Math 306 Section T

"Practice" Midterm Exam 1

February 17, 2015

Name: Solution Key

This is a practice exam. For the real exam:

- Nothing on your desk except writing instruments and UB ID card.
- No electronics! I will keep track of time on the board.
- Like practice exam, the real exam has six questions, half of which involve qualitative analysis, curve sketching, and/or applications (note, you are not expected to memorize any specific mathematical models, just to be able to do the math in the context of a given model).
- Questions on the real exam are **not** guaranteed to be easier or harder than the practice exam; it's just for the sake of review and practice.
- Like the real exam, give yourself 80 minutes.

1. Find all solutions to the differential equation:

$$y^2 y' + 2xy^3 = 6x$$

First assume $y \neq 0$, get: $y' + 2xy = 6x/y^2$

NOTE: This can be solved <u>two different ways</u>. While it is not necessary for the problem, we show both ways so you can check your work.

<u>Method 1: Bernoulli substitution</u>

$v = y^{1-(-2)} = y^3 \qquad y = v^{1/3}$

$y' = \frac{1}{3} v^{-2/3} v'$

$\frac{1}{3} v^{-2/3} v' + 2x v^{1/3} = 6x (v^{1/3})^{-2}$

$\Longrightarrow$

$v' + 6xv = 18x$ (linear)

$P(x) = 6x \qquad \int P(x)\,dx = 3x^2$

$\rho = e^{3x^2}$

$\underbrace{e^{3x^2} v' + 6x e^{3x^2} v}$ $= 18x e^{3x^2}$

$\frac{d}{dx}\left(e^{3x^2} v\right) = 18x e^{3x^2}$

$\Longrightarrow$

$e^{3x^2} v = \int 18x e^{3x^2}\,dx = 3e^{3x^2} + C$

$v = 3 + Ce^{-3x^2}$

$\boxed{y^3 = 3 + Ce^{-3x^2}}$

Finally, we return to our assumption that $y \neq 0$.

In the case $y = 0$,

$0 \cdot y' + 2x \cdot 0 = 6x \Rightarrow x = 0$

No more solutions arise from that case.

<u>Method 2: Sep. of Var's</u>

$y' = 2x\left(\frac{3}{y^2} - y\right) = 2x\left(\frac{3 - y^3}{y^2}\right)$

$\Rightarrow \underbrace{\int \frac{y^2}{3 - y^3}\,dy} = \int 2x\,dx = x^2 + C$

$\boxed{\text{Assume } 3 - y^3 \neq 0}$

$u = 3 - y^3$

$du = -3y^2$

$-\frac{1}{3} du = y^2\,dy$

$\int \frac{-1/3}{u}\,du = -\frac{1}{3}\ln|u|$

$= -\frac{1}{3}\ln|3 - y^3| = x^2 + C$

$\ln|3 - y^3| = -3x^2 + B$

$|3 - y^3| = e^B e^{-3x^2}$

$3 - y^3 = \pm e^B e^{-3x^2} = Ae^{-3x^2} \qquad A \neq 0$

$y^3 = 3 + Ae^{-3x^2}$ where $A \neq 0$

or

$3 - y^3 = 0 \Rightarrow y^3 = 3 \qquad y = 3^{1/3}$

$y' = 0$

$2y^2 y' + 2xy^3 =$

$2y^2(0) + 2x(3) = 6x$ ✓

Soln: $\boxed{y^3 = 3 + Ae^{-3x^2}}$ (no restriction on A)

2. Find all solutions to the differential equation:

$$x(x+y)y' = y(x-y)$$

This D.E. is homogeneous, so $v = y/x$ substition is possible.

First rewrite as $y' = F(\frac{y}{x})$:

$$y' = \frac{y(x-y)}{x(x+y)} = \left(\frac{y}{x}\right)\left(\frac{x-y}{x+y}\right)\cdot\frac{1/x}{1/x} = \left(\frac{y}{x}\right)\left(\frac{1-y/x}{1+y/x}\right)$$

$$v = y/x \Rightarrow y = xv \Rightarrow y' = v + xv'$$

$$v + xv' = v\left(\frac{1-v}{1+v}\right) \Rightarrow xv' = v\left(\frac{1-v}{1+v}\right) - v\left(\frac{1+v}{1+v}\right) = \frac{v - v^2 - v - v^2}{1+v}$$

$$\Rightarrow xv' = \frac{-2v^2}{1+v} \Rightarrow \int \frac{1+v}{v^2}\,dv = \int -\frac{2}{x}\,dx$$

$$\int \frac{1+v}{v^2}\,dv = \int \frac{1}{v^2} + \frac{1}{v}\,dv = -\frac{1}{v} + \ln|v| + C$$

$$\int -\frac{2}{x}\,dx = -2\ln|x| + C$$

$$\Rightarrow -2\ln|x| = -\frac{1}{v} + \ln|v| + C$$

$$\Rightarrow \frac{1}{v} + C = \ln|v| + 2\ln|x|$$

$$= \ln|v||x|^2 = \ln|vx^2|$$

$$\Rightarrow \frac{x}{y} + C = \ln\left|\frac{y}{x}x^2\right| = \ln|xy|$$

Along the way, we assumed $x \neq 0$, $y \neq 0$, and $x \neq -y$

If $y(x) = 0$, then $y' = 0$ and $x(x+0)\cdot 0 = 0(x-0)$ ✓

If $y(x) = -x$, then $y' = -1$ and $x(x-x)(-1) = -x(x+x)$ ✗

$$\boxed{\frac{x}{y} + C = \ln|xy|}$$

AND

$$\boxed{y(x) = 0}$$

NOTE general soln. in implicit form plus one singular soln.

3. Consider the differential equation $\frac{dx}{dt} = (x+2)(x-2)x$

(a) Find all equilibrium solutions and determine whether they are stable, unstable, or semistable.

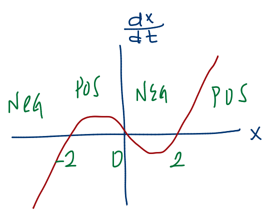
(b) Sketch a slope field for this differential equation.

(c) Find the solution corresponding to the initial condition x(0) = 1.

(d) Sketch the solution curve corresponding to your answer for (b).

$x' = (x+2)(x-2)x = 0$ if $x = -2, 2,$ or 0

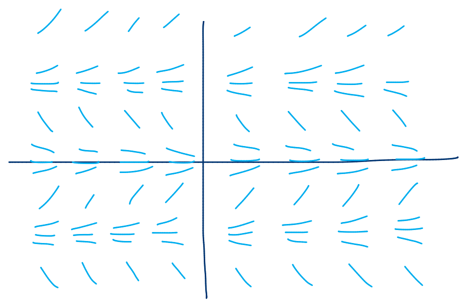
$= x^3 - 4x$



(a) $x = 2$ unstable

$x = 0$ stable

$x = -2$ unstable



(b) slope field

$$\int \frac{1}{(x+2)(x-2)x}\,dx = \int dt = t + C$$

$$\frac{1}{(x+2)(x-2)x} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{x}$$

$$1 = A(x-2)x + B(x+2)x + C(x+2)(x-2)$$

$$= Ax^2 - 2Ax + Bx^2 + 2Bx + Cx^2 - 4C$$

$$= \underbrace{(A+B+C)}_{0}x^2 + \underbrace{(2B-2A)}_{0}x \underbrace{- 4C}_{1}$$

$$\Rightarrow C = \frac{-1}{4}, \quad A = B$$

$$A + A - \frac{1}{4} = 0 \Rightarrow A = 1/8 \ \& \ B = 1/8$$

$$\int \frac{1/8}{x+2} + \frac{1/8}{x-2} - \frac{1/4}{x}\,dx =$$

$$\frac{1}{8}\ln|x+2| + \frac{1}{8}\ln|x-2| - \frac{1}{4}\ln|x|$$

$$= t + C \Rightarrow$$

$$\ln|x+2| + \ln|x-2| - \ln x^2 = 8t + 8C$$

$$\ln\left(\frac{|x^2-4|}{x^2}\right) = 8t + C_1$$

$$\frac{|x^2-4|}{x^2} = e^{C_1}e^{8t} = Ae^{8t}$$

$$x(0) = 1 \Rightarrow \frac{|-3|}{1} = Ae^0$$

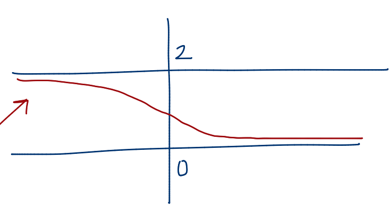
$$\Rightarrow A = 3$$

$$\frac{4-x^2}{x^2} = 3e^{8t} \Rightarrow 4 - x^2 = 3e^{8t}x^2$$

$$\Rightarrow x^2(3e^{8t}+1) = 4 \Rightarrow x^2 = \frac{4}{(3e^{8t}+1)}$$

$$\Rightarrow \boxed{x = \frac{2}{\sqrt{3e^{8t}+1}} \quad (c)}$$

(d) soln curve



Note on (b) and (c): it may seem strange that velocity is never zero yet the moving body only travels a finite distance. This is a good example of how mathematical models, which describe a theoretical ideal, differ from reality! In the model, the moving body heads ever more slowly towards position x_0, but never stops, and never quite reaches x_0---that's the difference between the limit of a function at infinity, and the value of the function at a given point. In typical reality, the moving body would stop (for example, due to forces of friction not accounted for in the model).

4. Suppose a body moves through a resisting medium with resistance proportional to velocity, so that $\frac{dv}{dt} = -kv$. Let $x(t)$ be the position of the body, so that $v(t) = \frac{dx}{dt}$.

   (a) Solve for the function $x(t)$ in terms of the initial conditions $x_0 = x(0)$ and $v_0 = v(0)$.

   (b) Show that the moving body only travels a finite distance, by computing $\lim_{t\to\infty} x(t)$.

   (c) According to the model given by the differential equation $\frac{dv}{dt} = -kv$, if $v_0 > 0$, is velocity ever zero?

$$\frac{dv}{dt} = -kv \Rightarrow \underbrace{\int \frac{1}{v}\,dv}_{=\ln|v|} = \int -k\,dt = -kt + C$$

$$\Rightarrow \ln|v| = -kt + C$$

$$|v| = e^C e^{-kt}$$

$$v = \pm e^C e^{-kt} = Ae^{-kt}$$

$v(0) = v_0 \Rightarrow$

$$v_0 = Ae^0 \Rightarrow v_0 = A \Rightarrow v(t) = v_0 e^{-kt}$$

$$\frac{dx}{dt} = v(t) = v_0 e^{-kt} \Rightarrow \underbrace{\int dx}_{=x} = \int v_0 e^{-kt}\,dt = \left(-\frac{v_0}{k}\right)e^{-kt} + C$$

$x(0) = x_0 \Rightarrow$ $$x_0 = \left(\frac{-v_0}{k}\right)e^0 + C \Rightarrow C = x_0 + \frac{v_0}{k}$$

$$\Rightarrow \boxed{x(t) = \left(-\frac{v_0}{k}\right)e^{-kt} + x_0 + \frac{v_0}{k}} \quad \text{a}$$

$$\boxed{\text{b} \quad \lim_{t\to\infty} x_0 + \frac{v_0}{k} - \frac{v_0}{k}e^{-kt} = x_0 + \frac{v_0}{k}}$$

(because the limit is finite, the body approaches a fixed position as $t \to \infty$)

{ optional explanation

c If $v_0 > 0$, $v(t)$ is NEVER zero.

There are two ways to argue this: (only one needed for answer)

1. $v(t) = v_0 e^{-kt}$ is never zero if $v_0 > 0$.
2. $\frac{dv}{dt} = -kv$ fulfills existence & uniqueness thm at all points & has equilibrium solution $v(t) = 0$ for all t. So if $v_0 > 0$, this soln does not intersect $v = 0$ soln!

5. Consider the initial value problem

$$\frac{dy}{dx} = 2\sqrt{y} \qquad y(0) = y_0$$

(a) Find all solutions to the differential equation.

(b) For which y_0 does a unique solution exist?

(c) Show that, if $y_0 = 0$, two solutions exist. Explain why this does not contradict the theorem on existence and uniqueness of solutions to first-order ordinary differential equations.

$\frac{dy}{dx} = 2\sqrt{y}$ implicitly requires $y \geq 0$

notice $\boxed{y(x) = 0}$ is a (equilibrium) solution

if $y \neq 0$, have $\int \frac{1}{2\sqrt{y}}\, dy = \int dx \Rightarrow \boxed{y^{1/2} = x + C}$ (a)

(b) If $y_0 > 0$, solutions are unique.
(This can be concluded by part (a) or existence & uniqueness theorem.)

(c) Both $y(x) = 0$ & $\sqrt{y} = x$ are solutions that contain the I.C. $(0,0)$.

There are two ways the existence & uniqueness thm fails to apply.

First let's recall the conditions of the thm (version for $y' = f(x,y)$ form D.E.)

Given I.V.P. $y' = f(x,y)$ w/ I.C. (x_0, y_0), if on some rectangle containing (x_0, y_0) in the interior, both

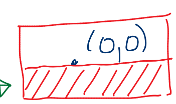
$$f(x,y) \;\&\; \frac{\partial}{\partial y} f(x,y)$$

are continuous, then the IVP has a unique soln defined on an interval containing x_0.

THE TWO REASONS
(only one needed for answer)

① $f(x,y) = 2\sqrt{y}$ not defined on rectangle w/ $(0,0)$ in interior

any such rectangle



$2\sqrt{y}$ not continuous here

② $\frac{\partial}{\partial y} f(x,y) = 2 \cdot \frac{1}{2} y^{-1/2} = \frac{1}{\sqrt{y}}$

not defined at $y = 0$

6. Use the substitution $p = \frac{dy}{dx}$ to solve the differential equation below. You may leave your answer in terms of an integral if you use the fundamental theorem of calculus correctly.

$$2\sqrt{1-x^2}y'' - (y')^3 e^{-2x} = \sqrt{4-4x^2}y' = 2\sqrt{1-x^2}\,y'$$

$p = y' \quad p' = y'' \rightsquigarrow 2\sqrt{1-x^2}\,p' - p^3 e^{-2x} = 2\sqrt{1-x^2}\,p \Rightarrow$

$p' - p^3 e^{-2x}/(2\sqrt{1-x^2}) = p$ (Restrict to $-1 < x < 1$)

$\Rightarrow p' - p = \dfrac{e^{-2x}}{2\sqrt{1-x^2}} p^3$ ← BERNOULLI

$v = p^{1-3} = p^{-2} \qquad p = v^{-1/2} \qquad p' = -\frac{1}{2} v^{-3/2} v'$

$-2v^{3/2}\left(-\frac{1}{2} v^{-3/2} v' - v^{-1/2}\right) = \dfrac{e^{-2x}}{2\sqrt{1-x^2}} (v^{-1/2})^3 = \dfrac{e^{-2x}\, v^{-3/2}}{2\sqrt{1-x^2}} \cdot (-2v^{3/2})$

$\Rightarrow v' + 2v = \dfrac{-e^{-2x}}{\sqrt{1-x^2}}$ ← LINEAR

$P(x) = 2 \qquad \int P(x)\,dx = 2x + C$

use $\rho = e^{\int P(x)dx} = e^{2x}$

$\Rightarrow \underbrace{e^{2x} v' + 2e^{2x} v}_{} = \dfrac{-1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}\left(e^{2x} v\right) = \dfrac{-1}{\sqrt{1-x^2}} \Rightarrow e^{2x} v + C = \displaystyle\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\sin^{-1}(x)$

by $x = \sin\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = \cos\theta\, d\theta$

$\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$ since $\cos\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\displaystyle\int \frac{-\cos\theta\, d\theta}{\cos\theta} = -\theta = -\sin^{-1}(x)$

Get $e^{2x} v = C - \sin^{-1}(x)$

$v = e^{-2x}\left(C - \sin^{-1}(x)\right)$

$\parallel$

$p^{-2} \Rightarrow p = \left(e^{-2x}\left(C - \sin^{-1}(x)\right)\right)^{-1/2} = \dfrac{e^x}{\sqrt{C - \sin^{-1}(x)}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^x}{\sqrt{C - \sin^{-1}(x)}}$

$\Rightarrow y(x) = \displaystyle\int \frac{e^x}{\sqrt{C - \sin^{-1}(x)}}\,dx = \int_0^x \frac{e^t}{\sqrt{C_1 - \sin^{-1}(t)}}\,dt + C_2$

Note: Important that final answer is two-parameter family (has C_1 & C_2)

using instruction in problem that it's OK to leave answer in terms of (unevaluated) integral (not true in general! special for this rather long problem.)

This last integral cannot be expressed in an elementary way, at least, not by me, nor by Maple!