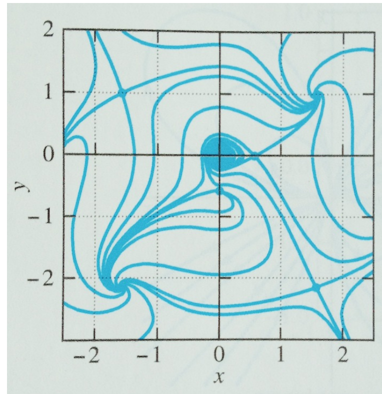
DUE: Start of recitation, 4/21/15 (T3) or 4/23/15 (T2).

For this assignment, please download sample Maple code[1] available on UBlearns or at http://www.nsm.buffalo.edu/∼mangahas/Math306/Samplecode.html.

This assignment involves the almost-linear system with phase portrait shown:

$$x'(t) = -y\cos(x + y - 1)$$
$$y'(t) = x\cos(x - y + 1)$$



1. Complete the sample code by including the Maple commands to determine eigenvalues and eigenvectors of the matrix corresponding to the critical point at (1.5708, -2.1416).

   Next, modify the **phaseportrait** command to display the local phase portrait about this critical point. Finally, classify this critical point: stable/unstable, asymptotically stable/unstable, and node/spiral/center/saddle point.

2. Modify the entire procedure in order to locate two additional critical points. (If Maple returns an error message regarding the parameter values, simply locate another point.) For each of these points, alter the code to display the corresponding matrix, eigenvalues, eigenvectors, and local phase portrait. Classify these points. (You should, of course, choose 2 points *other than* the one used for part (1) above!)

3. Modify the final block of sample code (following **restart:**) to display a phase portrait **as similar as possible** to the one above. This will not only require altering the system of differential equations, but also experimenting with different ranges for t and adding **numerous** initial conditions. It may be necessary to include 5-8 sets of ICs *in each quadrant* to obtain a fairly complete phase portrait.

Note: in the following pages I've attached scans of the textbook's Applications Manual, which includes more details about the system in this problem, and how to use Maple to analyze it. This is just for your reference and interest! You only need this first sheet to complete the assignment, but you are welcome to use the scanned pages to help your understanding.

[1] Code taken from Shared Software for 306, UB Department of Mathematics

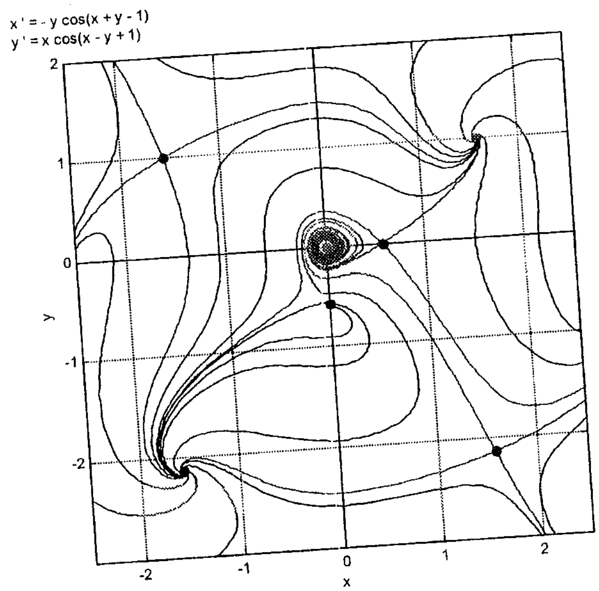
Application 6.2
Phase Plane Portraits of Almost Linear Systems

Interesting and complicated phase portraits often result from simple nonlinear perturbations of linear systems. For instance, the figure below shows a phase plane portrait for the almost linear system

$$\begin{aligned}\frac{dx}{dt} &= -y\cos(x+y-1) \\ \frac{dy}{dt} &= x\cos(x-y+1).\end{aligned} \tag{1}$$

Among the seven critical points marked with dots, we see

- Apparent spiral points in the first and third quadrants of the xy-plane;
- Apparent saddle points in the second and fourth quadrants, plus another one on the positive x-axis;
- A critical point of undetermined character on the negative y-axis; and
- An apparently "very weak" spiral point at the origin -- meaning one that is approached very slowly as t increases or decreases (according as it is a sink or a source).



Some ODE software systems can automatically locate and classify critical points. For instance, Fig. 6.2.22 in the text shows a screen produced by John Polking's MATLAB **pplane** program (cited in the Section 6.1 application). It indicates that the fourth-quadrant critical point in the figure above has approximate coordinates (1.5708, –2.1416), and that the coefficient matrix of the associated linear system has the positive eigenvalue $\lambda_1 \approx 2.8949$ and the negative eigenvalue $\lambda_2 \approx -2.3241$. It therefore follows from Theorem 2 in Section 6.2 that this critical point is, indeed, a saddle point of the almost linear system
in (1).

With a general computer algebra system, you may have to do a bit of work yourself — or tell the computer precisely what to do — in order to find and classify a critical point. In the sections below, we illustrate this procedure using *Maple*, *Mathematica*, and MATLAB. Once the critical-point coordinates $a = 1.5708$, $b = -2.1416$ indicated above have been found, the substitution $x = u + a, \ \ y = v + b$ yields the translated system

$$\begin{aligned} \frac{du}{dt} &= (2.1416 - v)\cos(1.5708 - u - v) \ = \ f(u,v) \\ \frac{dv}{dt} &= (1.5708 + u)\cos(4.7124 + u - v) \ = \ g(u,v). \end{aligned} \tag{2}$$

If we substitute $u = v = 0$ in the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{bmatrix} \tag{3}$$

we get the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 2.1416 & 2.1416 \\ 1.5708 & -1.5708 \end{bmatrix} \tag{4}$$

of the linear system corresponding to the almost linear system in (2).

Alternatively, one can circumvent the translated system in (2) by looking at the Taylor expansions

$$\begin{aligned} f(x,y) &= D_x f(a,b)(x-a) + D_y f(a,b)(y-b) + \cdots \\ g(x,y) &= D_x g(a,b)(x-a) + D_y g(a,b)(y-b) + \cdots \end{aligned} \tag{5}$$

of the right-hand side functions in the original system (1), and retaining only the linear terms in this expansion. We see from (5) that

$$\mathbf{A} = \begin{bmatrix} D_x f(a,b) & D_y f(a,b) \\ D_x g(a,b) & D_y g(a,b) \end{bmatrix} \tag{6}$$

is the coefficient matrix of the linearization of the system (1) that results when we substitute $u = x - a,\ v = y - b$ and retain only the terms that are linear in u and v.

In any event, we can then use our computer algebra system to find the eigenvalues $\lambda_1 \approx 2.8949$ and $\lambda_2 \approx -2.3241$ of the matrix **A**, thereby verifying that the critical point $(1.5708, -2.1416)$ of (1) is, indeed, a saddle point.

Use a computer algebra system to find and classify similarly the other critical points of (1) indicated in the figure above. Then investigate similarly an almost linear system of your own construction. One convenient way to construct such a system is to start with a linear or almost linear system and insert sine or cosine factors resembling the ones in (1). For instance:

1. $x' = x \cos y$, $\quad y' = y \sin x$

2. $x' = -y + y^2 \cos y$, $\quad y' = -x - x^2 \sin x$

3. $x' = y \cos(2x + y)$, $\quad y' = -x \sin(x - 3y)$

4. $x' = -x - y^2 \cos(x + y)$, $\quad y' = y + x^2 \cos(x - y)$

Using *Maple*

After we enter the right-hand side functions in (1),

```
f := -y*cos(x+y-1):
g :=  x*cos(x-y-1):
```

we can proceed to solve numerically for a solution near $(1.5, -2)$:

```
soln :=
fsolve({f=0,g=0}, {x,y}, x=1..2, y=-3..-1);
```

$$\{x = 1.570796327, y = -2.141592654\}$$

Thus our critical point (a, b) is given approximately by

```
a := rhs(soln[1]);
b := rhs(soln[2]);
```

a := 1.570796327
b := -2.141592654

To classify this critical point, we proceed to calculate first the partial derivatives

```
fx := evalf(subs(x=a,y=b,diff(f,x))):
fy := evalf(subs(x=a,y=b,diff(f,y))):
gx := evalf(subs(x=a,y=b,diff(g,x))):
gy := evalf(subs(x=a,y=b,diff(g,y))):
```

evaluated at (a, b), and then the Jacobian matrix in (6):

```
with(linalg):
A := matrix(2,2, [fx,fy,gx,gy]);
```

Finally, its eigenvalues are given by

```
eigenvals(A);
```

2.894893108, -2.324096781

Thus the eigenvalues $\lambda_1 \approx 2.8949$ and $\lambda_2 \approx -2.3241$ are real with opposite signs, so the critical point (1.5708, –2.1416) is, indeed, a saddle point of the system in (1).

Using *Mathematica*

After we enter the right-hand side functions in (1),

```
f = -y*Cos[x+y-1];
g =  x*Cos[x-y+1];
```

we can proceed to solve numerically for a solution near (1.5, –2):

```
soln =
FindRoot[{f == 0, g == 0}, {x,1.5}, {y,-2}]

{x -> 1.5708, y -> -2.14159}
```

Thus our critical point (a, b) is given approximately by

```
a = x /. soln
b = y /. soln

1.5708
-2.14159
```

To classify this critical point, we proceed to set up the Jacobian matrix in (6),