# Math 306 Section T

# "Practice" Midterm Exam 2

## April 7, 2015

Name: Solutions

This is a practice exam. For the real exam:

- Nothing on your desk except writing instruments and UB ID card.
- No electronics! I will keep track of time on the board.
- Like practice exam, the real exam has six questions, half of which involve qualitative analysis, curve sketching, and/or applications (note, you are not expected to memorize any specific mathematical models, just to be able to do the math in the context of a given model).
- Questions on the real exam are **not** guaranteed to be easier or harder than the practice exam; it's just for the sake of review and practice.
- Like the real exam, give yourself 80 minutes.

1. Consider the differential equation:

$$y^{(3)} + 2y'' + 2y' = 0$$

(a) How many linearly independent solutions do you expect, and why?

(b) Find the general solution.

(a) Three, since this is a 3rd-order linear DE w/ continuous (constant) coefficients.

(b) char. eqn $= r^3 + 2r^2 + 2r = r(r^2+2r+2)$

roots are $r = 0$ & $r = \frac{1}{2}\left(-2 \pm \sqrt{4-8}\right) = -1 \pm i$

general solution is $y(t) = c_1 + c_2 e^{-t}\cos t + c_3 e^{-t}\sin t$

CHECK:

$$y = \boxed{c_1 + c_2 e^{-t}\cos t + c_3 e^{-t}\sin t} = y$$

$$\begin{aligned} y' = {} & c_3 e^{-t}\cos t - c_3 e^{-t}\cos t \\ & - c_2 e^{-t}\cos t - c_2 e^{-t}\sin t \end{aligned}$$

$$= \boxed{(c_3 - c_2)e^{-t}\cos t - (c_2 + c_3)e^{-t}\sin t} = y'$$

$$y'' = \begin{bmatrix} -(c_2+c_3) \\ -(c_3-c_2) \end{bmatrix} e^{-t}\cos t - \begin{bmatrix} c_3 - c_2 \\ -c_2 - c_3 \end{bmatrix} e^{-t}\sin t$$

$$= \boxed{-2c_3 e^{-t}\cos t + 2c_2 e^{-t}\sin t} = y''$$

$$y^{(3)} = \boxed{(2c_2 + 2c_3)e^{-t}\cos t + (2c_3 - 2c_2)e^{-t}\sin t} = y^{(3)}$$

$y^{(3)} + 2y'' + 2y' =$

$$\underbrace{\begin{bmatrix} 2c_2 + 2c_3 \\ -4c_3 \\ -2c_2 + 2c_3 \end{bmatrix}}_{0} e^{-t}\cos t + \underbrace{\begin{bmatrix} 2c_3 - 2c_2 \\ +4c_2 \\ -2c_3 - 2c_2 \end{bmatrix}}_{0} e^{-t}\sin t = 0 \checkmark$$

2. Solve the initial value problem:

$$y^{(4)} - 16y = 0$$

$$y(0) = 1 \quad y'(0) = 4 \quad y''(0) = 4 \quad y^{(3)}(0) = 0$$

Char eqn: $r^4 - 16 = (r^2-4)(r^2+4) = (r-2)(r+2)(r-2i)(r+2i)$

Gen soln: $y = c_1e^{2t} + c_2e^{-2t} + c_3\cos(2t) + c_4\sin(2t)$

$y' = 2c_1e^{2t} - 2c_2e^{-2t} + 2c_4\cos(2t) - 2c_3\sin(2t)$

$y'' = 4c_1e^{2t} + 4c_2e^{-2t} - 4c_3\cos(2t) - 4c_4\sin(2t)$

$y^{(3)} = 8c_1e^{2t} - 8c_2e^{-2t} - 8c_4\cos(2t) + 8c_3\sin(2t)$

ICs: $y(0) = c_1 + c_2 + c_3 = 1$

$y'(0) = 2c_1 - 2c_2 + 2c_4 = 4$

$y''(0) = 4c_1 + 4c_2 - 4c_3 = 4$

$y^{(3)}(0) = 8c_1 - 8c_2 - 8c_4 = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 4 & 4 & -4 & 0 \\ 8 & -8 & 0 & -8 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 2 & -2 & 0 & 2 & 4 \\ 4 & 4 & -4 & 0 & 4 \\ 8 & -8 & 0 & -8 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & -4 & -2 & 2 & 2 \\ 0 & 0 & -8 & 0 & 0 \\ 0 & -16 & -8 & -8 & -8 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & -2 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -16 & -16 \end{array}\right]$$

$\Longrightarrow -16c_4 = -16 \Rightarrow c_4 = 1$

$-c_3 = 0$

$-2c_2 - \cancel{c_3} + \underbrace{c_4}_{=1} = 1 \Rightarrow c_2 = 0$

$c_1 + \cancel{c_2} + \cancel{c_3} = 1 \Rightarrow c_1 = 1$

ANS:

$\boxed{y(t) = e^{2t} + \sin(2t)}$ $\rightarrow$ $y(0) = 1$

CHECK: $y'(t) = 2e^{2t} + 2\cos(2t) \rightarrow y'(0) = 4$

$y''(t) = 4e^{2t} - 4\sin(2t) \rightarrow y''(0) = 4$

$y^{(3)}(t) = 8e^{2t} - 8\cos(2t) \rightarrow y^{(3)}(0) = 0$ ✓

3. Find the general solution to the differential equation:

$$y''(t) + 2y'(t) + y(t) = 2t\sin(t)$$

Char eqn $r^2 + 2r + 1 = (r+1)^2$

Complementary soln: $y_c = (c_1 + c_2 t)e^{-t}$

Method of undetermined coeffs:

$$y_p = At\sin t + Bt\cos t + C\sin t + D\cos t$$

$$y_p' = -Bt\sin t + At\cos t + (A-D)\sin t + (B+C)\cos t$$

$$y_p'' = -At\sin t - Bt\cos t + (-B-B-C)\sin t + (A+A-D)\cos t$$

$$y_p'' + 2y' + y = (\overbrace{A-2B-A}^{2})\,t\sin t + (\overbrace{B+2A-B}^{0})\,t\cos t$$
$$+ (\underbrace{C+2A-2D-2B-C}_{0})\sin t + (\underbrace{D+2B+2C+2A-D}_{0})\cos t$$

$$-2B = 2 \Rightarrow B = -1 \qquad 2A = 0 \Rightarrow A = 0$$

$$0 = 2A - 2D - 2B \Rightarrow 0 = -2D + 2 \Rightarrow D = 1$$

$$2(A+B+C) = 0 \Rightarrow -1 + C = 0 \Rightarrow C = 1$$

$$\boxed{y(t) = (c_1 + c_2 t)e^{-t} - t\cos t + \sin t + \cos t}$$

CHECK. $y' = c_2 e^{-t} - (c_1 + c_2 t)e^{-t} + t\sin t - \cos t + \cos t - \sin t$

$$y' = \left[(c_2 - c_1) - c_2 t\right]e^{-t} + t\sin t - \sin t$$

$$y'' = \left[\big(\underbrace{-c_2 - (c_2 - c_1)}_{(c_1 - 2c_2)}\big) + c_2 t\right]e^{-t} + t\cos t + \sin t - \cos t$$

$$y'' + 2y' + y = \cancel{\begin{bmatrix} c_1 - 2c_2 \\ -2c_1 + 2c_2 \\ +c_1 \end{bmatrix}} e^{-t} + \cancel{\begin{bmatrix} c_2 + \\ -2c_2 \\ +c_2 \end{bmatrix}} e^{-t} + 2t\sin t + \cancel{t\cos t - t\cos t} + \cancel{\sin t - 2\sin t + \sin t}$$
$$\cancel{-\cos t + \cos t}$$

4. Find the general solution to the system of DEs:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{vmatrix} 9-\lambda & 4 \\ -1 & 5-\lambda \end{vmatrix} = 45 + \lambda^2 - 14\lambda + 4 = \lambda^2 - 14\lambda + 49 = (\lambda-7)^2$$ eigenval $\lambda = 7$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x + 4y = 0$$ let $y = 1$ then $x = -2$ eigen vec $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

defect = 1 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ not a multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

create length-2 chain: $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = u_2$$

ANS:

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{7t} + c_2 \left( \begin{bmatrix} 2 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{7t}}$$

CHECK: $$\begin{bmatrix} x' \\ y' \end{bmatrix} = c_1 \begin{bmatrix} 14 \\ -7 \end{bmatrix} e^{7t} + c_2 \left( \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{7t} + \left( \begin{bmatrix} 14 \\ -7 \end{bmatrix} t + \begin{bmatrix} 7 \\ 0 \end{bmatrix} \right) e^{7t} \right)$$

$$= c_1 \begin{bmatrix} 14 \\ -7 \end{bmatrix} e^{7t} + c_2 \left( \begin{bmatrix} 14 \\ -7 \end{bmatrix} t + \begin{bmatrix} 9 \\ -1 \end{bmatrix} \right) e^{7t}$$

$$\begin{bmatrix} 9 & 4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix} \qquad \begin{bmatrix} 9 & 4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix} \checkmark$$

5. Consider the system of DEs:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

(a) Find the general solution.

(b) Find the solution satisfying ICs $x(0) = 1$ and $y(0) = 5$.

(c) Sketch the phase portrait for this system.

(a) $\begin{vmatrix} 1-\lambda & 4 \\ 12 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 48 = \lambda^2 - 1 - 48 = \lambda^2 - 49 = (\lambda-7)(\lambda+7)$

$\lambda = 7 \rightarrow \begin{bmatrix} -6 & 4 \\ 12 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -6x + 4y = 0$, $\quad y = \left(\frac{6}{4}\right)x = \frac{3x}{2}$ $\quad$ evec $\begin{bmatrix} \cancel{1}\,2 \\ \cancel{3/2}\,3 \end{bmatrix}$

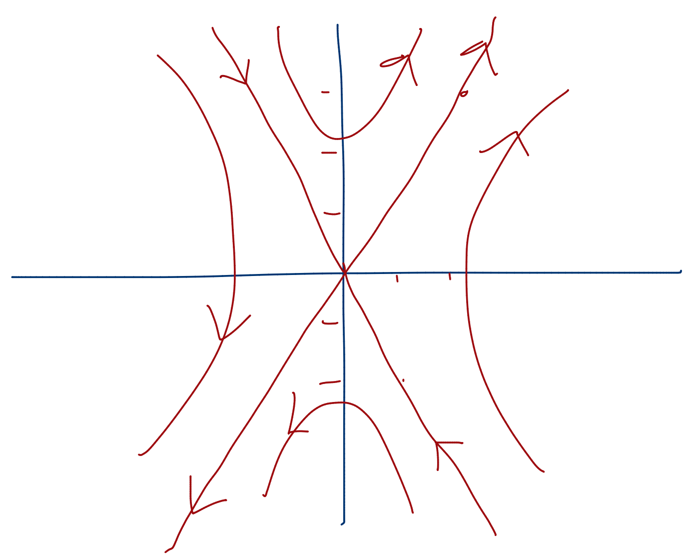
$\lambda = -7 \rightarrow \begin{bmatrix} 8 & 4 \\ 12 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad$ evec $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-7t}$$

(b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix} c_1 + \begin{bmatrix} 1 \\ -2 \end{bmatrix} c_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4-3}\begin{bmatrix} -2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \frac{-1}{7}\begin{bmatrix} -7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{7t} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-7t}$$

(c)



CHECK:

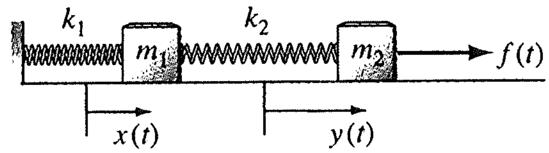
(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \end{bmatrix} e^{7t} + \begin{bmatrix} 7 \\ -14 \end{bmatrix} e^{-7t}$

$\begin{bmatrix} 1 & 4 \\ 12 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 \\ 12 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -14 \end{bmatrix}$ ✓

6. Consider the mass-spring system on the left, derived from the diagram on the right:

$$x'' = -3x + 2(y - x),$$
$$y'' = -2(y - x)$$




$$x'' = -5x + 2y$$
$$y'' = 2x - 2y$$

Set $m_{1,2} = 1$, $k_1 = 3$, $k_2 = 2$, and $f(t) \equiv 0$.

(a) Rewrite this system of two 2nd-order DEs system as a system of four 1st-order DEs.

(b) Use the eigenvalue method to find the general solution to your answer for (a).

(a) Let $z_1 = x$ $\quad z_2 = x'$ $\quad z_3 = y$ $\quad z_4 = y'$ $\quad$ let $\vec{z} = [z_1, z_2, z_3, z_4]^T$

ANS. $$\boxed{\begin{aligned} z_1' &= z_2 \\ z_2' &= -5z_1 + 2z_3 \\ z_3' &= z_4 \\ z_4' &= 2z_1 - 2z_3 \end{aligned}}$$

(b) $$\vec{z}\,' = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}}_{A} \vec{z}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -5 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 1 \\ 2 & 0 & -2 & -\lambda \end{vmatrix}$$

$$= -\lambda\left[-\lambda\left(\lambda^2 - (-2)\right)\right] - 1 \begin{vmatrix} -5 & 2 & 0 \\ 0 & -\lambda & 1 \\ 2 & -2 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2 + 2) - \left[-5\left(\lambda^2 - (-2)\right) - 2(0 - 2)\right] = \lambda^4 + 2\lambda^2 - \left[-5(\lambda^2 + 2) + 4\right]$$

$$= \lambda^4 + 7\lambda^2 + 6 = (\lambda^2 + 1)(\lambda^2 + 6) = 0 \Rightarrow \boxed{\lambda = \pm i, \ \pm\sqrt{6}\, i}$$

eigenvalues!

Case $\lambda = i$: solve for eigenvector $\vec{v}$:

$$\begin{bmatrix} -i & 1 & 0 & 0 \\ -5 & -i & 2 & 0 \\ 0 & 0 & -i & 1 \\ 2 & 0 & -2 & -i \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[\text{elim.}]{\text{Gauss.}}$$

scale by $i$
$+ 5i$ (old top row)
scale by $i$
$- 2i$ (old top row)

$$\left[\begin{array}{cccc|c} 1 & i & 0 & 0 & 0 \\ 0 & 4i & 2 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \\ 0 & -2i & -2 & -i & 0 \end{array}\right] \rightarrow$$

same
scale by $-i/4$
same
$+ \frac{1}{2}$ (old 2nd row)

$$\left[\begin{array}{cccc|c} 1 & i & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \\ 0 & 0 & -1 & -i & 0 \end{array}\right]$$

× same info

$$\rightarrow \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z_3 = -i z_4$$
$$z_2 = \frac{i}{2} z_3$$
$$z_1 = -i z_2$$

continued on next page

Eigenvector $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$ for $\lambda = i$ satisfies

$z_4 = \text{anything}$
$z_3 = -i z_4$
$z_2 = i/_2\, z_3$
$z_1 = -i z_2$

eigenvector takes form $\begin{bmatrix} -is/2 \\ s/2 \\ -is \\ s \end{bmatrix} = s \begin{bmatrix} -i/2 \\ 1/2 \\ -i \\ 1 \end{bmatrix}$ use $s = 1$, have evec $\bar{v} = \begin{bmatrix} -i/2 \\ 1/2 \\ -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1/2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

Have complex solution $s(t) = \bar{v} e^{it} =$

$$\left( \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1/2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right) \left( \cos(t) + i \sin(t) \right)$$

$$= \underbrace{\left( \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} -1/2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \sin(t) \right)} + i \underbrace{\left( \begin{bmatrix} -1/2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \cos(t) + \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \sin(t) \right)}$$

$\mathrm{Re}(s(t)) = \begin{bmatrix} \frac{1}{2}\sin(t) \\ \frac{1}{2}\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}$ $\qquad \mathrm{Im}(s(t)) = \begin{bmatrix} -1/2 \cos(t) \\ 1/2 \sin(t) \\ -\cos(t) \\ \sin(t) \end{bmatrix}$

Two linearly indep solns are $\mathrm{Re}(s(t))$ & $\mathrm{Im}(s(t)) =$

(b) ANS: $\begin{bmatrix} \frac{1}{2}\sin(t) \\ \frac{1}{2}\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}$ & $\begin{bmatrix} -1/2 \cos(t) \\ 1/2 \sin(t) \\ -\cos(t) \\ \sin(t) \end{bmatrix}$

for shortened version of problem: find two linearly independent solutions.

Optional: complete solution, see next page

Case $\lambda = \sqrt{6}\,i$ : solve $(A - \sqrt{6}i\,I)\bar{v} = 0$ :

$$\frac{5}{\sqrt6} - \sqrt6 = \frac{5}{\sqrt6} - \frac{6}{\sqrt6} = -\frac{1}{\sqrt6}$$

$$\left[\begin{array}{cccc|c} -\sqrt6 i & 1 & 0 & 0 & 0 \\ -5 & -\sqrt6 i & 2 & 0 & 0 \\ 0 & 0 & -\sqrt6 i & 1 & 0 \\ 2 & 0 & -2 & -\sqrt6 i & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & i/\sqrt6 & 0 & 0 & 0 \\ 0 & i\left(\frac{5}{\sqrt6} - \sqrt6\right) & 2 & 0 & 0 \\ 0 & 0 & 1 & i/\sqrt6 & 0 \\ 0 & -2i/\sqrt6 & -2 & -\sqrt6 i & 0 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{cccc|c} 1 & i/\sqrt6 & 0 & 0 & 0 \\ 0 & -i/\sqrt6 & 2 & 0 & 0 \\ 0 & 0 & 1 & i/\sqrt6 & 0 \\ 0 & 1 & -\sqrt6 i & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & i/\sqrt6 & 0 & 0 & 0 \\ 0 & 1 & 2\sqrt6 i & 0 & 0 \\ 0 & 0 & 1 & i/\sqrt6 & 0 \\ 0 & 0 & -3\sqrt6 i & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & i/\sqrt6 & 0 & 0 & 0 \\ 0 & 1 & i2\sqrt6 & 0 & 0 \\ 0 & 0 & 1 & i/\sqrt6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$z_4$ = anything ; $z_3 = -i/\sqrt6\, z_4$ ; $z_2 = -i2\sqrt6\, z_3$ ; $z_1 = -i/\sqrt6\, z_2$

eigenvec $= s\begin{bmatrix} 2i/\sqrt6 \\ -2 \\ -i/\sqrt6 \\ 1 \end{bmatrix}$ choose $s=1$, eigenvec : $\begin{bmatrix} i\sqrt{\frac{2}{3}} \\ -2 \\ -i\sqrt{\frac16} \\ 1 \end{bmatrix} = \left(\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} \sqrt{2/3} \\ 0 \\ -\sqrt{1/6} \\ 0 \end{bmatrix}\right) = \bar{v}$

soln $\bar{v}e^{i\sqrt6 t} = \left(\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} \sqrt{2/3} \\ 0 \\ -\sqrt{1/6} \\ 0 \end{bmatrix}\right)\left(\cos\sqrt6 t + i\sin\sqrt6 t\right) =$

$$\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}\cos\sqrt6 t - \begin{bmatrix} \sqrt{2/3} \\ 0 \\ -\sqrt{1/6} \\ 0 \end{bmatrix}\sin\sqrt6 t + i\left(\begin{bmatrix} \sqrt{2/3} \\ 0 \\ -\sqrt{1/6} \\ 0 \end{bmatrix}\cos\sqrt6 t + \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}\sin\sqrt6 t\right)$$

$$\text{Re} = \begin{bmatrix} -\sqrt{2/3}\sin\sqrt6 t \\ -2\cos\sqrt6 t \\ +\sqrt{1/6}\sin\sqrt6 t \\ \cos\sqrt6 t \end{bmatrix} \qquad \text{Im} = \begin{bmatrix} \sqrt{2/3}\cos\sqrt6 t \\ -2\sin\sqrt6 t \\ -\sqrt{1/6}\cos\sqrt6 t \\ \sin\sqrt6 t \end{bmatrix}$$

general soln:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = c_1\begin{bmatrix} \frac12\sin(t) \\ \frac12\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix} + c_2\begin{bmatrix} -\frac12\cos(t) \\ \frac12\sin(t) \\ -\cos(t) \\ \sin(t) \end{bmatrix} + c_3\begin{bmatrix} -\sqrt{2/3}\sin\sqrt6 t \\ -2\cos\sqrt6 t \\ \sqrt{1/6}\sin\sqrt6 t \\ \cos\sqrt6 t \end{bmatrix} + c_4\begin{bmatrix} \sqrt{2/3}\cos\sqrt6 t \\ -2\sin\sqrt6 t \\ -\sqrt{1/6}\cos\sqrt6 t \\ \sin\sqrt6 t \end{bmatrix}$$

In particular, $x(t) = z_1 = \frac{c_1}{2}\sin(t) - \frac{c_2}{2}\cos(t) - c_3\sqrt{\frac23}\sin(\sqrt6 t) + c_4\sqrt{\frac23}\cos(\sqrt6 t)$

$y(t) = z_3 = c_1\sin(t) - c_2\cos(t) + c_3\sqrt{\frac16}\sin(\sqrt6 t) - c_4\sqrt{\frac16}\cos(\sqrt6 t)$

CHECK:

$$x' = \left(\frac{c_2}{2}\right)\sin(t) + \left(\frac{c_1}{2}\right)\cos(t) - 2c_4\sin(\sqrt{6}t) - 2c_3\cos\sqrt{6}t$$

$$x'' = -\frac{c_1}{2}\sin(t) + \frac{c_2}{2}\cos(t) + 2\sqrt{6}c_3\sin\sqrt{6}t - 2\sqrt{6}\cos\sqrt{6}t$$

$-5x + 2y =$

$$\underbrace{\left(-\frac{5}{2}+2\right)}_{-1/2}c_1\sin(t) + \underbrace{\left(\frac{5}{2}-2\right)}_{1/2}c_2\cos(t) + \underbrace{\left(5\sqrt{\frac{2}{3}} + 2\sqrt{\frac{1}{6}}\right)}_{5\sqrt{\frac{2}{3}}+\sqrt{\frac{2}{3}}}c_3\sin(\sqrt{6}t) + \underbrace{\left(-5\sqrt{\frac{2}{3}} - 2\sqrt{\frac{1}{6}}\right)}_{=-2\sqrt{6}}c_4\cos\sqrt{6}t$$

$$= 6\sqrt{\frac{2}{3}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{12}{\sqrt{6}} = 2\sqrt{6}$$

✓

$$y' = c_2\sin(t) + c_1\cos(t) + c_4\sin(\sqrt{6}t) + c_3\cos(\sqrt{6}t)$$

$$y'' = -c_1\sin(t) + c_2\cos(t) - \sqrt{6}c_3\sin(\sqrt{6}t) + \sqrt{6}c_4\cos(\sqrt{6}t)$$

$2x - 2y =$

$$\underbrace{(1-2)}_{-1}c_1\sin(t) + \underbrace{(-1+2)}_{+1}c_2\cos(t) + \underbrace{\left(-2\sqrt{\frac{2}{3}} - 2\sqrt{\frac{1}{6}}\right)}_{\frac{-4}{\sqrt{6}}-\frac{2}{\sqrt{6}} = -\sqrt{6}}c_3\sin(\sqrt{6}t)$$

$$+ \underbrace{\left(2\sqrt{\frac{2}{3}} + 2\sqrt{\frac{1}{6}}\right)}_{\sqrt{6}}c_4\cos(\sqrt{6}t)$$

✓