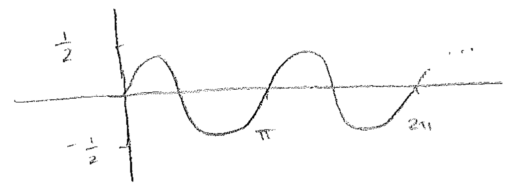
7.6

① $x'' + 4x = \delta(x) \qquad x(0) = x'(0) = 0$

$\big\downarrow \mathcal{L}$

$s^2 X(s) - \underbrace{s\,x(0)}_{=0} - \underbrace{x'(0)}_{=0} + 4X(s) = 1$



$\frac{1}{2}\sin(2t)$

$X(s)(s^2+4) = 1$

$X(s) = \dfrac{1}{s^2+4} \implies x(t) = \frac{1}{2}\sin(2t)$

$= \dfrac{1}{2}\cdot\dfrac{2}{s^2+4}$

③ $x'' + 4x' + 4x = 1 + \delta(t-2) \qquad x(0) = x'(0) = 0$

$\big\downarrow \mathcal{L}$

$s^2 X(s) - \cancel{s\,x(0)} - \cancel{x'(0)} = \dfrac{1}{s} + e^{-2s}$

$+\, 4\left[s X(s) - \cancel{x(0)}\right]$

$+\, 4X(s)$

$\implies X(s)\left[s^2 + 4x + 4\right] = \dfrac{1}{s} + e^{-2s}$

$\implies X(s) = \dfrac{1}{s(s+2)^2} + \dfrac{e^{-2s}}{(s+2)^2}$

$x(t) = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t} + u(t-2)(t-2)e^{-2(t-2)}$

see textbook solutions for graph: notice sharp corner at $t=2$ (where delayed start term gets added in)

Recognize as delayed start:

$e^{-2s}\dfrac{1}{(s+2)^2} = e^{-2s}F(s)$ where $f(t) = \mathcal{L}^{-1}\{F(s)\} = te^{-2t}$

$\big\downarrow \mathcal{L}^{-1}$

$u(t-2)(t-2)e^{-2(t-2)}$

$= \dfrac{A}{s} + \dfrac{B}{s+2} + \dfrac{C}{(s+2)^2}$

$\mathcal{L}^{-1} \big\downarrow$

$A + Be^{-2t} + Cte^{-2t}$

$A(s+2)^2 + Bs(s+2) + Cs = 1$

$s=0 \implies A = 1/4$

$s=-2 \implies C = -1/2$

$As^2 + Bs^2 = 0s^2 \implies B = -A = -1/4$

⑥ $x'' + 9x = \delta(t-3\pi) + \cos 3t \qquad x(0) = x'(0) = 0$

$\big\downarrow \mathcal{L}$

$$s^2 X(s) + 9X(s) = e^{-3\pi s} + \frac{s}{s^2+9} \Rightarrow X(s) = \frac{e^{-3\pi s}}{s^2+9} + \frac{s}{(s^2+9)^2} \quad (*)$$

Skipping steps:
many terms drop since
$x(0) = x'(0) = 0$

Recognize as
$\mathcal{L}\{u(t-3\pi)\frac{1}{3}\sin(3(t-3\pi))\}$
delayed start

$$\frac{s}{(s^2+9)^2} = \frac{1}{3}\left(\frac{3}{s^2+9}\right)\left(\frac{s}{s^2+9}\right)$$

$$\frac{1}{3}\mathcal{L}\{\sin 3t\} \quad \mathcal{L}\{\cos 3t\}$$

$$\mathcal{L}\{\sin 3t\}\mathcal{L}\{\cos 3t\} = \mathcal{L}\left\{\int_0^t \underbrace{\sin(3\tau)\cos(3(t-\tau))}\,d\tau\right\}$$

$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$$= \frac{1}{2}[\sin(3t) - \sin(3t-6\tau)]$$

$$\int_0^t \frac{1}{2}[\sin(3t) - \sin(3t-6\tau)]\,d\tau =$$

$$\frac{1}{2}\left[\tau\sin(3t) - \frac{\cos(3t-6\tau)}{6}\right]_{\tau=0}^{\tau=t}$$

$$= \frac{1}{2}\left[t\sin(3t) - \frac{1}{6}\cos(-3t) - \left(0 - \frac{1}{6}\cos(3t)\right)\right]$$

$$= \frac{1}{2}t\sin(3t) \underbrace{- \frac{1}{12}\cos(-3t) + \frac{1}{12}\cos(3t)}_{=0}$$

$$\frac{1}{3}\mathcal{L}\{\sin 3t\}\mathcal{L}\{\cos 3t\} = \frac{1}{3}\mathcal{L}\{\tfrac{1}{2}t\sin(3t)\}$$

$$= \frac{1}{6}\mathcal{L}\{t\sin(3t)\}$$

$$\rightarrow \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \frac{1}{6}t\sin(3t)$$

apply to $(*)$: $\quad x(t) = u(t-3\pi)\frac{1}{3}\underbrace{\sin(3(t-3\pi))}_{\text{note: this is equivalent to } -\sin(3t)} + \frac{1}{6}t\sin(3t)$

see textbook for graph

⑨ $x'' + 4x = f(t) \qquad x(0) = x'(0) = 0$

$$W(s) = \frac{1}{s^2+4} \qquad w(t) = \tfrac{1}{2}\sin(2t)$$

$$x(t) = \int_0^t \tfrac{1}{2}\sin(2\tau) f(t-\tau)\, d\tau$$

⑪ $x'' + 6x' + 8x = f(t) \qquad x(0) = x'(0) = 0$

$$W(s) = \frac{1}{s^2+6s+8} = \frac{1}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4} = \frac{A(s+4)+B(s+2)}{(s+2)(s+4)}$$

$s = -2 \Rightarrow 1 = 2A \Rightarrow A = \tfrac{1}{2}$

$As + Bs = 0 \Rightarrow B = -A = -\tfrac{1}{2}$

$$= \frac{\frac{1}{2}}{s+2} - \frac{\frac{1}{2}}{s+4} \Longrightarrow w(t) = \tfrac{1}{2}\left(e^{-2t} - e^{-4t}\right)$$

NOTE $= \frac{1}{2} e^{-3t}\left[e^{t} - e^{-t}\right] = e^{-3t}\sinh t$

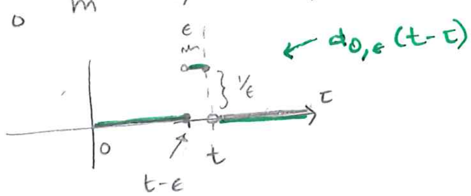
$$x(t) = \int_0^t \tfrac{1}{2}\left(e^{-2\tau} - e^{-4\tau}\right) f(t-\tau)\, d\tau$$

⑬ ⓐ $mx'' = p\, d_{0,\epsilon}(t) \qquad x(0) = x'(0) = 0$

Using Duhamel: $W(s) = \frac{1}{ms^2} \Rightarrow w(t) = \frac{t}{m}$

$$x_\epsilon(t) = \int_0^t \frac{p}{m}\tau\, d_{0,\epsilon}(t-\tau)\, d\tau = \int_{t-\epsilon}^{t} \frac{p}{m\epsilon}\tau\, d\tau = \frac{p}{2m\epsilon}\tau^2 \Big|_{\tau=t-\epsilon}^{\tau=t}$$

assume $t > \epsilon$ since we will take $\epsilon \to 0$



$$= \frac{p}{2m\epsilon}\left[t^2 - (t^2 - 2\epsilon t + \epsilon^2)\right] = p\left(\frac{2\epsilon t - \epsilon^2}{2m\epsilon}\right) = \frac{p(2t-\epsilon)}{2m}$$

ⓑ $\lim_{\epsilon \to 0} x_\epsilon(t) = \frac{pt}{m}$. On the other hand, $mx'' = p\delta(t)$ has soln, using Duhamel, $\int_0^t \frac{p\tau}{m}\delta(t-\tau)\, d\tau = \int_0^\infty \frac{p\tau}{m}\delta(\tau - t)\, d\tau$

(since $\delta(t-\tau) = 0$ for $\tau > t$ & $\delta(x) = \delta(-x)$)

$$= \int_0^\infty \frac{p\tau}{m}\delta_t(\tau)\, d(\tau) = \frac{pt}{m} \quad \text{(by definition)}$$

SAME ✓

(13) (c) Assuming an ϵ-brief impulse, where $\epsilon < t$, we saw

$$x_\epsilon(t) = \frac{pt}{m} - \frac{p\epsilon}{2m}$$

An instaneous impulse corresponds to solution

$$x(t) = \lim_{\epsilon \to 0} x_\epsilon(t) = \frac{pt}{m}$$

In either case, $v = x'(t) = \frac{p}{m}$ so $p = mv$

(15) $mx'' + kx = 0 \quad x(0) = 0 \quad x'(0) = v_0$

let's find soln:

$$m\Big(s^2 X(s) - s\cancel{x(0)} - \underbrace{x'(0)}_{v_0}\Big) + k X(s) = \mathcal{L}\{0\} = 0$$

$$\Rightarrow X(s)(ms^2+k) - mv_0 = 0 \Rightarrow X(s) = \frac{mv_0}{ms^2+k} = \frac{v_0}{s^2 + k/m}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{m}{k}}\,t\right)$$

Alternately, $mx'' + kx = p_0\delta(t) \quad x(0) = 0 \quad x'(0) = 0$

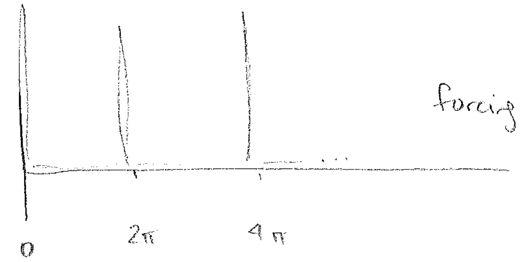
$$\Rightarrow X(s)(ms^2+k) = p_0 \cdot 1 \Rightarrow X(s) = \frac{p_0}{ms^2+k}$$

but $p_0 = mv_0$

so answer for $x(t)$ is same as before ✓

$$= \mathcal{L}^{-1}\{X(s)\}$$

(22)



forcing by hammer blows (spikes represent δ-functions)

forcing = $f(t) = \delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \ldots$

$$= \sum_{n=0}^{\infty} \delta(t-2n\pi)$$

initially at rest $\Rightarrow$ $x(0) + x'(0) = 0$

$m=1,\ k=1 \Rightarrow x'' + x = f(t) = \sum_{n=0}^{\infty} \delta(t-2n\pi)$

$$\Big\downarrow \mathcal{L}$$

$$X(s)\left[s^2+1\right] = \sum_{n=0}^{\infty} e^{-2n\pi s}$$

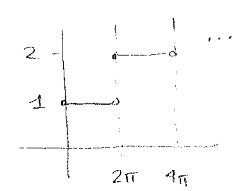
$$X(s) = \sum_{n=0}^{\infty} e^{-2n\pi s}\left(\frac{1}{s^2+1}\right)$$

Recognize translation

$$\Big\downarrow \mathcal{L}^{-1}$$

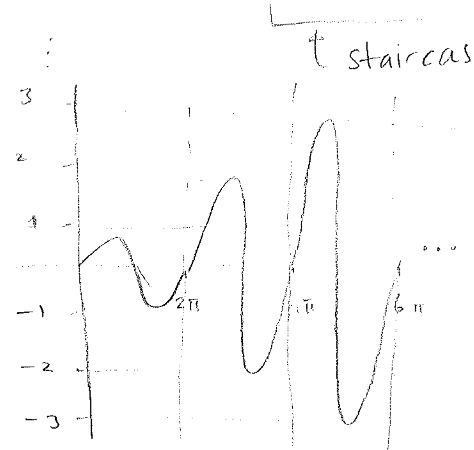
$$x(t) = \sum_{n=0}^{\infty} u(t-2n\pi)\underbrace{\sin(t-2n\pi)}_{=\sin t}$$

$$= \sin t \cdot \underbrace{\sum_{n=0}^{\infty} u(t-2n\pi)}_{\text{staircase function}}$$



amplitude increases in steps

graph of x(t):



$x(t) = (n+1)\sin t$

where $2n\pi < t < 2(n+1)\pi$