Math 306 Section T

Midterm Exam 1

February 19, 2015

Name: Solution Key

Problem	Points	Maximum		Problem	Points	Maximum
1		6		4		10
2		8		5		8
3		8		6		10
Subtotal		22		Subtotal		28
				Total		50

Please read the problems carefully and indicate your solutions clearly!
No credit awarded for unclear answers or unclear work.

1. (6 points) Find the general solution to the differential equation :

$$\frac{dy}{dx} = e^x - 2y$$

$y' + 2y = e^x \longleftarrow$ LINEAR

$P(x) = 2 \qquad \int P(x)\,dx = 2x + C \qquad \rho(x) = e^{\int P(x)dx} = e^{2x}$

$$\underbrace{e^{2x}y' + 2e^{2x}y}_{} = e^{3x}$$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{3x}$$

$$e^{2x}y + C = \int e^{3x}dx$$

$$e^{2x}y = \frac{1}{3}e^{3x} - C$$

$$\boxed{y = \frac{e^x}{3} - \frac{C}{e^{2x}}}$$

2. (8 points) The temperature $y(t)$ of an object placed in a warm water bath at time t is modeled by the equation below, where $k > 0$ is a constant.

$$\frac{dy}{dt} = k(80 - y)$$

(a) Find the solution $y(t)$ in terms of the initial condition $y(0) = T_0$.

(b) Sketch the solution curve for $T_0 = 100$.

Tip: though not required, sketching a slope field may help with part (b).

Observe $y(x) = 80$ is an equilibrium solution.

$$\int \frac{1}{80-y}\,dy = \int k\,dt \implies -\ln|80-y| = kt + C \qquad \text{assumes } y \neq 80$$

$$\implies \ln|80-y| = -kt - C$$

$$\implies |80-y| = e^{-kt}e^{-C}$$

$$\implies 80 - y = Ae^{-kt}$$

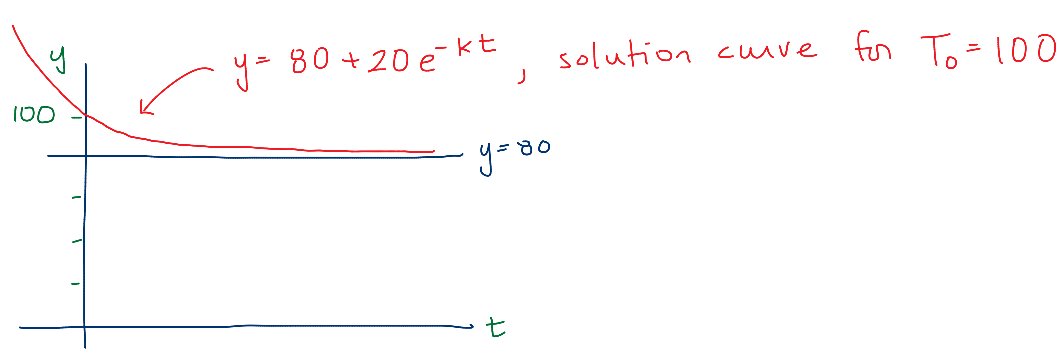
assumption can be removed here

$$A = \begin{cases} e^{-C} & \text{if } 80 > y \\ -e^{-C} & \text{if } 80 < y \\ 0 & \text{if } y = 80 \end{cases}$$

$$y(0) = T_0 \implies 80 - T_0 = Ae^0 = A$$

(a) $\boxed{y = 80 - (80 - T_0)e^{-kt}}$

(b)



3. (8 points) Find all solutions to the differential equation:

$$xy + y^2 - x^2y' = 0 \qquad (x > 0)$$

First observe that $y(x) = 0$ is a solution. There are two ways to find the general sol'n.:

METHOD 1: DE is homogeneous

$$\frac{y}{x} + \left(\frac{y}{x}\right)^2 - y' = 0$$

$$v = \frac{y}{x} \qquad y = vx \qquad y' = v + v'x$$

$$v + v^2 - (v + v'x) = 0 \Longrightarrow$$

$$v^2 = v'x = x\frac{dv}{dx} \Longrightarrow$$

$$\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx \Longrightarrow$$

$$-\frac{1}{v} = \ln(x) + C \qquad (\text{since } x > 0)$$

$$-\frac{x}{y} = \ln(x) + C$$

$$\boxed{y = \frac{-x}{\ln(x) + C} \quad \text{or} \quad y = 0}$$

METHOD 2: DE is Bernoulli

$$y' - \frac{1}{x}y = \frac{1}{x^2}y^2$$

$$v = y^{1-2} = \frac{1}{y} \qquad y = \frac{1}{v} \qquad y' = -\frac{1}{v^2}v'$$

$$-\frac{1}{v^2}v' - \frac{1}{x}\cdot\frac{1}{v} = \frac{1}{x^2}\cdot\frac{1}{v^2} \Longrightarrow$$

$$v' + \frac{1}{x}\cdot v = -\frac{1}{x^2}$$

$$\rho = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x \qquad (\text{since } x > 0)$$

$$xv' + v = x\left(\frac{-1}{x^2}\right) = \frac{-1}{x}$$

$$\frac{d}{dx}(xv) = -\frac{1}{x}$$

$$xv + C = \int -\frac{1}{x}\,dx$$

$$xv = -\ln|x| + C = -\ln(x) + C$$

$$\frac{x}{y} = -\ln(x) + C \Longrightarrow \boxed{y = \frac{-x}{\ln(x) + C} \quad \text{or} \quad y = 0}$$

4. (10 points) Consider the initial value problem

$$y' = \sqrt{1-y^2} \qquad y(0) = y_0$$

(a) Suppose $y_0 = 0$. Find the corresponding solution $y(x)$ **that is defined for all x.**

(b) For which y_0 does a solution exist?

(c) For which y_0 does a unique solution exist?

Tip: sketching a slope field may help you answer correctly.

First observe that y' is defined for $-1 \le y \le 1$ only.

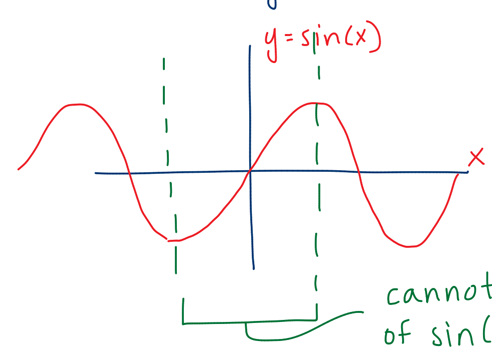
Also notice $y(x) = -1$ & $y(x) = 1$ are (equilibrium) solutions.

Assuming $y \neq \pm 1$, we may write:

$$\int \frac{1}{\sqrt{1-y^2}}\, dy = \int 1\, dx = x + C \implies \arcsin(y) = x + C$$

Recall arcsine is defined for $-1 \le y \le 1$ and has range $-\frac{\pi}{2} \le \arcsin(y) \le \frac{\pi}{2}$:

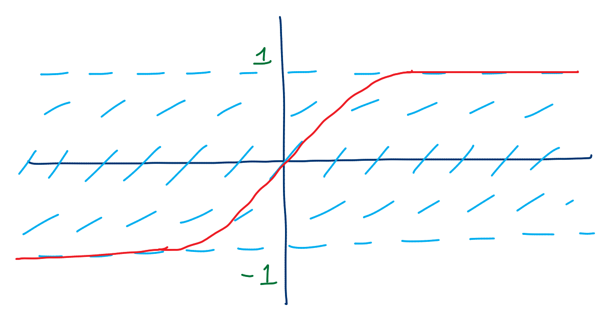
Therefore the solution $y = \sin(x + C)$ is valid only on the interval $-\frac{\pi}{2} \le x + C \le \frac{\pi}{2}$



cannot define inverse of $\sin(x)$ beyond this interval.

The solution valid for all x is:

$$y(x) = \begin{cases} -1 & x + C \le -\frac{\pi}{2} \\ \sin(x+C) & -\frac{\pi}{2} \le x + C \le \frac{\pi}{2} \\ 1 & x + C \ge \frac{\pi}{2} \end{cases}$$



Through $y(0) = 0$, the solution is:

above, a sketch of $y(x)$ defined in (a). Notice slopes y' are never negative, another clue that $y = \sin(x+C)$ does not work as a solution for all values of x.

(a)
$$y(x) = \begin{cases} -1 & x \le -\frac{\pi}{2} \\ \sin(x) & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1 & x \ge \frac{\pi}{2} \end{cases}$$

(b) We found solutions for I.C. $y(0) = y_0$ where $-1 \le y_0 \le 1$

(c) By theorem or observation, these are unique for $-1 < y_0 < 1$

5. (8 points) Use the substitution $p = \frac{dy}{dx}$ to solve the differential equation:

$$y'' + (y')^2 = 0$$

There are multiple ways to solve this problem.

Let $p = \frac{dy}{dx} = y'$. Then $y'' = \frac{d}{dx} p = p'$. It is also true $p' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy} \cdot p$.

METHOD 1: $p' + p^2 = 0 \Rightarrow \frac{dp}{dx} = -p^2 \Rightarrow \int \frac{-1}{p^2}\, dp = \int dx = x + C$ (if $p \neq 0$)

$\Rightarrow -\frac{1}{p} = x + C \Rightarrow p = \frac{-1}{x+c} \Rightarrow \frac{dy}{dx} = \frac{-1}{x+C}$

$\Rightarrow y = \int \frac{-1}{x+c}\, dx = \boxed{-\ln|x+c| + B}$ *

* note this $= \ln|x+c| + \ln(e^B)$
$= \ln|e^B x + e^B c$
$= \ln|Ax + D|$

METHOD 2: $\frac{dp}{dy} p + p^2 = 0 \Rightarrow \frac{dp}{dy} = -p$ (if $p \neq 0$) $\Rightarrow \int \frac{1}{p}\, dp = \int -1\, dy = -y + C$

$\Rightarrow \ln|p| = -y + C \Rightarrow |p| = e^{-y} e^{C} \Rightarrow p = Ae^{-y}$ where $A = \pm e^{C} \neq 0$

Method 2a: $x = \int \left(\frac{dx}{dy}\right) dy = \int \frac{1}{p}\, dy$ since $p = \frac{dy}{dx}$

$= \int \frac{e^y}{A}\, dy = \frac{1}{A} e^y + B \Rightarrow Ax - AB = e^y$ or $\boxed{y = \ln(Cx + D)}$

Method 2b: $\frac{dy}{dx} = Ae^{-y} \Rightarrow \int \frac{e^y}{A}\, dy = \int x\,dx = x + B \Rightarrow$

$\frac{1}{A} e^y = x + B \Rightarrow Ax + AB = e^y$ or $\boxed{y = \ln(Cx + D)}$

Any of the boxed answers is acceptable. In full generality, the solution is:

$y(x) = \ln(Ax + B)$ where $A > 0$, for $x > -B/A$,

$y(x) = \ln(Ax + B)$ where $A < 0$, for $x < -B/A$, or

$y(x) = C$ (note, the last solution is equivalent to $y = \ln(0x + B)$ where $B = e^C$)

6. (10 points) Suppose P people live in an isolated community. Let $y(t)$ represent the number of these people infected at time t by a new virus. The rate of infection is modeled by

$$\frac{dy}{dt} = y(P - y)$$

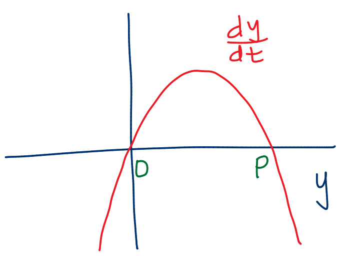
(a) Find all equilibrium solutions and indicate whether they are stable, unstable, or semistable.

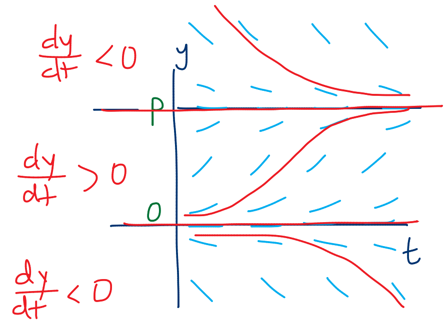
(b) Given the initial condition $y(0) = P/4$, determine $y(t)$. This models the situation where 25% of people are infected at the start.

(c) According to this model, if $y(0)$ is positive, what is the limit of $y(t)$ as $t \to \infty$?

Tip: though not required, sketching a slope field may help with this problem.

$y(t) = 0$ & $y(t) = P$ are equilibrium solutions.





By checking the sign of $\frac{dy}{dt}$ for $y < P$, $0 < y < P$, & $y > P$ we see

(a) $y = P$ is STABLE & $y = 0$ is UNSTABLE

$$\int \frac{1}{y(P-y)}\,dy = \int 1\,dt = t + C$$

$$\frac{1}{y(P-y)} = \frac{A}{y} + \frac{B}{P-y}$$

$$1 = A(P-y) + By = AP + (B-A)y$$

$$\Rightarrow AP = 1 \Rightarrow A = 1/P$$
$$B - A = 0 \qquad B = A = 1/P$$

$$\int \frac{\frac{1}{P}}{y} + \frac{\frac{1}{P}}{P-y}\,dy = t + C$$

$$\int \frac{1}{y} + \frac{1}{P-y}\,dy = Pt + C_2$$

$$\ln|y| - \ln|P-y| = Pt + C_2$$

$$\left|\frac{y}{P-y}\right| = e^{C_2} e^{Pt}$$

If $y(0) = P/4$,

$$\left|\frac{P/4}{P - P/4}\right| = e^{C_2} e^{P \cdot 0} = e^{C_2} \cdot 1$$

$$\left|\frac{P/4}{3P/4}\right| = \left|\frac{1}{3}\right| = \frac{1}{3}$$

$$\Rightarrow \left|\frac{y}{P-y}\right| = \frac{1}{3} e^{Pt}.$$

For $y(0) = P/4$, $\left|\frac{y}{P-y}\right| = \frac{1}{3} > 0$.
Because solutions are unique, solution curve stays within $0 < y < P$.
Thus $y > 0$ & $P - y > 0$, so $\left|\frac{y}{P-y}\right| = \frac{y}{P-y}$.

$$\frac{y}{P-y} = \frac{1}{3} e^{Pt} \Rightarrow y = \frac{1}{3} e^{Pt}(P - y) \Rightarrow y\left(1 + \frac{e^{Pt}}{3}\right) = \frac{P}{3} e^{Pt} \Rightarrow \boxed{\text{(b) } y(t) = \frac{\frac{P}{3} e^{Pt}}{1 + e^{Pt}/3}} = \frac{P}{3e^{-Pt} + 1}$$

(c) $\lim\limits_{t \to \infty} y(t) = \lim\limits_{t \to \infty} \dfrac{\frac{P}{3} e^{Pt}}{\frac{1}{3} e^{Pt}} = P$. (This can also be inferred from sketch of slope field/soln curve.)