DUE: Start of recitation, 3/3/15 (T3) or 3/5/15 (T2)

1. Download Maple code[1] for implementing Euler's method: this code is available online at http://www.nsm.buffalo.edu/∼mangahas/Math306/Samplecode.html and also on UBlearns (under Course Documents).

   Modify the code in order to display output for parts (a) and (b) below, for $y' = t^2 + \frac{1}{y}$, starting at $y(0) = 1$, over the interval $0 \leq t \leq 2$. Be sure to **vary the range so that the output is displayed clearly.** Print all your output for parts (a) and (b) below.

   a. For both $h = 0.5$ and $h = 0.2$, display polygonal approximations for the solution curve.

   b. For both $h = 0.1$ and $h = 0.02$, display the output of Euler's method as a set of points in two ways: listed as the points (t_i, y_i) in *datalist*, and also graphed as points on the plane.

2. This problem is Application 3.1 from the Applications Manual for our textbook. The relevant pages are included in this PDF. Submit work for parts (a) and (c) on a separate sheet of paper, and attach printed Maple output for parts (b) and (c).

   a. Derive (by hand, not using Maple) Equations (3)–(5) on Application 3.1, page 69 (scanned page attached), which give particular solutions for the differential equation $y'' + 3y' + 2y = 0$, using the general solution shown in Equation (2) (you learn how to obtain Equation (2) in 3.1, see 3.1 Theorem 5).

   b. Following the Maple code on page 70, generate Figures 3.1.6 and 3.1.7 in our textbook (scanned page attached), which illustrate some of the solutions you found for $y'' + 3y' + 2y = 0$ in part (a).

   c. For the differential equation $y'' + 2y' + 2y = 0$ (Application 3.1 #5), construct both a family of different solution curves satisfying $y(0) = 1$ and a family of different solution curves satisfying the initial condition $y'(0) = 1$.

      That is, obtain a version of part (b) above, but for $y'' + 2y' + 2y = 0$. **Each family should have 8-10 solution curves.**

      To accomplish this, you need equations similar to Equations (2)–(5), but solving the new differential equation. The general solution is already given to you as $y(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$ (you learn how to obtain this solution in 3.3, see 3.3 Theorem 3). From the general solution you should be able to derive particular solutions corresponding to (3)–(5).

[1] Code taken from Shared Software for 306, UB Department of Mathematics

Chapter 3

Linear Equations of Higher Order

Application 3.1
Plotting Second-Order Solution Families

This application deals with the computer-plotting of solution families like those illustrated in Figs. 3.1.6 and Fig. 3.1.7 in the text. Show first that the general solution of the differential equation

$$y'' + 3y' + 2y = 0 \tag{1}$$

is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}. \tag{2}$$

Then show that the particular solution of Eq. (1) satisfying the initial conditions $y(0) = a,\ y'(0) = b$ is

$$y(x) = (2a + b)e^{-x} - (a + b)e^{-2x} \tag{3}$$

- For Fig. 3.1.6, substitution of $a = 1$ in (3) gives

$$y(x) = (b + 2)e^{-x} - (b + 1)e^{-2x}. \tag{4}$$

for the solution curve through the point $(0,1)$ with initial slope $y'(0) = b$.

- For Fig. 3.1.7, substitution of $b = 1$ in (3) gives

$$y(x) = (2a + 1)e^{-x} - (a + 1)e^{-2x}. \tag{5}$$

for the solution curve through the point $(0,a)$ with initial slope $y'(0) = 1$.

In the technology-specific sections following the problems below, we illustrate the use of computer systems like *Maple*, *Mathematica*, and MATLAB to plot simultaneously a family of solution curves like those defined by (4) or (5). Start by reproducing Figs. 3.1.6 and 3.1.7 in the text. Then, for each of the following differential equations, construct both a family of different solution curves satisfying $y(0)=1$ and a family of different solution curves satisfying the initial condition $y'(0)=1$.

1. $y''-y=0$
2. $y''-3y'+2=0$
3. $2y''+3y'+y=0$
4. $y''+y=0$ (with general solution $y(x)=c_1\cos x+c_2\sin x$)
5. $y''+2y'+2y=0$ (with general solution $y(x)=e^{-x}(c_1\cos x+c_2\sin x)$)

Using *Maple*

Using Eq. (4), the particular solution with $y(0)=1,\ y'(0)=b$ is defined by

```
partSoln := (b+2)*exp(-x)-(b+1)*exp(-2*x);
```

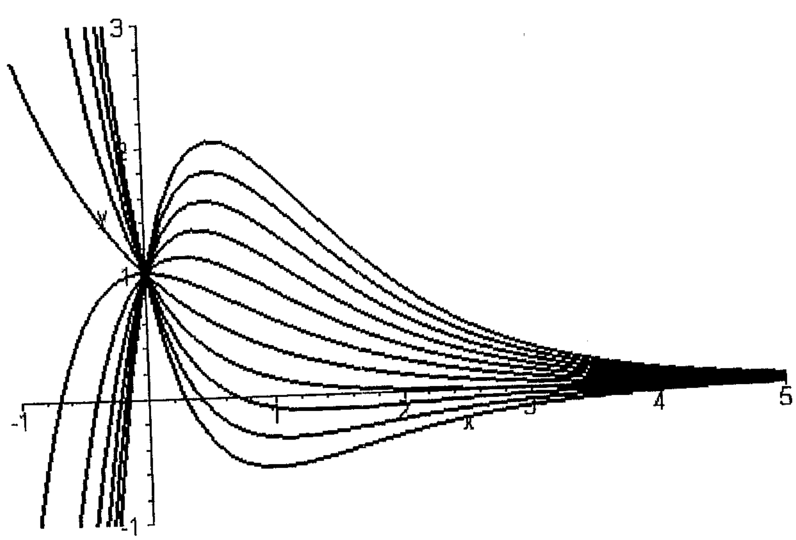
$$partSoln := (b+2)e^{(-x)}-(b+1)e^{-(2x)}$$

The set of such particular solutions with initial slopes $b=-5,-4,-3,\ldots,4,\ 5$ is then defined by

```
curves := {seq(partSoln, b =-5..5)}:
```

We plot these 11 curves simultaneously on the x-interval $(-1, 5)$ with the single command

```
plot(curves, x =-1..5, y =-1..3);
```



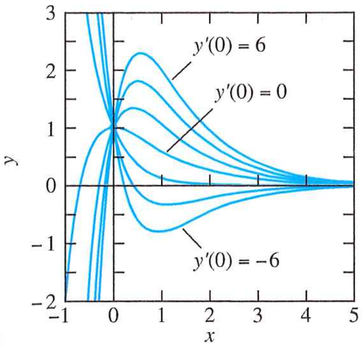


FIGURE 3.1.6. Solutions of $y'' + 3y' + 2y = 0$ with the same initial value $y(0) = 1$ but different initial slopes.

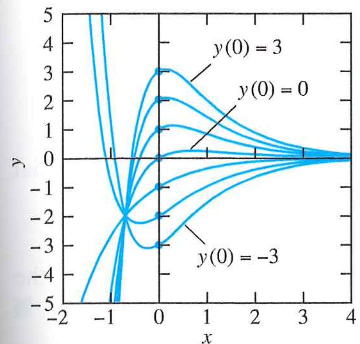


FIGURE 3.1.7. Solutions of $y'' + 3y' + 2y = 0$ with the same initial slope $y'(0) = 1$ but different initial values.

THEOREM 2 Existence and Uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{8}$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1. \tag{11}$$

Remark 1: Equation (8) and the conditions in (11) constitute a second-order linear **initial value problem.** Theorem 2 tells us that any such initial value problem has a unique solution on the *whole* interval I where the coefficient functions in (8) are continuous. Recall from Section 1.3 that a *nonlinear* differential equation generally has a unique solution on only a smaller interval.

Remark 2: Whereas a *first-order* differential equation $dy/dx = F(x, y)$ generally admits only a single solution curve $y = y(x)$ passing through a given initial point (a, b), Theorem 2 implies that the *second-order* equation in (8) has infinitely many solution curves passing through the point (a, b_0)—namely, one for each (real number) value of the initial slope $y'(a) = b_1$. That is, instead of there being only one line through (a, b_0) tangent to a solution curve, *every* nonvertical straight line through (a, b_0) is tangent to some solution curve of Eq. (8). Figure 3.1.6 shows a number of solution curves of the equation $y'' + 3y' + 2y = 0$ all having the same initial value $y(0) = 1$, while Fig. 3.1.7 shows a number of solution curves all having the same initial slope $y'(0) = 1$. The application at the end of this section suggests how to construct such families of solution curves for a given homogeneous second-order linear differential equation. ■

Example 1 *Continued*

We saw in the first part of Example 1 that $y(x) = 3\cos x - 2\sin x$ is a solution (on the entire real line) of $y'' + y = 0$. It has the initial values $y(0) = 3$, $y'(0) = -2$. Theorem 2 tells us that this is the *only* solution with these initial values. More generally, the solution

$$y(x) = b_0 \cos x + b_1 \sin x$$

satisfies the *arbitrary* initial conditions $y(0) = b_0$, $y'(0) = b_1$; this illustrates the *existence* of such a solution, also as guaranteed by Theorem 2. ■

Example 1 suggests how, given a *homogeneous* second-order linear equation, we might actually find the solution $y(x)$ whose existence is assured by Theorem 2. First, we find two "essentially different" solutions y_1 and y_2; second, we attempt to impose on the general solution

$$y = c_1 y_1 + c_2 y_2 \tag{12}$$

the initial conditions $y(a) = b_0$, $y'(a) = b_1$. That is, we attempt to solve the simultaneous equations

$$\begin{aligned} c_1 y_1(a) + c_2 y_2(a) &= b_0, \\ c_1 y_1'(a) + c_2 y_2'(a) &= b_1 \end{aligned} \tag{13}$$

for the coefficients c_1 and c_2.

Further Investigation

Have you noticed anything common to all three of the plots of solution families shown above (as well as several of the plots you made for differential equations **1-5** above)? In this section we will examine this phenomenon more closely—and try to identify the circumstances under which it occurs.

First, review the plots you made for differential equations **1-5** for the case where $y'(0)$ is held fixed at 1, and answer the following question:

For which of these solution families does it seem that *all the solution curves meet at a common point in the plane*?

(Be prepared to change the "viewing rectangle" if you think something significant might be going on off-screen.)

To see what makes this phenomenon occur with one differential equation and not another, list the *characteristic roots* for each of the equations **1-5** listed at the beginning. Can you make a conjecture about when the phenomenon occurs?

To test your conjecture (or to help you form one), plot solution families like the ones above for the following differential equations (of course with the same initial conditions $y'(0)=1$, $y(0)=a$, and with a ranging from -5 to 5):

- $y''+y'-2y=0$
- $y''-y=0$
- $y''-y'=0$

By now you are probably convinced that there is a theorem in here somewhere, and indeed there is! Can you prove that (as one example) the phenomenon we have observed always occurs when the characteristic roots are *real, distinct, and of the same sign*? As a bonus, your proof should also predict for you—in terms of the characteristic roots—the *value* of x at which the solution curves meet; compare this prediction with the graphs you found above. (*Hint*: Call the roots r_1 and r_2 and write the solution y of the initial value problem explicitly in terms of $a=y(0)$. There is a point at which the solution curves meet if and only if there is an x at which y does not depend upon a, that is, at which $\frac{\partial y}{\partial a}=0$.)