Math 461/561 Week 5 Solutions

5.1

(i). Let \( v_1, v_2 \in V_\lambda \). Then

\[ a(cv_1 + v_2) = c(a(v_1) + a(v_2)) = c\lambda(a)v_1 + \lambda(a) = \lambda(a)(cv_1 + v_2) \]

so \( cv_1 + v_2 \in V_\lambda \) and so it is a subspace.

(ii). It is obvious that \( V \) is the direct sum of the spans of the \( e_i \) as a vector space, so if we can show that \( \text{Span}\{e_i\} = V_\epsilon \) then we are done. It is clear that \( e_i \in V_\epsilon \). Now suppose \( v \) is not in the span of \( e_i \), so \( v = \sum a_k e_k \) where there is some \( a_s \neq 0 \) for \( s \neq i \). Consider the elementary matrix \( E_{ss} \). Then \( E_{ss} v = a_s e_s \) but \( \epsilon_i(E_{ss}) = 0 \) so \( v \not\in V_\epsilon \). Thus \( \text{Span}\{e_i\} = V_\epsilon \) as desired.

5.5

(i). The \( m = 1 \) case is obvious. For \( m \geq 2 \) we compute:

\[ z^m = z(xy - yx) = xzy - zyx \]

since \( y \) commutes with \( z \).

Now \( \text{tr}(AB) = \text{tr}(BA) \) so \( \text{tr}(zyx) = \text{tr}(xzy) \). Thus \( \text{tr}(xzy - zyx) = 0 \) as desired.

(ii). Recall that if the eigenvalues of \( A \) are \( \{\lambda_i\} \) then the eigenvalues of \( A^k \) are \( \{\lambda_i^k\} \) and further that the trace of a matrix is the sum of the eigenvalues. Thus from part (i) we have:

\[ \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k = 0 \]

for each \( k \geq 1 \). Now choose just the nonzero eigenvalues \( \mu_1, \mu_2, \ldots, \mu_t \) where \( \mu_i \) ha multiplicity \( m_i \). Then the equation above can be written as:

\[ \sum_{i=1}^{t} m_i \mu_i^k = 0, k = 1, 2, 3, \ldots. \]

Consider this as a system of equations in the variables \( m_i \). It has corresponding matrix:

\[
A = \begin{bmatrix}
\mu_1 & \mu_2 & \cdots & \mu_t \\
\mu_1^2 & \mu_2^2 & \cdots & \mu_t^2 \\
\vdots & \vdots & \ddots & \vdots \\
\mu_1^t & \mu_2^t & \cdots & \mu_t^t
\end{bmatrix}
\]

This is known as a Vandermonde determinant and it is nonzero unless some \( \mu_i = \mu_j \) for \( i \neq j \). But we chose distinct \( \mu \)'s, so the determinant is nonzero. But (1) tells us \((1,1,\cdots,1)^T\) lies in the kernel, a contradiction. Thus there can be no nonzero eigenvalues for \( z \), so \( z \) is nilpotent.

5.6
(i) Since $A$ is a Lie subalgebra, $[A, A] \subseteq A$ so $A \subseteq N_L(A)$. Suppose $U$ is a subalgebra of $L$ in which $A$ is an ideal. Then $[U, A] \subseteq A$ which means, by definition, that $U$ lies in the normaliser, and thus $N_L(A)$ is the largest subalgebra in which $A$ is an ideal.

Finally to show $N_L(A)$ is a subalgebra, let $n_1, n_2 \in N_L(A)$ and let $a \in A$. Then:

$[[n_1, n_2], a] = [n_1, [n_2, a]] + [n_2, [a, n_1]].$

Notice $[n_2, a] \in A$ since $n_2 \in N_L(A)$, and so $[n_1, [n_2, a]] \in A$. Similarly $[n_2, [a, n_1]] \in A$ so $[n_1, n_2] \in N(A)$ and we have a subalgebra.

(ii) Now let $L := \mathfrak{gl}(n, \mathbb{C})$ and $A$ be the subalgebra of all diagonal matrices. Now apply the invariance lemma to the subalgebra $N_L(A)$ and its ideal $A$. The lemma says that for each weight of $A$, the corresponding weight space is $N_L(A)$ invariant. However each standard basis vector is a common eigenvector for $A$ so a matrix in $N_L(A)$ must preserve $\text{span}(e_i)$ for each $i$. This forces it to be a diagonal matrix, so $N_L(A)$ is just $A$.

6.1

(i) Let $x : V \to V$ be nilpotent. If $x = 0$ then any nonzero $v$ will suffice. Otherwise choose the largest $m$ so that $x^m = 0$ and $x^{m-1} \neq 0$. Any $v \neq 0$ in the image of $x^{m-1}$ will do the trick.

(ii) Now let $U = \text{span}(v)$. Since $xU = 0$ we get a well-defined map $\pi : V/U \to V/U$ defined by $\pi(a + U) = x(a) + U$. Now by induction choose a basis $\{v_1 + U, \ldots, v_{n-1} + U\}$ in which $\pi$ is strictly upper triangular. Now consider the basis $v, v_1, v_2, \ldots, v_{n-1}$ of $V$. Notice that $x(v_i)$ is a linear combination of $v_{i+1}, v_{i+2}, \ldots, v_n$ and $v$ while $xv = 0$. So the matrix for $x$ in this basis is indeed strictly upper triangular.

6.3 Let $L$ be a complex Lie algebra. Show that $L$ is nilpotent if and only if every 2-dimensional subalgebra of $L$ is abelian.

**Proof:** Let $L$ be nilpotent and suppose $L$ has a 2-dimensional nonabelian subalgebra. By Theorem 3.1 we know it has a basis $\{x, y\}$ with $[x, y] = x$, so $[y, x] = -x$ and

$[y, [y, x]] = [y, -x] = x$

Clearly $y$ is not ad-nilpotent, contradicting Theorem 6.3. Thus if $L$ is nilpotent then every 2-dimensional subalgebra of $L$ is abelian.

Conversely suppose every two-dimensional subalgebra is abelian. Let $x \in L$ and consider $\text{ad}x : L \to L$. Since the field is complex, the linear map $\text{ad}x$ has an eigenvector $y$, i.e. an element $y \in L$ such that $[x, y] = \lambda y$. If $\lambda \neq 0$, then $(x, y)$ is a two-dimensional, nonabelian subalgebra. Thus $\lambda = 0$ is the only eigenvalue of $\text{ad}x$, so $\text{ad}x$ is nilpotent. By Engel’s theorem, $L$ is nilpotent.
6.4 Let $p$ be a prime and let $F$ be a field of characteristic $p$. Let $x$ and $y$ be the $p \times p$ matrices given. One easily checks that $xy$ has $1, 2, 3, \ldots, p-1$ on the superdiagonal, with all other entries 0. Similarly $yx$ has $0, 1, 2, \ldots, p-2$ on the superdiagonal and $p-1$ in the lower left corner, so $[x, y] = x$. (Using that we are in characteristic $p$ so $1 - p = 1$)

Thus $x, y$ span a two-dimensional subalgebra $L$, which is solvable since $L' = \langle x \rangle$ and $L^{(2)} = 0$.

Since $y$ is diagonal one easily sees it has the standard basis $\{e_1, e_2, \ldots, e_p\}$ as eigenvectors with eigenvalues $\{0, 1, 2, \ldots, p-1\}$. Clearly none of these are eigenvectors for $x$, so the conclusion of Proposition 6.6, and thus of Lie’s theorem, fails in characteristic $p$. The first part of 6.5 also fails.

Notice that $x \in L'$ and $x^p = Id$ so $x$ is clearly not nilpotent.

6.5 (i) Let $L$ be a solvable subalgebra of $gl(V)$ where $V$ is a complex vector space. By Lie’s theorem, we can choose a basis of $\beta$ of $V$ so that every $x \in L$ has corresponding matrix $[x]_\beta \in b(n, \mathbb{C})$. We worked out in class that for any two matrices $A, B \in b(n, \mathbb{C})$, the bracket $[A, B] \in n(n, \mathbb{C})$. But any strictly upper triangular matrix corresponds to a nilpotent linear map, so every element of $L'$ is nilpotent.

(ii). Suppose $L$ is solvable and $F = \mathbb{C}$. Then $ad L$ is a solvable subalgebra of $gl(L)$. In particular, since $[ad x, ad y] = ad [x, y]$, we have by part (i) that $ad z$ is a nilpotent endomorphism of $V$ for every $z \in L'$. This means every element of $L'$ is ad-nilpotent. So by the second version of Engel’s theorem, $L'$ is nilpotent. Conversely suppose $L'$ is nilpotent. Then certainly $L'$ is solvable, so $(L')^{(m)} = 0$ for some $m$. But $(L')^{(m)} = (L)^{(m+1)}$ so $L$ is solvable.

6.6 Use Lie’s Theorem to give another proof of

**Proposition 5.7** Let $x, y : V \to V$ be linear maps from a complex vector space $V$ to itself. Suppose that $x$ and $y$ both commute with $[x, y]$. Then $[x, y]$ is a nilpotent map.

**Proof:** Think of $x$ and $y$ as elements of the Lie algebra $gl(V)$. Let $z = [x, y]$. By assumption

$$[x, z] = [y, z] = 0.$$ 

Thus the span $\langle x, y, z \rangle$ gives a subalgebra $L$ of $gl(V)$. (at most 3-dimensional).

Notice $L' = \langle z \rangle$ so $L^{(2)} = 0$ and $L$ is solvable. Then by 6.5(ii) above (which used Lie’s theorem) we see every element of $L'$ is nilpotent, that is $z = [x, y]$ is nilpotent.

**Prove a finite dimensional Lie algebra has a unique maximal nilpotent ideal.**

**Proof:** Let $I$ and $J$ be nilpotent ideals. It is enough to show that $I + J$ is also nilpotent, then our ideal will be the sum of all nilpotent ideals of $L$. We claim by induction that:

$$(I + J)^{n+1} \subseteq I^{n+1} + J^{n+1} + \sum_{s=0}^{n} I^s \cap J^{n-s}.$$
The $n=0$ case is obvious:

$$(I + J)^1 = [I + J, I + J] = [I, I] + [I, J] + [J, I] + [J, J] \subseteq I^1 + J^1 + J \cap I.$$

Suppose then the result holds for $n = k - 1$.

$$(I + J)^{k+1} = [I + J, (I + J)^k] = [I, (I + J)^k] + [J, (I + J)^k] \subseteq [I, I^k + J^k + \sum_{s=0}^{k-1} I^s \cap J^{k-1-s}] + [J, I^k + J^k + \sum_{s=0}^{k-1} I^s \cap J^{k-1-s}]$$

by inductive hypothesis.

$$\subseteq I^{k+1} + J^{k+1} + \sum_{s=0}^{k} I^s \cap J^{k-s}$$

by expanding out.

Now suppose $I^s = J^t = 0$, since $I$ and $J$ are nilpotent. WLOG suppose $s \geq t$. Then $(I + J)^{2s+1} = 0$ since every term in (3) will have an $I^l$ for $l \geq s$ or a $J^w$ for $w \geq s \geq t$, and thus will be zero. Thus $I + J$ is nilpotent.