2.13 First we show $B$ is an ideal. Clearly it is a subspace. So let $b_1, b_2 \in B$ and $x \in L$. Then

$$[[x, b_1], b_2]] = [0, b_2] = 0$$

so $[x, b_1] \in B$ as desired.

Suppose further that $Z(I) = 0$ and every derivation of $I$ is of the form $ad x$ for some $x \in I$. From the definitions we see that $I \cap B = Z(I)$ so $I \cap B = 0$. Take an arbitrary element $x \in L$. Then $ad x : I \to I$ since $I$ is an ideal. By the hypothesis then $ad x = ad i$ for some $i \in I$, i.e. for any $y \in I$ we have $[x, y] = [i, y]$, i.e. $x - i \in B$. This implies that $L = I + B$. But their intersection is trivial and both are ideals, so this suffices to show $L = I \oplus B$.

3.1 Let $V$ be a vector space and let $\phi$ be an endomorphism of $V$. Let $L$ have underlying vector space $V \oplus \text{Span}\{x\}$. Show that if we define the Lie bracket on $L$ by $[y, z] = 0$ and $[x, y] = \phi(y)$ for $y, z \in V$, then $L$ is a Lie algebra and $\dim L' = \text{rank } \phi$.

**Proof:** Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $V$. As usual think of $V$ as a subspace of $L$ in the first coordinate. Since $[V, V] = 0$ and $[x, v_i] = \phi(v_i)$, it’s clear that $\{\phi(v_1), \ldots, \phi(v_n)\}$ span $L'$, so its dimension is just the dimension of the image of $\phi$, i.e. the rank of $\phi$.

Checking the Jacobi identity is, as usual, annoying. Let $a = w_1 + \lambda_1 x, b = w_2 + \lambda_2 x, c = w_3 + \lambda_3 x$ be three arbitrary elements of $L$, so $w_i \in V$.

$$[a, [b, c]] = [a, \lambda_2 \phi(w_3) - \lambda_3 \phi(w_2)]$$
$$= \lambda_1 \lambda_2 \phi(\phi(w_3)) - \lambda_1 \lambda_3 \phi(\phi(w_2))$$

$$[b, [c, a]] = [b, \lambda_3 \phi(w_1) - \lambda_1 \phi(w_3)]$$
$$= \lambda_2 \lambda_3 \phi(\phi(w_1)) - \lambda_2 \lambda_1 \phi(\phi(w_3))$$

$$[c, [a, b]] = [c, \lambda_1 \phi(w_2) - \lambda_2 \phi(w_1)]$$
$$= \lambda_3 \lambda_1 \phi(\phi(w_2)) - \lambda_3 \lambda_2 \phi(\phi(w_1))$$

Thus

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

3.3

(i) A little linear algebra shows that $gl_3(3, C)$ is all matrices of the form:

$$\begin{pmatrix}
0 & b & c \\
-b & 0 & f \\
c & f & 0
\end{pmatrix}$$
Choosing the obvious basis and taking brackets we see $L'$ is also 3-dimensional, so this is a copy of $sl(2, C)$.

(ii) We find that $[v, w] = 0$. Also $[u, v]$ and $[u, w]$ are zero if $l = n$ or $m = n$ respectively. So we have several cases. If all 3 parameters are equal we have an abelian 3-dimensional lie algebra. If all 3 are different then we have a two-dimensional $L'$ spanned by $v$ and $w$. Also $ad u$ is diagonalisable so we are are in the case $L_\tau$ on page 23 where $\tau = \frac{u-v}{\lambda-\nu}$.

If $\mu = \nu \neq \lambda$ then we have a center spanned by $w$ and a one-dimensional derived algebra spanned by $v$. This is the case of a direct sum of a one-dimensional abelian and a 2-dimensional nonabelian. Similarly if $\lambda = nm \neq \mu$.

The final case is if $\lambda = \mu \neq \nu$. We get a two-dimensional derived algebra and a one-dimensional center, where we once again get some $L_\tau$.

3.5 We know that $ad h$ is diagonalizable for $h$ in $sl(2, R)$. Now consider an arbitrary element of $R^3$ in the basis $i, j, k$. So let $x = \alpha i + \beta j + \gamma k$. The matrix of $ad x$ is:

$$
\begin{pmatrix}
0 & -\gamma & \beta \\
\gamma & 0 & -\alpha \\
-\beta & \alpha & 0
\end{pmatrix}
$$

This has characteristic polynomial $(-t)(t^2 + \alpha^2 + \beta^2 + \gamma^2)$, so only one real eigenvalue. Thus $ad x$ is not diagonalizable over $R$. An isomorphism of Lie algebras preserves all brackets, and in particular the ad maps. So the two algebras cannot be isomorphic over $R$.

3.7 Suppose $L$ is a nonabelian Lie algebra and choose $x, y$ so $[x, y] \neq 0$. The $x$ and $y$ are linearly independent and neither is in the center. Thus the center has dimension at most $\dim L - 2$.

3.8 Recall the Heisenberg algebra has dimension 3:

$$H = \{ x, y, z \mid [x, y] = z, [x, z] = 0, [y, z] = 0 \}.$$

An arbitrary linear map from $H$ to $H$ is given by a $3 \times 3$ matrix, so potentially we have 9 dimensions. However if $T$ is a derivation we need $T[a, b] = [Ta, b] + [a, Tb]$ for all $a, b \in H$, which imposes relations. So in the basis $\{x, y, z\}$ let $T$ be given by the matrix:

$$
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}
$$

So the relation $[x, y] = z$ means $T([x, y]) = [Tx, y] + [x, Ty]$, i.e.

$$
\begin{pmatrix}
c \\
f \\
i
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$
The relation \([z, x] = 0\) means \(0 = [Tz, x] + [z, Tx]\):
\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
-f
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Finally the relation \([z, y] = 0\) means \(0 = [Tz, y] + [z, Ty]\):
\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
c
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Thus we can write our matrix as:

\[
\begin{pmatrix}
a & b & 0 \\
d & e & 0 \\
g & h & a+e
\end{pmatrix}
\]

so the derivation space is six-dimensional. Notice the inner derivations are spanned by \(\text{ad} x\) with matrix
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]
and by \(\text{ad} y\) with matrix
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]

Notice that multiplying two of the matrices from (1) corresponds to multiplying the upper left \(2 \times 2\) matrices together, although the \((3, 1)\) and \((3, 2)\) entries are more complicated. But when we quotient out by the \(\text{Inn} L\), we just have \(gl(2, F) \cong \text{Der} L/\text{Inn} L\).

3.9: (i).
\[
\Theta([s,t])(x) = [[s,t],x] = -[[t,x],s] - [[x,s],t] \quad \text{by Jacobi}
\]
\[
= [s,[t,x]] - [t,[s,x]]
\]
\[
= (\Theta(s) \circ \Theta(t) - \Theta(t) \circ \Theta(s))(x)
\]
\[
= [\Theta(s),\Theta(t)](x).
\]
So \(\Theta\) is a Lie algebra homomorphism. We checked in class that \([s, -]\) is a derivation, so the image of \(\Theta\) lies in \(\text{Der} I\).

(ii). We just show that the bracket is bilinear and satisfies \([x,x] = 0\) and the Jacobi identity. Also we just show \(S\) is a subalgebra and \(I\) is an ideal, where we identify \(S\) with \(\{(s,0)\}\) and \(I\) with \(\{(0,i)\}\).

Bilinearity follows from that of the brackets on \(S\) and \(I\) and the fact that \(\Theta\) is linear.

\[
([s,x],(s,x)) = ([s,s],[x,x] + \Theta(s)x - \Theta(s)x) = (0,0).
\]

Finally the Jacobi identity. Since the bracket is bilinear we can split the Jacobi identity verification into 4 cases, depending whether zero, one, two or all three of the coordinates are in \(S\).
Case 1 The Jacobi identity including \([s_1, 0, [s_2, 0, [s_3, 0]]\) follows from the Jacobi identity in \(S\).

Case 4 The Jacobi identity including \([0, x_1, [0, x_2, (0, x_3)]\) follows from the Jacobi identity in \(I\) since \(\Theta(0) = 0\).

Case 2

\[
\begin{align*}
[(s_1, 0), [s_2, 0], (0, x_3)] &= [(s_1, 0), (0, \Theta(s_2)(x_3))] \\
&= (0, \Theta(s_1)(\Theta(s_2)(x_3))
\end{align*}
\]

\[
\begin{align*}
[(s_2, 0), [0, x_3], (s_1, 0)] + [(0, x_3), [(s_1, 0), (s_2, 0)]] &= [(s_2, 0), (0, -\Theta(s_1)(x_3))] + [(0, x_3), ([s_1, s_2], 0)] \\
&= (0, -\Theta(s_2)\Theta(s_1)(x_3)) + (0, -\Theta([s_1, s_2])(x_3)) \\
&= (0, -\Theta(s_2)\Theta(s_1)(x_3) - \Theta([s_1, s_2])(x_3))
\end{align*}
\]

Adding the results of the two previous arrays we get 0, because \(\Theta\) is a Lie algebra homorphism so

\[
\Theta([s_1, s_2]) = \Theta(s_1) \circ \Theta(s_2) - \Theta(s_2) \circ \Theta(s_1).
\]

Case 3

\[
\begin{align*}
[(s_1, 0), [0, x_2], (0, x_3)] &= [(s_1, 0), (0, [x_2, x_3])] \\
&= (0, \Theta(s_1)([x_2, x_3]))
\end{align*}
\]

\[
\begin{align*}
[(0, x_2), [(0, x_3), (s_1, 0)] + [(0, x_3), [(s_1, 0), (0, x_2)]] &= [(0, x_2), (0, -\Theta(s_1)(x_3))] + [(0, x_3), (0, \Theta(s_1)(x_2))] \\
&= (0, [x_2, -\Theta(s_1)(x_3)]) + (0, [x_3, \Theta(s_1)(x_2)])
\end{align*}
\]

The Jacobi identity now follows since \(\Theta(s_1)\) is a derivation so

\[
\Theta(s_1)([x_2, x_3]) = [x_2, \Theta(s_1)(x_3)] + [\Theta(s_1)(x_2), x_3].
\]

(iii). In exercise 3.1 our vector space is \(V \oplus \text{Span}(x)\) where \(V\) is an abelian Lie algebra structure. The role of \(I\) is played by \(V\) and the role of \(S\) is played by \(\text{Span}(x)\). Let \(\Theta(x) = \phi\). On an abelian Lie algebra any linear map is trivially a derivation, so we just check the brackets agree. In the semidirect product we have:

\[
[(x, 0), (0, v)] = (0, \Theta(x)(v)) = (0, \phi(v)
\]

which is precisely the definition from 3.1.