. For example, some of these problems concern packs of cards *dealt at random*. This is intended to mean that any of the 52! ways in which the pack of 52 cards can be arranged is as likely as any other.

Hence the set E will consist of these 52! arrangements or will be derived from it in some straightforward way.

The codes used in this way to set up probability problems are analogous to the coded way of describing problems in mechanics, where such phrases as *a light string* or *a frictionless pulley* are intended to indicate what assumptions can be made in devising the mathematical model.

We are now in a position to answer Problem 2A from Chapter 1.

PROBLEM 2.7

In the British national lottery, 6 balls are drawn at random from a set of 49 balls, numbered 1, 2, …, 49. To play, you need to buy one or more tickets. Each player selects six numbers from 1 to 49 on each of his or her tickets. You win the *jackpot* if on one of your tickets you have chosen the six numbers on the balls that are drawn. What is the probability that a particular ticket wins the jackpot in the national lottery?

Solution

Here, the set E consists of all possible ways of drawing 6 balls from 49. Thus $\#(E) = C(49, 6) = 13{,}983{,}816$, since the order in which the balls are drawn is irrelevant. The set E_1 consists of just the one case where the six numbers selected by the player correspond to the numbers on the six balls that are drawn. So $\#(E_1) = 1$. Thus the probability of a ticket winning the jackpot is 1/13,983,816.

PROBLEM 2.8

You win a prize of £10 in the national lottery if precisely three of the six numbers that you select are included among the six numbers on the balls that are drawn. What is the probability that you win a prize of £10?

Solution

Here, E is again the set of all possible ways that 6 balls can be drawn from 49, so, as before, #(E) = C(49,6). E_1 is the number of ways that 6 balls can be drawn so that the numbers on 3 of them are included among your 6 numbers, and the numbers on the other 3 are included among the 43 numbers that you did not select. So to get a set of 6 numbers that is in E_1 we first need to choose 3 numbers from the 6 numbers on the chosen balls and then 3 numbers from the other 43 numbers. Thus $\#(E_1) = C(6,3) \times C(43,3)$, and hence the required probability is $[C(6,3) \times C(43,3)]/C(49,6) = 8{,}815/499{,}422$. This is approximately a 1 in 56.7 chance, or 0.01765 to five decimal places.

Card games are a rich source of probability problems of this type. To understand the examples that follow, you do not need to know the rules of the card games that are mentioned. All you need to know is that a standard pack contains 52 cards, divided up into 4 *suits*: spades (for which the symbol ♠ is used), hearts (♥), diamonds (♦), and clubs (♣). There are 13 cards in each suit. The *ranks* of the cards in each suit are 2,3,4,5,6,7,8,9,10, jack (often denoted by J), queen (Q), king (K), and ace (A).

In games such as bridge and whist there are four players, often called North, South, East, and West. In the initial deal, each player is dealt a hand of 13 cards. Thus the number of different hands a player may receive is the number of ways of choosing 13 cards from 52, that is, $C(52,13) = 635{,}013{,}559{,}600$. In bridge and whist the distribution of the cards between the different suits is important.

PROBLEM 2.9

How many different bridge hands are there with

i. Four spades, four hearts, four diamonds, and one club?
ii. Four spades, four hearts, three diamonds, and two clubs?

Solution

i. We can easily solve this problem by using the methods of this chapter. To choose a hand of the kind described, we first choose 4 spades from the 13 spades in the pack, which we can do in $C(13,4)$ ways, then 4 hearts, which can also be done in $C(13,4)$ ways, then 4 diamonds, also in $C(13,4)$ ways, and finally 1 club, which can be chosen in $C(13,1)$ ways. Hence using the principle of multiplication of choices, the total number of hands with 4 spades, 4 hearts, 4 diamonds, and 1 club is $C(13,4) \times C(13,4) \times C(13,4) \times C(13,1) = 4{,}751{,}836{,}375$.
ii. Similarly, the answer here is $C(13,4) \times C(13,4) \times C(13,3) \times C(13,2) = 11{,}404{,}407{,}300$.

A hand with four cards in each of three suits and one card in the fourth suit is said to have a 4–4–4–1 *suit distribution*. In general, a hand with an a–b–c–d suit distribution, with $a \geq b \geq c \geq d$, is one with a cards in one suit, b cards in a second suit, c cards in a third suit, and d cards in the fourth suit, irrespective of which suits these are. Of course, for a bridge or whist hand of 13 cards, we require that $a + b + c + d = 13$.

PROBLEM 2.10

What is the probability that a bridge hand dealt at random has the following suit distributions?

i. 4–4–4–1
ii. 4–4–3–2

Solution

i. We have calculated in Problem 2.9(i) the number of bridge hands with four cards in each of three specified suits and one card in a fourth suit. To get the number of all hands with a 4–4–4–1 suit distribution we need to multiply the answer to that problem by the number of ways in which we can specify the three suits with four cards each and the one suit with just one card. We can choose the three 4-card suits in $C(4,3) = 4$ ways, and having chosen these, the suit with one card is automatically determined. (Equivalently, the one suit containing one card may be chosen in $C(4,1) = 4$ ways, and the three suits each containing three cards are then automatically determined.) So there are $4 \times 4{,}751{,}836{,}375 = 19{,}007{,}345{,}500$ hands with a 4–4–4–1 suit distribution. To get the probability that a hand dealt at random has this suit distribution, we need to divide this number by the total number of bridge hands. Thus the required probability is 19,007,345,500/635,013,559,600. This probability is 0.030 to three decimal places.

ii. We need to multiply the answer to Problem 2.9(ii) by the number of ways of choosing the two suits with four cards each and the one suit with three cards, after which the suit with two cards is automatically determined. This multiplier is $C(4,2) \times C(2,1) = 12$. So the total number of such hands is $12 \times 11{,}404{,}407{,}300 =$ 136,852,887,600, and hence the probability of dealing such a hand at random is 136,852,887,600/635,013,559,600, which is 0.216 to three decimal places.

PROBLEM 2.11

How many poker hands of five cards are there in which there is at least one suit with no cards in it?

Solution

We list the suit distributions with at least one suit with zero cards in it, and then use the method of Problem 2.10 to work out the total number of hands with these suit distributions, as follows:

Suit Distribution	Number of Hands
5–0–0–0	5,148
4–1–0–0	111,540
3–2–0–0	267,696
3–1–1–0	580,008
2–2–1–0	949,104
Total	1,913,496

You might think that there is a quicker way to solve Problem 2.11. It is easy to count the number of 5-card poker hands with, for example, no spades. Such a hand is obtained by choosing 5 cards from the 39 cards in the pack that are not spades. So there are $C(39,5) = 575{,}757$ poker hands with no spades. There is a similar number of hands with no hearts, with no diamonds, and with no clubs, respectively. So it seems that the total number of 5-card hands with a missing suit is $4 \times 575{,}757 = 2{,}303{,}028$. Unfortunately this quick solution gives an answer that is different from that we obtained in our solution to Problem 2.11. Where have we gone wrong?

It is not difficult to see where our mistake lies. Let V be the set of poker hands with at least one missing suit, and let V_S, V_H, V_D, V_C be those hands with no spades, hearts, diamonds, and clubs, respectively. Clearly, $V = V_S \cup V_H \cup V_D \cup V_C$. Our second calculation assumed that $\#(V) = \#(V_S) + \#(V_H) + \#(V_D) + \#(V_C)$, but this overlooks the fact that some of the hands in V are in more than one of the sets V_S, V_H, V_D, V_C For example, a hand made up of three diamonds and two clubs but no spades and no hearts is in both V_S and V_H. Thus the sum $\#(V_S) + \#(V_H) + \#V_D + \#(V_C)$ counts some of the hands with a missing suit more than once. Thus it is no wonder that our "quick answer" is higher than the correct answer that we obtained in the solution to Problem 2.11.

To sum up this point, it is correct that if we have a collection of sets $X_1, \ldots, X_k$ *that are pairwise disjoint*, that is, $X_i \cap X_j = \varnothing$ for $1 \le i < j \le k$, then

$$\#\left(\bigcup_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} \#(X_i), \tag{2.2}$$

but this is not true if the sets, X_i, are not pairwise disjoint. We discuss in Chapter 4 the modification that we need to make to Equation 2.2 in the cases where these sets are not pairwise disjoint.

Exercises

2.4.1A A ticket wins a fourth prize in the national lottery if precisely four of the six numbers selected on it are included among the six numbers on the balls that are drawn. What is the probability that a ticket wins a fourth prize?

2.4.1B In his autobiography, *What I Remember*, Adolphus Trollope* describes the Italian lottery as follows:

> Ninety numbers, 1–90, are always put into the wheel. Five only of these are drawn out. The player bets that a number named by him shall be one of these (*semplice estratto*); or that it shall be the first drawn (*estratto determinato*); or that two numbers named by him shall be two of the five drawn (*ambo*); or that three so named shall be drawn (*terno*). It will be seen, therefore, that the winner of an *estratto determinato*, ought, if the play were quite even, to receive ninety times his stake. But, in fact, such a player would receive only 75 times his stake, the profit of the Government consisting of this pull of 15 per 90 against the player. Of course, what he ought to receive in any of the other cases is easily (not by me, but by experts) calculable.

What would be fair odds for the *semplice estratto*, *ambo*, and *terno* bets?

2.4.2A A bag contains 50 red balls and 50 blue balls. Ten balls are drawn at random from the bag and not replaced. What is the probability that this sample will contain five red balls and five blue balls? (This question is connected with the reliability of opinion polls. See the solution for more about this.)

2.4.2B A bag contains $2n$ red balls and $2n$ blue balls. What is the probability that if $2n$ balls are drawn at random, the sample will consist of n red balls and n blue balls?

2.4.3A (This is Problem 2B from Chapter 1.) If there are n people in a room, what is the probability that at least two of them share a birthday? How large does n have to be before this probability becomes more than a half? (By "sharing a birthday" we mean that two people were born in the same month and on the same day in that month but not necessarily in the same year. For the purpose of this problem you should ignore leap years, so that there are 365 possible birthdays for each person. You should also assume that all 365 birthdays are equally likely. In fact, as the answer to Exercise 2.4.3B

* Thomas Adolphus Trollope, *What I Remember*, abridged by Herbert van Thal, William Kimber, London, 1973, pp. 189–190 (originally published in three volumes by R. Bentley & Son, London, 1887–1889).

indicates, if, as is actually the case, some birthdays are more likely than others, the probability of a coincidence increases.)

2.4.3B Suppose there are $2n$ balls in a bag of which a are red and b are blue, where $a + b = 2n$. One ball is removed at random from the bag and then replaced. Then a second ball is drawn at random from the bag and then replaced. Calculate the probability that either a red ball is drawn twice or a blue ball is drawn twice. Show that this probability is a minimum when $a = b = n$.

2.4.4A What is the probability that a bridge hand dealt at random has the following suit distributions?

i. 5–4–3–1   ii. 5–4–4–0   iii. 4–3–3–3

2.4.4B Suppose that in a given bridge deal North and South between them have nine spades. What is the probability that the remaining four spades in the other two hands are divided two-two?

2.4.5A *Poker Hands.* Poker hands, which in standard poker games consist of 5 cards drawn from the full pack of 52 cards, are classified as follows:

i. *Flush:* five cards all of the same suit but not forming a sequence of consecutive ranks.
For example, 5♣, 7♣, J♣, Q♣, K♣ is a flush.

ii. *Four of a kind:* four cards of one rank and one other card, for example, 3♠, 3♥, 3♦, 3♣, J♠.

iii. *Full house:* three cards of one rank and two cards of another rank, for example, 7♠, 7♥, 7♣, 10♦, 10♣.

iv. *One pair:* two cards of one rank and three cards of three different ranks, for example, J♦, J♣, 4♠, 7♥, K♠.

v. *Straight:* five cards of consecutive ranks but not all in the same suit (note that for this purpose an ace may count either low or high, so that both A, 2, 3, 4, 5 and 10, J, Q, K, A count as consecutive ranks), for example, 7♦, 8♦, 9♥, 10♠, J♥.

vi. *Straight flush:* five cards of consecutive ranks (again, an ace may count either low or high) and in the same suit, for example, 4♥, 5♥, 6♥, 7♥, 8♥.

vii. *Three of a kind:* three cards of one rank and two cards of two different ranks, for example, 9♥, 9♣, 9♦, 4♥, Q♣.

viii. *Two pairs:* two cards of one rank, two cards of another rank, and a fifth card of a third rank, for example, 5♦, 5♣, 8♥, 8♦, J♦.

ix. *Other hands:* all other hands that do not fall into any of the preceding categories.

Calculate how many poker hands there are in each of the categories (a) to (e) above. Hence work out the probability that a poker hand dealt at random falls into each of the these categories.

2.4.5B Complete the calculation of Exercise 2.4.5A by working out how many poker hands there are that fall into the categories (f) to (i) above. Also work out the probability that a poker hand dealt at random falls into each of these categories.

2.4.6A We see, from the solution to Problem 2.6, that if a fair coin is tossed 10 times, then the probability of getting four heads is $C(10,4)/2^{10}$. More generally, if a fair coin is tossed n times, the probability of getting r heads, for $0 \le r \le n$ is $C(n,r)/2^n$. We note that the different probabilities, as k ranges from 0 to n, add up to 1, as we expect. This follows from the result of Exercise 2.3.2A, as we can deduce from this result that

$$\sum_{r=0}^{n} \frac{C(n,r)}{2^n} = \frac{1}{2^n}\sum_{r=0}^{n} C(n,r) = \frac{1}{2^n} \times 2^n = 1.$$

It can be shown that if the coin is not necessarily fair, so that the probability of getting a head is p and hence the probability of getting a tail is $1-p$, where p need not be equal to $\frac{1}{2}$, then the probability of getting r heads is $C(n,r)p^r\,(1-p)^{n-r}$. Show that these probabilities also add to 1, that is, show that for all values of p, $\sum_{r=0}^{n} C(n,r)p^r(1-p)^{n-r} = 1$.

2.4.6B A coin is biased so that the probability of getting a head is 0.6. If the coin is tossed five times, what is the probability of getting three heads?

2.5 THE MULTINOMIAL THEOREM

The multinomial theorem is a generalization of the binomial theorem, and, as with the binomial theorem, it can be approached either algebraically or combinatorially. We will look at it from a combinatorial viewpoint. We begin with the following, which generalizes Problem 2.6.

PROBLEM 2.12

A football team plays 38 games in a season. It has equal probabilities of winning, drawing, or losing each game. What is the probability that the team wins 20 games, draws 11, and loses only seven games?

Solution

The results obtained by the team can be regarded as a sequence of 38 symbols, each of which is either W, D, or L, indicating a win, a draw, and a loss, respectively. As the team has equal probabilities of achieving each of the three possible results, each sequence of 38 Ws, Ds, and Ls is equally likely. Hence, the required probability is the number of these sequences made up of 20 Ws, 11 Ds, and 7 Ls divided by the total number of sequences of 38 symbols. The second number is the easier to calculate. Since there are 3 choices for each symbol, there are altogether 3^{38} sequences of 38 symbols each of which is W, D, or L.

Next we count the number of these sequences in which there are 20 Ws, 11 Ds, and 7 Ls. We can construct such a sequence by first choosing the 20 positions in which the symbol W occurs. This involves choosing 20 positions from 38 and so may be done in $C(38,20)$ ways. This leaves 18 positions for the 11 Ds, which may therefore be chosen in $C(18,11)$ ways, and the remaining seven positions are then automatically filled by the Ls, that is, in just one way. Hence the total number of sequences is

$$C(38,20) \times C(18,11) = \frac{38!}{20!18!} \times \frac{18!}{11!7!} = \frac{38!}{20!11!7!},$$

and the required probability is $[38!/(20!11!7!)]/3^{38}$, which is 0.0008 to four decimal places.

We see from the form, 38!/(20!11!7), of our final expression for the number of sequences made up of 20, 11, and 7, respectively, of the three symbols that we would also obtain this answer if we had, for example, first placed the 11 Ds, then the 7 Ls, and finally the 20 Ws. It also suggests the general result, given as Theorem 2.10.

First, we generalize Problem 2.12. As we know, if we have k different objects, they can be arranged in order in $k!$ ways. However, if all the k objects are indistinguishable, all these arrangements look the same, and, from this point of view there is just one way to arrange the objects in order. In general, whenever we have a set comprising both distinguishable and indistinguishable objects, by "different" arrangements, we mean arrangements of those objects that can be distinguished from one another. For example, the letters A,B,C,D, and E may be arranged in order in 5! different ways. However, if all we are interested in is whether a letter is a vowel (V) or a consonant (C), then the sequences A,B,C,D,E and E,B,C,D,A would both be recorded as V,C,C,C,V. Regarding the two vowels as indistinguishable and the three consonants as indistinguishable, there are just 10, that is, $C(5,2)$, different ways of arranging these five letters, corresponding to the number of ways of choosing the positions of the two Vs.

THEOREM 2.10

Suppose that for $1 \le r \le k$, we have n_r objects of type T_r, where the objects of any given type are indistinguishable but may be distinguished from the objects of any other type. Suppose also that we have n objects in total, so that $\sum_{r=1}^{k} n_r = n$. Then the number of different ways of arranging the n objects in order is $n!/(n_1!n_2!\ldots n_k!)$.

Proof

We have n positions to fill. We can choose n_1 of these for the objects of type T_1 in $C(n,n_1)$ ways. When these positions have been chosen, there remain $n-n_1$ positions to be filled, and hence positions for the n_2 objects of type T_2 may be chosen in $C(n-n_1,n_2)$ ways. Then the positions for the n_3 objects of type T_3 may be chosen in $C(n-n_1-n_2,n_3)$ ways, and so on. Therefore, the total number of different ways of choosing positions for the n objects is

$$C(n,n_1) \times C(n-n_1,n_2) \times C(n-n_1-n_2,n_3) \times \ldots \times C(n-n_1-n_2-\ldots-n_{k-1},n_k)$$

$$= \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times \frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!} \times \ldots \times \frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k!0!}$$

$$= \frac{n!}{n_1!n_2!\ldots n_k!},$$

after a lot of canceling. Having obtained the simplified formula by an algebraic argument, we should look for a combinatorial argument that shows why it is correct and that gives us a better understanding of why it is true.

Ignoring first the fact that the objects of each type are indistinguishable, we see that the n objects may be arranged in order in $n!$ different ways. However, since the n_1 objects of type T_1 are indistinguishable, we need to divide by $n_1!$ to allow for the fact that, once the positions of these objects have been chosen, the order in which each object of type T_1 is chosen does not matter. Likewise, we need also to divide by $n_2!$, $n_3!,\ldots,n_k!$ and hence the number of indistinguishable arrangements is $n!/(n_1!n_2!\ldots n_k!)$.

THEOREM 2.11
The Multinomial Theorem

The coefficient of the term $a_1^{n}a_2^{n_2}\ldots a_k^{n_k}$ in the expansion of $(a_1 + a_2 + \ldots + a_k)^n$, where $n_1 + n_2 + \ldots + n_k = n$, is $n!/(n_1!n_2!\ldots n_k!)$.

Proof

Each term in the expansion of $(a_1 + a_2 + \ldots + a_k)^n$ has the form $t_1t_2\ldots t_n$, where each t_r, for $1 \leq r \leq n$, is one of the symbols $a_1,a_2,\ldots,a_k$. The coefficient of $a_1^{n_1}a_2^{n_2}\ldots a_k^{n_k}$ is the number of such sequences in which, for $1 \leq r \leq n$, there are n_r occurrences of the symbol a_r. By Theorem 2.10 there are $n!/(n_1!n_2!\ldots n_k!)$ such sequences. This completes the proof.

Exercises

2.5.1A If 12 dice are thrown simultaneously, what is the probability that each of the faces from one to six comes up twice?

2.5.1B If 21 dice are thrown simultaneously, what is the probability that 1 comes up once, 2 comes up twice, 3 comes up three times, 4 comes up four times, 5 comes up five times, and 6 comes up six times?

2.5.2A In how many different ways can one arrange the sequence of letters in the word ABRACADABRA?

2.5.2B In how many different ways can one arrange the sequence of letters in the word PROPERISPOMENON ?

2.6 PERMUTATIONS AND CYCLES

In the final section of the chapter, we look at permutations in a new way that will turn out to be very fruitful later on. We have defined a *permutation* of n objects to be a way of choosing these objects in order. For example, one permutation of the set {1,2,3,4,5,6} is the choice of these numbers in the order 6,1,3,5,4,2. We can think of this permutation as a reordering of the set.

In this way, we regard this permutation as a bijection (a one-one onto function) mapping the set {1,2,3,4,5,6} to itself. In general, for any set X, by a *permutation* of X, we mean a bijection $f: X \rightarrow X$. Usually, however, we will be considering permutations of the set $\{1,2,3,\ldots,n\}$ consisting of the first n positive integers.

There are two commonly used notations for permutations. The first is a minor variant of the diagram of Figure 2.4. We drop the arrows and put the numbers inside brackets. So the permutation above would, in this notation, be written as

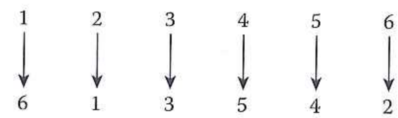

FIGURE 2.4

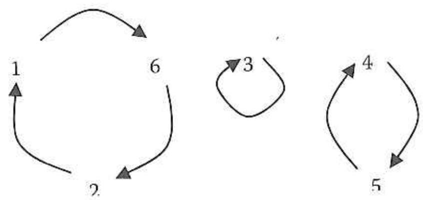

FIGURE 2.5

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 5 & 4 & 2 \end{pmatrix}.$$

We call this the *bracket notation* for permutations. It is a rather cumbersome notation, and it is not very helpful when it comes to answering questions about permutations. For example, the bracket notation does not make it very clear how many times we need to repeat a permutation to get each number back to its starting point. So we introduce a second, more useful notation.

Since the domain and codomain of a permutation are the same set, we can represent them in a single diagram showing how the elements of the set are mapped by the permutation. For example, the above permutation may be represented by the diagram in Figure 2.5.

We can see from this diagram that the effect of the permutation is to cycle the numbers 1, 6, 2 in this order, to leave 3 fixed, and to cycle (or interchange) 4 and 5. Thus, the permutation is made up of three parts, namely, cycles of lengths 2 and 3 and one fixed point. We can represent this using a notation in which we write the numbers in three separate pairs of brackets, showing which cycle they are in, with the order of the numbers in each bracket showing how they are mapped. Thus in *cycle notation* we can write the permutation as

$$(1\ \ 6\ \ 2)(3)(4\ \ 5).$$

This is sometimes called the *disjoint cycle form* of the permutation because the numbers making up the different cycles form disjoint sets.

From the disjoint cycle form we see that the numbers 1, 6, and 2 are in a cycle of length 3 and so return to their initial position after we have carried out the permutation 3, 6, 9 … times and, in general, any multiple of three times. Likewise, the numbers 4 and 5 form a cycle of length 2 and so return to their original position after we have carried out the permutation a multiple of two times. The number 3 is not moved, but we can think of it as forming a cycle of length 1. Thus, all the numbers are returned to their original positions for the first time after we have carried out the permutation six times.

Although the cycle notation for permutations makes it easy to answer such questions, we need to be careful with it. We are using a linear notation to represent cycles. For example, the bracket (1 6 2) represents the first cycle in Figure 2.5. We need to remember that it means that 2 is mapped back to 1, as this is not immediately obvious from the notation. Also, there is some arbitrariness in this notation. We could equally well have written it as (6 2 1) or (2 1 6), as these both represent the same cycle. Likewise, the cycle (4 5) of length 2 could also be written as (5 4). Although it is conventional to have first in each bracket the lowest number in the cycle, this is not essential. Furthermore, we could also have written the disjoint cycles in a different order, for example, as (4 5)(1 6 2)(3). We will need to remember this when counting permutations.

There is another notational point. The cycle (3), of length 1, that occurs in our permutation tells us that it maps 3 to itself. It is usual to omit cycles of length 1 when writing permutations in cycle notation. So normally we would write the above permutation as (1 6 2) (4 5). Although it is convenient to omit cycles of length 1, this does introduce another ambiguity. Thus, (1 6 2)(4 5) could be a permutation of the set {1,2,3,4,5,6} that fixes 3, but it could also be, for example, a permutation of {1,2,3,4,5,6,7,8} that fixes 3, 7, and 8. So it is really safe to omit cycles of length 1 only if it is clear from the context which set of numbers we are permuting.

We will make good use of the cycle notation for permutations later in the book, especially in Chapters 11, 12, 13, and 14.

Exercises

2.6.1A Write the following permutation in cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 8 & 9 & 7 & 4 & 10 & 2 & 6 & 5 \end{pmatrix}.$$

2.6.1B Write the following permutation in cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 2 & 8 & 10 & 7 & 6 & 4 & 9 & 3 & 1 \end{pmatrix}.$$

2.6.2A How many different permutations are there of the numbers {1,2,3,4,5,6,7,8,9,10} made up of

i. Three disjoint cycles of lengths 2, 3, and 5?

ii. Three disjoint cycles of which two are of length 3 and one is of length 4?

2.6.2B How many different permutations are there of the numbers {1,2,3,4,5,6,7,8,9,10} made up of

i. Four disjoint cycles of lengths 1, 2, 3, and 4?

ii. Four disjoint cycles of which three are of length 2 and one is of length 4?

2.6.3A If a permutation of the set $\{1,2,3,\ldots,n\}$ is chosen at random, what is the probability that it consists of a single cycle of length n?

2.6.3B If a permutation of the set $\{1,2,3,\ldots,n\}$ is chosen at random, what is the probability that it includes exactly one cycle of length 1?

CHAPTER 3

Occupancy Problems

3.1 COUNTING THE SOLUTIONS OF EQUATIONS

When we discussed Problem 3C in Chapter 1 we noted that attempting to determine even the *number* of solutions in nonnegative integers of an equation of the form $x + y + z = n$ by listing them would become impractical if the number of variables were substantially increased. Here, we first show how a simple *reinterpretation* of the problem allows us to solve *all* such problems instantly: Indeed we can simply write down the answer! To help you follow the method we begin by solving the problem in a particular case, "large" enough to make the listing method at best tiresome but "small" enough to fit the reinterpretation easily on the page. It should be fairly clear that the method employed is perfectly general,* that is, applicable in all circumstances.

PROBLEM 3.1

How many solutions are there in nonnegative integers of the equation $x + y + z + w + t = 14$?

Solution

It is easy to write down many solutions (x, y, z, w, t) of this equation, for example, (1,2,2,7,2) or (2,0,6,5,1) or (2,6,5,0,1) or (0,3,0,3,8). With each of these solutions we can associate a diagram of dots (•) and lines (|) , as follows:

With (1,2,2,7,2) associate the diagram • | • • | • • | • • • • • • • | • •.

With (2,0,6,5,1) associate the diagram • • | | • • • • • • | • • • • • | •.

With (2,6,5,0,1) associate the diagram • • | • • • • • • | • • • • • | | •.

With (0,3,0,3,8) associate the diagram | • • • | | • • • | • • • • • • • •.

In the first diagram the four lines split the 14 dots into five groups of sizes 1, 2, 2, 7, and 2. In the second diagram the four lines split the 14 dots into five groups of sizes 2, 0,

* We commented on this attitude in Chapter 1, Section 1.3.

6, 5, and 1. In the third diagram the four lines split the 14 dots into five groups of sizes 2, 6, 5, 0, and 1. In the fourth diagram the four lines split the 14 dots into five groups of sizes 0, 3, 0, 3, and 8.

We readily see that each diagram faithfully represents the solution associated with it. It is equally clear that, conversely, any such diagram comprising four lines and 14 dots gives rise to a unique solution of the given equation; for example, the diagram

| | • • • • • • • • • • | | • • • •

corresponds to the solution (0,0,10,0,4). In this way we clearly obtain a matching of the set of solutions of the given equation with the set of all diagrams comprising four lines and 14 dots. So to solve Problem 3.1 we only need to work out how many diagrams of this type there are.

Now each diagram has 18 "places" each to be filled with a line or a dot. So the total number of diagrams is just the number of ways of choosing (out of 18 places) the 4 places in which to put a line, the remaining 14 places being filled automatically with dots. It follows that the number of different solutions, in nonnegative integers, of the given equation is just the binomial coefficient $C(18,4)$. Perhaps you are thinking that we could, instead, have put 14 dots in the 18 places, filling the remaining 4 places with lines. Then the number of solutions of our equation would, by the same argument, be $C(18,14)$. Fortunately, by Theorem 2.5, $C(18,4) = C(18,14)$.

Despite our general wish to impress on you that checking one example doesn't provide a general proof, we claim here that the method applied to our particular example is, transparently, just as valid when applied generally. We can therefore give a brief proof of the general result.

THEOREM 3.1

For each positive integer k and each nonnegative integer n, the number of nonnegative integer solutions, $(x_1, x_2,\ldots, x_k)$, of the equation $x_1 + x_2 + \ldots + x_k = n$ is $C(n + (k-1),k-1)$ or, equivalently, $C(n + (k-1),n)$.

Proof

As we have seen from the example above, the nonnegative integer solutions of the equation $x_1 + x_2 + \ldots + x_k = n$ are in one–one correspondence with sequences made up of n dots and $k-1$ lines, and there are $C(n + (k-1),k-1)$ such sequences.

We are now able to solve the following two problems from Chapter 1.

PROBLEM 3.2

(This is Problem 3A of Chapter 1.) A manufacturer of high-quality (and therefore high-priced!) chocolates makes just six different flavors of chocolate and sells them in boxes of 10. He claims he can offer more than 3000 different "selection boxes." If he is wrong, he will fall foul of the advertizing laws. Should he fear prosecution?

Solution

To make up a box of 10 chocolates the manufacturer has to decide how many chocolates of each of the six flavors to include. So the number of different selections is the number of nonnegative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$. By Theorem 3.1, this is $C(15,5)$, which is equal to 3003. The manufacturer can therefore sleep soundly.

To solve the next problem we need a slight modification of Theorem 3.1 to cover the case where we consider only *positive* integer solutions.

THEOREM 3.2

Let k, n be positive integers with $n \geq k$. Then the number of *positive* integer solutions, $(x_1, x_2,\ldots, x_k)$, of the equation $x_1 + x_2 + \ldots + x_k = n$ is $C(n-1,k-1)$, or, equivalently, $C(n-1,n-k)$.

Proof

Let $X_i = x_i - 1$ for $1 \leq i \leq k$. Then, $X_1 + X_2 + \ldots + X_k = x_1 + x_2 + \ldots + x_k - k$ and, for $1 \leq i \leq k$, $x_i \geq 1 \Leftrightarrow X_i \geq 0$. Therefore, $(x_1, x_2,\ldots, x_k)$ is a solution in positive integers of the equation

$$x_1 + x_2 + \ldots + x_k = n \tag{3.1}$$

if and only if $(X_1, X_2,\ldots, X_k)$ is a solution in nonnegative integers of the equation

$$X_1 + X_2 + \ldots + X_k = n - k. \tag{3.2}$$

So the number of positive integer solutions of Equation 3.1 is the same as the number of nonnegative integer solutions of Equation 3.2. Hence, by Theorem 3.1, the number of positive integer solutions of Equation 3.1 is $C((n-k) + (k-1),k-1)$, that is, $C(n-1,k-1)$.

PROBLEM 3.3

(This is Problem 3B of Chapter 1.) There is a widely held view that, in a truly random selection of six distinct numbers from among the numbers 1 to 49 (as in the British national lottery), the chance that two consecutive numbers will be chosen is extremely small. Has this opinion any validity?

Solution

The total number of ways of choosing six distinct numbers from the numbers 1 to 49 is, of course, $C(49,6)$. Rather than count the number of selections in which consecutive numbers occur, we count those in which consecutive numbers do *not* occur. Thus, we count the number of sextuples of positive integers, $(t_1,t_2,t_3,t_4,t_5,t_6)$, satisfying

$$0 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < 50$$

and

$$t_2 - t_1,\ t_3 - t_2,\ t_4 - t_3,\ t_5 - t_4, \text{ and } t_6 - t_5 \text{ are all greater than 1.} \tag{3.3}$$

We see that the number of such sextuples is the same as the number of sextuples satisfying $-1 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < 51$, where each of the numbers $t_1 - (-1)$, $t_2 - t_1$,

$t_3 - t_2$, $t_4 - t_3$, $t_5 - t_4$, $t_6 - t_5$, and $51 - t_6$ is at least 2. (*All* the differences now being on equal footing, namely, "greater than or equal to 2," makes life a little easier!) Now, putting $T_1 = t_1 - (-1)$, $T_2 = t_2 - t_1$, $T_3 = t_3 - t_2$, $T_4 = t_4 - t_3$, $T_5 = t_5 - t_4$, $T_6 = t_6 - t_5$, and $T_7 = 51 - t_6$, we see that $T_1 + T_2 + \ldots + T_7 = 52$. Hence the number of sextuples $(t_1, t_2, t_3, t_4, t_5, t_6)$ satisfying Equation 3.3 is the same as the number of positive integer solutions, $(T_1, T_2, T_3, T_4, T_5, T_6, T_7)$, of the equation $T_1 + T_2 + \ldots + T_7 = 52$ with $T_i \geq 2$ for $1 \leq i \leq 7$. Copying the method used in the proof of Theorem 3.2, by putting $S_i = T_i - 2$, for $1 \leq i \leq 7$, this is the same as the number of nonnegative integer solutions of $S_1 + S_2 + \ldots + S_7 = 52 - (7 \times 2) = 38$. By Theorem 3.1, this number is $C(38 + (7-1), 7-1)$, that is, $C(44,6)$.

Hence the probability that a lottery draw (of 6 balls from 49) will contain *no* pair of successively numbered balls is

$$\frac{C(44,6)}{C(49,6)} = \frac{44!}{6!38!} \bigg/ \frac{49!}{6!43!} = \frac{44 \times 43 \times 42 \times 41 \times 40 \times 39}{49 \times 48 \times 47 \times 46 \times 45 \times 44} = \frac{22{,}919}{45{,}402} = 0.505$$

to three decimal places. So the probability of a pair of consecutive numbers appearing when six (different) numbers are drawn from numbers 1 to 49 is, approximately, $1 - 0.505 = 0.495$. In other words, it is only slightly *less* likely than not!

Another variation on Problem 3.1 is the following.

PROBLEM 3.4

How many nonnegative integer solutions are there of the inequality $x + y + z + w + t < 14$?

Solution

An unthinking attempt at this problem might say: "Let us find the number of solutions to each of the equations $x + y + z + w + t = n$ for $n = 0,1,2,\ldots,13$ and then add up the answers." There is nothing actually wrong with this approach, and, by Theorem 3.1, this gives the answer as $C(4,4) + C(5,4) + \ldots + C(17,4)$.

However, the problem can be answered more briefly as follows. To each solution, say (a,b,c,d,e), of the equation $x + y + z + w + t = n$, with $0 \leq n \leq 13$, there is a solution, (a,b,c,d,e,f), of the equation $x + y + z + w + t + u = 13$ with $f = 13 - n$, and vice versa, with the newly introduced term, f, taking up the slack. Hence the number of nonnegative integer solutions of $x + y + z + w + t < 14$ is the same as the number of nonnegative solutions of $x + y + z + w + t + u = 13$. It follows immediately from Theorem 3.1 that this number is $C(18,5)$.

The reader who wonders if (or, perhaps better, *why*) these answers are the same should reread Theorem 2.8.

The second method used in the solution of Problem 3.4 clearly leads to the following further generalization of Theorem 3.1.

THEOREM 3.3

Let k, n be positive integers. Then the number of nonnegative integer solutions of the inequality $x_1 + x_2 + \ldots + x_k < n$ is $C(n + k - 1, k)$ or, equivalently, $C(n + k - 1, n - 1)$.

Exercises

3.1.1A How many solutions are there in nonnegative integers of the inequality $x+y+z+t \leq 20$?

3.1.1B How many solutions are there in nonnegative integers of the inequality $x+y+z+t < 20$?

3.1.2A How many integer solutions are there of the equation $x+y+z+t+w=14$ with $x \geq 5$, $y \geq -3$, $z \geq 2$, $t \geq -7$, and $w \geq 4$?

3.1.2B How many integer solutions are there to the equation $x+y+z=20$ with $x > -4$, $y > 1$, and $z > 4$?

3.1.3A A manufacturer makes marbles that are identical except for their color, which can be red, blue, green, or yellow. In how many different ways can the manufacturer make up a pack of 50 marbles?

3.1.3B
i. Each soccer team in the English Premier League plays 38 matches during the season. At the end of the season, the league table shows how many of these matches each team has won, drawn, and lost. How many combinations of these results are possible for any one team?
ii. What is the answer to the above question if the league table shows the results of the 19 matches that the team played at home separately from the 19 matches that it played away?

3.1.4A In a certain cricket match the 11 batsmen of one team between them scored 200 runs. In how many different ways could these runs be distributed between the batsmen?

3.1.4B How many different boxes can our chocolate manufacturer of Problem 3.2 supply if he guarantees that there is at least one chocolate of each flavor in each box?

3.1.5A In how many ways can 20 identical balls be placed in four distinct cups such that each cup has an even number of balls? (Count 0 as even.)

3.1.5B In how many ways can 20 identical balls be placed in four distinct cups such that each cup has an odd number of balls?

3.1.6A In the *Everwin* lottery, 10 numbers are drawn at random from the set $\{1,2,...,100\}$. What is the probability that at least two consecutive numbers are drawn?

3.1.6B What is the smallest positive integer k such that if k numbers are drawn at random from the set $\{1,2,\ldots,100\}$, it is more likely than not that at least two of the numbers drawn are consecutive?

3.2 NEW PROBLEMS FROM OLD

The diagrams we drew in connection with Problem 3.1 were very helpful. But the solution to Problem 3.2 suggests a reinterpretation that leads to host of related and intriguing questions, several of which might have remained hidden, and some of which are a little more tricky to answer!

To reinterpret Problem 3.1 in the light of the solution to Problem 3.2, imagine five (distinct) boxes labeled x, y, z, w, and t. The solution (4,0,1,3,6) of the equation $x+y+z+w+t=14$ corresponds to placing 14 (identical) balls in the five boxes so that there are 4 balls in box x, 0 balls in box y, 1 ball in box z, 3 balls in box w, and 6 balls in box t. See Figure 3.1.

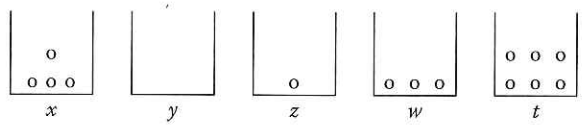

FIGURE 3.1

Consequently, Problem 3.1 can be recast as: In how many ways can 14 identical balls be placed in five distinct boxes?

In calling the balls "identical" and the boxes "distinct" we mean that we are interested only in how many balls there are in each box, but we want to distinguish, for example, having four balls in box x and zero in box y, from having zero balls in box x and four balls in box y. This problem has a number of variations according to whether we regard the balls as being identical or distinct and whether we regard the boxes as identical or distinct. We can also vary the problem by deciding whether we wish to include cases where some of the boxes may be empty. In this way we have eight different problems about placing balls in boxes: In how many ways can n (identical or distinct) balls be placed in k (identical or distinct) boxes with at least one ball in each box or with some boxes allowed to be empty?

We now give examples of all the eight possible cases of occupancy problems. It would be nice if, in the following examples, we could use the same values of n and k throughout. But, in order that our examples be neither too large to list easily nor too small to appear general enough, we shall choose n and k suitably in each case.

1a. *Placing* three *distinct balls in* two *distinct boxes*

We use a, b, c to represent the three distinct balls, and { }, [ ] to represent the two distinct boxes. Since for each ball there are two choices of boxes, there are $2 \times 2 \times 2 = 2^3 = 8$ different ways to allocate the balls to the boxes:

$\{a,b,c\}$, [ ]; $\{a,b\}$, $[c]$; $\{a,c\}$, $[b]$; $\{b,c\}$, $[a]$; $\{a\}$, $[b,c]$; $\{b\}$, $[a,c]$; $\{c\}$, $[a,b]$; { }, $[a,b,c]$.

1b. *Placing* three *distinct balls in* two *distinct boxes, with no box empty*

We see from the above list that of the eight ways of assigning three distinct balls to two distinct boxes, there are two where a box is empty, and hence there are six cases with no empty boxes.

2a. *Placing* four *distinct balls in* two *identical boxes*

Here we use a, b, c, d for the distinct balls and { } for each of the identical boxes. We see that there are eight ways the balls can be placed in the two boxes:

$\{a,b,c,d\}$,{ }; $\{a,b,c\},\{d\}$; $\{a,b,d\},\{c\}$; $\{a,c,d\},\{b\}$; $\{b,c,d\},\{a\}$; $\{a,b\},\{c,d\}$; $\{a,c\},\{b,d\}$; $\{a,d\},\{b,c\}$.

2b. *Placing* four *distinct balls in* two *identical boxes, with no box empty*

We see from (2a) that we need to discard the case $\{a,b,c,d\}$,{ }, leaving seven ways of placing the balls.

3a. *Placing* six *identical balls in* three *distinct boxes*

As the balls are identical, all that matters is the number of balls in each box. As we have seen, the number of ways to do this is the same as the number of nonnegative integer solutions of the equation $x + y + z = 6$. So Theorem 3.1 tells us that there are $C(8,2) = 28$ ways to place the balls in the boxes.

3b. *Placing* six *identical balls in* three *distinct boxes, with no box empty*

The number of ways to do this is the same as the number of positive integer solutions of the equation $x + y + z = 6$. So by Theorem 3.2 there are $C(5,2) = 10$ ways to place the balls in the boxes.

4a. *Placing* six *identical balls in* three *identical boxes*

Because the balls are identical, we need only indicate the *number* of balls in each box—and this we can do most conveniently via Figure 3.2. There are therefore seven solutions to this problem.

4b. *Placing* six *identical balls in* three *identical boxes, with no box empty*

We see from Figure 3.2 that there are just three solutions in this case.

It is convenient, at this point, to introduce some notation. As is standard, we use $S(n,k)$ for the number of (different) ways of placing n distinct balls in k identical boxes so that no box is empty. We use $p_k(n)$ for the number of ways of placing n identical balls in k identical boxes, where some boxes may be empty.* It would be more consistent to use, say, $P(n,k)$, but there are historical reasons why these notations follow different patterns.

The letter S used in the notation $S(n,k)$ commemorates the Scottish mathematician James Stirling who introduced these numbers in a different context, which is explained in Chapter 5 where we give some biographical information about Stirling. So we call the numbers $S(n,k)$ *Stirling numbers.* We shall have to wait until Chapter 4 to determine an explicit formula giving their values for all values of n and k.

We now present a summary of the eight cases.

[ ]	[ ]	[ ]
6	0	0
5	1	0
4	2	0
4	1	1
3	3	0
3	2	1
2	2	2

FIGURE 3.2

* Beware: Some authors use the same notation for the case where no box can be empty.

TABLE 3.1

Case	Balls	Boxes	Empty Boxes?	Number of Arrangements
1a	Distinct	Distinct	Yes	k^n
1b	Distinct	Distinct	No	$k!S(n,k)$
2a	Distinct	Identical	Yes	$S(n,1)+S(n,2)+\ldots+S(n,k)$
2b	Distinct	Identical	No	$S(n,k)$
3a	Identical	Distinct	Yes	$C(n+k-1,k-1)$
3b	Identical	Distinct	No	$C(n-1,k-1)$
4a	Identical	Identical	Yes	$p_k(n)$
4b	Identical	Identical	No	$p_k(n)-p_{k-1}(n)$

We now explain where the entries in the last column of Table 3.1 come from.

Case 1a. With n distinct balls to be placed in k distinct boxes, there are k boxes in which we could place each ball. Hence the total number of ways to place the n balls in the k boxes is, using the principle of multiplication of choices, $k \times k \times \ldots \times k = k^n$.

Case 1b. We can view the problem of finding the number of ways of placing n distinct balls in k distinct boxes, with no box empty, in the following way. First, place n distinct balls in k *identical* boxes, with no box empty. We have used $S(n,k)$ for the number of ways of doing this. Then, distinguish the boxes by placing k different labels on the boxes. There are k choices of box for the first label, k–1 for the second label, and so on. So the k labels may be placed on the boxes in $k \times (k-1) \times \ldots = k!$ ways. Combining the $S(n,k)$ choices with $k!$ choices, we see that the total number of choices is $S(n,k) \times k!$ Of course, it still remains to find a formula for $S(n,k)$.

Case 2a. If we have k identical boxes, some of which are allowed to be empty, then we may place all balls in one box, or share them between two boxes, or three boxes, and so on. (Since the boxes are identical, exactly which boxes we use or leave empty is irrelevant.) Clearly, then, the total number of ways to do this $S(n,1) + S(n,2) + \ldots + S(n,k)$.

Case 2b. This is just the notation we introduced above, with the formula for $S(n,k)$ yet to be determined.

Cases 3a and 3b. As we have already noted, these cases are covered by Theorems 3.1 and 3.2.

Case 4a. At this stage this is just a matter of notation, as introduced above. We shall discuss the problem of counting the number of ways to place identical balls in identical boxes in Chapter 6.

Case 4b. The number of ways of placing n identical balls in k identical boxes so that no box is empty is clearly obtained by taking the total number of ways of placing the balls in the k boxes and subtracting the number of cases where at most $k-1$ boxes are used. Thus the number of such arrangements is therefore $p_k(n)-p_{k-1}(n)$.

We have already observed that, if the balls are identical and the boxes are identical, all that matters is how many balls there are in each box. So deciding how to place n balls in

k boxes, with some boxes possibly empty, is the same as deciding how to write n as the sum of k nonnegative integers. As the boxes are identical, the order in which we put these k nonnegative integers is irrelevant. Thus we can view the seven rows in Figure 3.2 as corresponding to the following seven ways of writing 6 as the sum of three nonnegative integers, namely, $6+0+0$, $5+1+0$, $4+2+0$, $4+1+1$, $3+3+0$, $3+2+1$, and $2+2+2$. Since the zeros are somewhat redundant, we usually omit them. We view the problem in this way in Chapter 6.

In Table 3.1 we have carefully distinguished the cases according to whether the balls are regarded as distinct or identical and likewise for the boxes, and whether or not the boxes can be empty. In counting problems arising from real situations, you may need to think carefully about which category the situation falls into, and hence which line of Table 3.1 is relevant.

Exercises

3.2.1A A teacher has 30 identical chocolate bars and 30 nonidentical pupils. She gives the pupils an algebra test each day for six school weeks, that is, for 30 days in all, and gives a chocolate bar to the pupil with the highest score in each test. In how many different ways can the chocolate bars be distributed between the pupils?

3.2.1B There were 20 different birds and animals in the Caucus race. When the race had finished the Dodo said, "*Everybody* has won and all must have prizes." Alice had a box of 40 identical comfits in her pocket. In how many ways could she distribute these to the birds and animals so that each of them received at least one?*

3.2.2A In how many ways can we place four identical black marbles and six distinct nonblack marbles in five distinct boxes, some of which might be empty?

3.2.2B In how many ways can eight identical black marbles and ten distinct non-black marbles be placed in five distinct boxes if there is to be at least one black and one nonblack marble in each box?

3.2.3A A manufacturer makes identical transparent marbles and also marbles in 20 different colors. In how many ways can he make up a bag of 20 marbles, given that the bag may contain up to 20 transparent marbles but not two nontransparent marbles that have the same color?

3.2.3B A manufacturer makes white chalk and also chalk in 12 other colors. In how many ways can he make up a box containing 12 sticks of chalk of which at least six must be white and in which there must not be two non-white sticks of chalk with the same color?

3.3 A "REDUCTION" THEOREM FOR THE STIRLING NUMBERS

Before we obtain a general formula for the Stirling numbers, $S(n,k)$, we can, at least, readily find the values for small values of n and k, not by laboriously listing all the ways of placing distinct balls in identical boxes, but by establishing a relationship between $S(n,k)$

* Based on *Alice's Adventures in Wonderland* by the mathematician Lewis Carroll (1832–1898). Comfit is an old-fashioned term for what in England is now called a *sweet* and in the United States a *piece of candy*. Also, Carroll used *animals* to mean "mammals."