MATH 306 SECTION T

"PRACTICE" FINAL EXAM

MAY 7, 2015

NAME: SOLUTION KEY
(1-6 ONLY)

This is a practice exam. For the real exam:

- Nothing on your desk except writing instruments and UB ID card.
- No electronics! I will keep track of time on the board.
- Like the practice exam, the final has six questions on content from Chapters 6-8. In addition, there are two "cumulative" questions.
- Questions on the real exam are **not** guaranteed to be easier/harder than the practice exam; this is just for review and practice.
- Give yourself 80 minutes for the first six questions. The two "cumulative" questions do not require a time limit; shorter versions will appear on the exam.

(*) Case $T = 0 \Rightarrow$

$\lambda = \dfrac{0 \pm \sqrt{-4D}}{2}$ if $D < 0$

$= 2\sqrt{D} = 2\sqrt{-D}\, i$

pure imaginary $\Rightarrow$ center!

1. Consider the linear system:

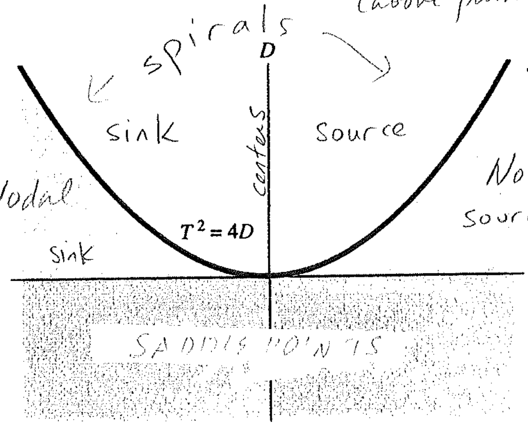
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Let $T = a + d$ and $D = ad - bc$. For this problem, you will show how the values of T and D determine the type of critical point at $(0, 0)$.

(a) Show that the eigenvalues for this system are

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

(b) On the (T, D) plane shown, label the regions containing values (T, D) that imply the critical point at $(0, 0)$ is a: nodal source, nodal sink, spiral source, spiral sink, center, and saddle point.



Eigenvalues λ are roots of char. eqn: $\det(A - \lambda I) =$

$$0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= ad - a\lambda - d\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

$$\lambda^2 - T\lambda + D$$

Roots via quadratic formula:

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

$\sqrt{T^2 - 4D} = 0 \Rightarrow D = T^2/4 \Rightarrow \lambda$ repeated root (borderline case)

$T^2 - 4D < 0 \Rightarrow \lambda$ complex $\Rightarrow$ spiral ; also $T^2 - 4D < 0 \iff \frac{T^2}{4} < D$

Otherwise $D \leq \frac{T^2}{4}$. Three cases: $T > 0 \Rightarrow T + \sqrt{T^2 - 4D} > 0 \Rightarrow$ at least one pos. λ

(*) continued top of page $T < 0 \Rightarrow T - \sqrt{T^2 - 4D} < 0 \Rightarrow$ at least one neg λ

The other λ has same sign if $|T| > \sqrt{T^2 - 4D} \iff T^2 > T^2 - 4D \iff 4D > 0 \iff D > 0$

So $D < 0 \Rightarrow$ opposite signs $\Rightarrow$ saddle point!

Otherwise $D > 0$, same sign, then $T > 0 \Rightarrow$ both $\lambda > 0 \Rightarrow$ source. else $D > 0$ & $T < 0 \Rightarrow$ sink

2. Find and classify all critical points of the almost linear system:

$$\begin{aligned} x' &= 2xy - 4x &= x(y-2)\cdot 2 \\ y' &= xy - 3y &= (x-3)y \end{aligned}$$

$x' = 0 \Rightarrow x = 0$ OR $y = 2$

$y' = 0 \Rightarrow x = 3$ OR $y = 0$

Compatible combinations: $(0,0)$ & $(3,2)$

Jacobian: $\begin{bmatrix} 2(y-2) & 2x \\ y & x-3 \end{bmatrix}$

at $(0,0)$: $\begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}$ $T = -7$, $D = 12$

$T^2 - 4D = 49 - 48 = 1 > 0$

Two real eigenval's

$D = 12 > 0 \Rightarrow$ same sign

$T < 0 \Rightarrow$ negative

all together $\Rightarrow$ nodal sink

or compute λ directly:

$\dfrac{-7 \pm \sqrt{1}}{2} = -4, -3$

$\begin{vmatrix} -4-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix} = (-4-\lambda)(-\lambda-3) = 0$

$(\lambda+4)(\lambda+3)$ roots $-4, -3$

at $(3,2)$: $\begin{bmatrix} 0 & 6 \\ 2 & 0 \end{bmatrix}$ $\begin{vmatrix} -\lambda & 6 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 12 = (\lambda - \sqrt{12})(\lambda + \sqrt{12})$

saddle point

3. Consider the differential equation

$$3y'' + xy' - 4y = 0$$

(a) Use power series to find the particular solution corresponding to initial conditions $y(0) = 1$ and $y'(0) = 0$.

(b) Find the degree-3 polynomial that, for x near zero, best approximates the general solution.

$$y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} (n-1)(n) c_n x^{n-2}$$

$$3\sum_{n=2}^{\infty} (n-1)(n) c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - 4\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} 3(n+1)(n+2) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 4 c_n x^n = 0$$

notice zero when $n=0$ so

$$= \sum_{n=0}^{\infty} n c_n x^n$$

Together: $$\sum_{n=0}^{\infty} \Big[\underbrace{3(n+1)(n+2)c_{n+2} + nc_n - 4c_n}_{=0 \text{ so}}\Big] x^n = 0$$

$$c_{n+2} = \frac{-nc_n + 4c_n}{3(n+1)(n+2)} = \frac{(n-4)c_n}{3(n+1)(n+2)}$$

$y(0) = c_0 = 1$

$y'(0) = c_1 = 0$

n	$c_{n+2} = \ldots$
0	$c_2 = \dfrac{4}{3\cdot1\cdot2} c_0 = \dfrac{4}{3\cdot1\cdot2}$
1	$c_3 = \dfrac{3}{3\cdot2\cdot3} c_1 = 0$
2	$c_4 = \dfrac{2}{3\cdot3\cdot4} c_2 = \dfrac{2}{3\cdot3\cdot4}\cdot\dfrac{4}{3\cdot1\cdot2}$
3	$c_5 = (\quad) c_3 = 0$

all odd c_n will be zero

n	
4	$c_6 = \dfrac{0\cdot c_4}{3\cdot4\cdot5} = 0$
6	$c_8 = \dfrac{2}{(\quad)} c_6 = \dfrac{2}{(\quad)}\cdot 0 = 0$
8	

all higher c_n will be zero!

$\boxed{\text{ⓐ soln} = 1 + \tfrac{2}{3}x^2 + \tfrac{1}{27}x^4}$

ⓑ $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 =$

$c_0 + c_1 x + \left(\frac{4}{6} c_0\right)x^2 + \left(\frac{3}{18} c_1\right)x^3$

$\boxed{= c_0\left[1 + \tfrac{2}{3}x^2\right] + c_1\left[x + \tfrac{x^3}{6}\right]}$ ⓑ

4. Use theorems about Laplace transforms to complete the table.

Function	Transform
$t\sin(kt)$	(a) $\frac{2ks}{s^2+k^2}$
$t\cos(kt)$	(b) $\frac{s^2-k^2}{(s^2+k^2)^2}$
(c) $\frac{1}{2k^2}\left[\frac{\sin kt}{k} - t\cos kt\right]$	$1/(s^2+k^2)^2$

Hint:

$$\frac{1}{2k^2}\left[\frac{1}{s^2+k^2} - \frac{s^2-k^2}{(s^2+k^2)^2}\right] = \frac{1}{(s^2+k^2)^2}$$

(a) $t\,f(t) \xrightarrow{\mathcal{L}} -F'(s)$

let $f(t) = \sin(kt)$ so $F(s) = \frac{k}{s^2+k^2}$

Then $-F'(s) = (-1)\cdot(-1)(s^2+k^2)^{-2}(2s)k = \frac{2ks}{s^2+k^2}$

(b) let $f(t) = \cos(kt)$ so $F(s) = \frac{s}{s^2+k^2}$

$-F'(s) = (-1)\cdot\left[(-1)\frac{s\cdot 2s}{(s^2+k^2)^2} + \frac{1}{(s^2+k^2)}\right] = \frac{2s^2}{(s^2+k^2)^2} - \frac{(s^2+k^2)}{(s^2+k^2)^2}$

$= \frac{s^2-k^2}{(s^2+k^2)^2}$

(c) $\frac{1}{(s^2+k^2)^2} = \frac{1}{2k^2}\left[\frac{1}{k}\frac{k\cdot 1}{s^2+k^2} - \frac{(s^2-k^2)}{(s^2+k^2)^2}\right]$

$\downarrow \mathcal{L}^{-1}$ $\quad\downarrow \mathcal{L}^{-1}$ $\quad\downarrow \mathcal{L}^{-1}$

$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^2}\left[\frac{1}{k}\sin(kt) - t\cos kt\right]$

5. Use Laplace transforms and the previous problem to solve the initial value problem:

$$x^{(4)}(t) + 18x''(t) + 81x(t) = 0$$

$$x(0) = 1; \quad x'(0) = 0; \quad x''(0) = -18; \quad x^{(3)}(0) = 0$$

$\mathcal{L}$

$$s^4 X(s) - s^3 \overbrace{x(0)}^{1} - s^2 \cancel{x'(0)} - s \overbrace{x''(0)}^{-18} - \overset{0}{\cancel{x'''(0)}}$$

$$+ 18\left[s^2 X(s) - s \underbrace{x(0)}_{1} - \cancel{x'(0)} \right] + 81 [X(s)] = 0$$

$$\Rightarrow \quad s^4 X(s) - s^3 \underbrace{+ 18s} + 18s^2 X(s) \underbrace{- 18s} + 81 X(s) = 0$$

$$\Rightarrow \quad X(s)\left[s^4 + 18s^2 + 81\right] - s^3 = 0$$

$$X(s) = \frac{s^3}{s^4 + 18s^2 + 81} = \frac{s^3}{(s^2+9)^2} = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{(s^2+9)^2}$$

form $(s^2 + k^2)^2$

Solve for A, B, C, D: $(As + B)(s^2 + 9) + Cs + D = s^3$ (clear denominators)

Substitute $s = 3i$: $0 + C(3i) + D = -27i \Rightarrow D = 0 \quad C = -9$

$(As + B)(s^2 + 9) - 9s = s^3$ solve for A, B:

$As^3 + A \cdot s \cdot 9 + Bs^2 + B \cdot 9 - 9s = s^3 \Rightarrow A = 1$ (both $As^3 = s^3$ & $A9s = 9s$)

$$X(s) = \frac{s}{s^2 + 9} + \frac{-9s}{(s^2+9)^2} = \frac{-9}{2 \cdot 3} \underbrace{\left(\frac{2 \cdot 3 \cdot s}{(s^2+3^2)^2} \right)}_{\mathcal{L}\{t \sin 3t\}}$$

$Bs^2 = 0$ & $9B = 0$

either $\Rightarrow B = 0$

$$x(t) = \cos(3t) + \frac{-9}{2 \cdot 3} t \sin 3t$$

$$= \cos(3t) - \frac{3}{2} t \sin 3t$$

6. Solve the initial value problem

$$x'' + \pi^2 x = \sum_{n=0}^{\infty} \delta(t-n); \quad x(0) = x'(0) = 0$$

and **sketch the solution curve** for $0 \le t \le 5$, showing numbers on each axis.

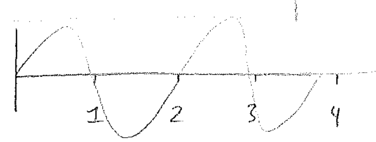
$$s^2 X(s) - s x(0) - x'(0) + \pi^2 X(s) = \sum_{n=0}^{\infty} e^{-ns}$$

$$X(s)\left[s^2 + \pi^2\right] = \sum_{n=0}^{\infty} e^{-ns} \Rightarrow X(s) = \sum_{n=0}^{\infty} \underbrace{e^{-ns}\left(\frac{1}{s^2+\pi^2}\right)}_{\frac{1}{\pi} e^{-ns}\left(\frac{\pi}{s^2+\pi^2}\right)}$$

$\Downarrow \mathcal{L}^{-1}$

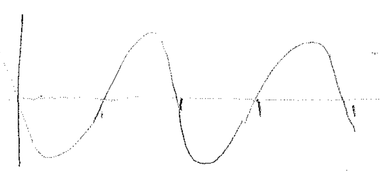
$$\boxed{x(t) = \sum_{n=0}^{\infty} \frac{1}{\pi} u(t-n) \overbrace{\sin(\pi(t-n))}^{\sin(\pi t - n\pi)}}$$

$n=0$



$\sin(\pi t - n\pi)$ same for all even n

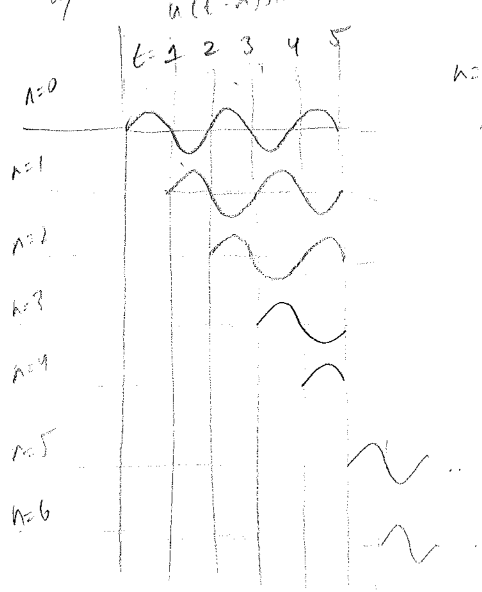
$n=1$



$\sin(\pi t - n\pi)$ same for all odd n

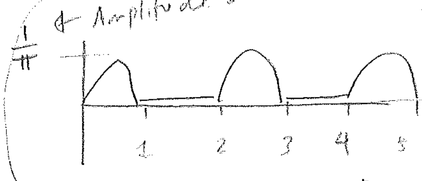
$n=2$

w/ step fcn: Graphing $u(t-n)\sin(\pi t - n\pi)$



+ Amplitude!

Total:



SKETCH

Note cancellations!

in general

$$x(t) = \begin{cases} \dfrac{\sin \pi t}{\pi} & n < t \le n+1, \ n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$