Math 306 Section T

"Practice" Final Exam

May 7, 2015

Name: ______________________________

This is a practice exam. For the real exam:

- Nothing on your desk except writing instruments and UB ID card.
- No electronics! I will keep track of time on the board.
- Like the practice exam, the final has six questions on content from Chapters 6-8. In addition, there are two "cumulative" questions.
- Questions on the real exam are **not** guaranteed to be easier/harder than the practice exam; this is just for review and practice.
- Give yourself 80 minutes for the first six questions. The two "cumulative" questions do not require a time limit; shorter versions will appear on the exam.

1. Consider the linear system:

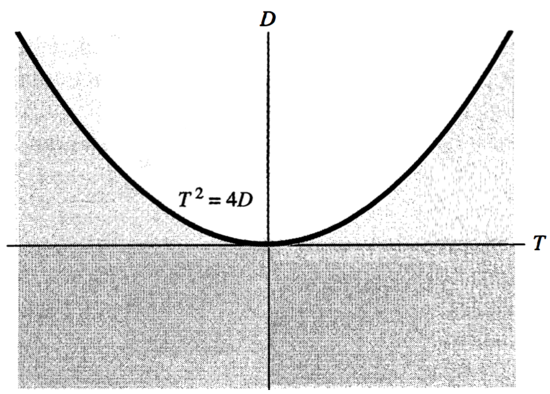
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Let $T = a + d$ and $D = ad - bc$. For this problem, you will show how the values of T and D determine the type of critical point at $(0, 0)$.

(a) Show that the eigenvalues for this system are

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

(b) On the (T, D) plane shown, label the regions containing values (T, D) that imply the critical point at $(0, 0)$ is a: nodal source, nodal sink, spiral source, spiral sink, center, and saddle point.



2. Find and classify all critical points of the almost linear system:

$$\begin{aligned} x' &= 2xy - 4x \\ y' &= xy - 3y \end{aligned}$$

3. Consider the differential equation

$$3y'' + xy' - 4y = 0$$

(a) Use power series to find the particular solution corresponding to initial conditions $y(0) = 1$ and $y'(0) = 0$.

(b) Find the degree-3 polynomial that, for x near zero, best approximates the general solution.

4. Use theorems about Laplace transforms to complete the table.

Function	Transform
$t\sin(kt)$	(a)
$t\cos(kt)$	(b)
(c)	$1/(s^2+k^2)^2$

Hint:

$$\frac{1}{2k^2}\left[\frac{1}{s^2+k^2} - \frac{s^2-k^2}{(s^2+k^2)^2}\right] = \frac{1}{(s^2+k^2)^2}$$

5. Use Laplace transforms and the previous problem to solve the initial value problem:

$$x^{(4)}(t) + 18x''(t) + 81x(t) = 0$$

$$x(0) = 1; \quad x'(0) = 0; \quad x''(0) = -18; \quad x^{(3)}(0) = 0$$

6. Solve the initial value problem

$$x'' + \pi^2 x = \sum_{n=0}^{\infty} \delta(t-n); \quad x(0) = x'(0) = 0$$

and **sketch the solution curve** for $0 \le t \le 5$, showing numbers on each axis.

Cumulative Part I. The final will have a shortened version of this question, so if you can solve all of it you will be prepared for the corresponding question on the final. Remember, you have to justify answers and show all work on the exam, so submitting memorized answers will not be sufficient!

Investigate the following mass-spring-dashpot system with damping coefficient $0 \le c < 2$:

$$x''(t) + cx'(t) + x(t) = f(t); \quad x(0) = 0; \quad x'(0) = 1$$

(a) First assume $f(t) = 0$, so no forcing. Find the solution $x_c(t)$, which depends on the parameter c. Let $x_0(t)$ be the solution when $c = 0$ (no damping), and let $x_1(t)$ be the solution when $c = 1$ (some underdamping). What is the difference in the behavior of the two solutions, as $t \to \infty$?

Next investigate the effect of forcing:

(b) Show that

$$x''(t) + x(t) = x_1(t); \quad x(0) = x'(0) = 0$$

and

$$x''(t) + x'(t) + x(t) = x_0(t); \quad x(0) = x'(0) = 0$$

have the same solution. (This can be done without solving either system). That is, forcing the $c = 0$ system with the solution of the $c = 1$ system has the same result as switching the roles of the two systems, assuming neutral initial conditions.

(c) Find the common solution to the two initial value problems in (b). Notice the solution can be divided into a steady periodic part, and an exponentially decaying part.

(d) Find the solution to the system:

$$x''(t) + cx'(t) + x(t) = x_c(t); \quad x(0) = x'(0) = 0$$

This corresponds to applying a force that mimics the system's behavior under our first (non-neutral) set of initial conditions.

(e) What is the qualitative difference between the solution to (d) when $c = 0$ compared to when $c = 1$? Compare coefficients of the trig terms, and their behavior as $t \to \infty$.

(f) Lastly, consider the initial value problem corresponding to a constant force $F_0 \neq 0$:

$$x''(t) + cx'(t) + x(t) = F_0; \quad x(0) = x'(0) = 0$$

For this last part also assume $c \neq 0$. Show that the solution tends to a fixed nonzero value as $t \to \infty$, that is, the spring settles into a stretched position. Find this value; that is, compute $\lim_{t \to \infty} x(t)$. Note that the limit may be determined without explicitly computing the solution $x(t)$, but you are free to use any method you prefer.

Cumulative Part II. The final will have a shortened version of this question, so if you can solve all of it you will be prepared for the corresponding question on the final. Remember, you have to justify answers and show all work on the exam, so submitting memorized answers will not be sufficient!

The system of differential equations below models a predator-prey system: $x(t)$ counts the population of prey (in hundreds, perhaps), and $y(t)$ counts the population of predators. Without predators, the prey population would grow logistically; without prey, the predator population would decline exponentially.

$$\begin{aligned} x' &= (M - x - y)x \\ y' &= (-2 + x)y \end{aligned}$$

For all parts below, **assume the parameter M is positive (and therefore nonzero).**

(a) Find all critical points.

(b) For which values of M are there exactly three critical points?

(c) For the values of M found for part (b), determine whether *each* critical point is stable or unstable. Make a table, because stability depends on the value of M.

(d) For which values of M does there exist an equilibrium solution (critical point) (x_0, y_0) where x_0 and y_0 are **both** positive (and therefore nonzero)?

(e) For the values of M found for part (d), classify the critical point (x_0, y_0) (i.e. is it a saddle point, center, or spiral/nodal source/sink). Again, the answer depends on M.

(f) Find explicitly the solution $x(t), y(t)$ for initial conditions $x(0) = 0$ and $y(0) = y_0 > 0$.

(g) Find explicitly the solution $x(t), y(t)$ for initial conditions $x(0) = x_0 > 0$ and $y(0) = 0$.

* Hint: parts (f) and (g) are made easier by the fact that, with these initial conditions, the pair of equations can be *decoupled*. That is, for each case, justify (mathematically! not just using the real-life interpretation of the model) that $x'(t)$ does not depend on y, and $y'(t)$ does not depend on x. Then solve each differential equation independently.

** For certain values of M, the population $y(t)$ tends to zero from any positive initial conditions. How would you interpret this in terms of the natural setting being modeled?

	Function	Transform		Function	Transform
	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$		1	$1/s$
*	$f'(t)$	$sF(s) - f(0)$		t^n	$n!/(s^{n+1})$
**	$e^{at}f(t)$	$F(s-a)$		e^{at}	$1/(s-a)$
**	$u(t-a)f(t-a)$	$e^{-as}F(s)$		$e^{at}coskt$	$\frac{s-a}{(s-a)^2+k^2}$
**	$tf(t)$	$-F'(s)$		$e^{at}sinkt$	$\frac{k}{(s-a)^2+k^2}$
**	$f(t)/t$	$\int_s^\infty F(\sigma)d\sigma$		$u(t-a)$	$\frac{e^{-as}}{s}$
**	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$		$\delta(t-a)$	e^{-as}
**	$f(t) = f(t+p)$ for all t	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)dt$			

* This theorem requires $f(t)$ be continuous, piecewise smooth, and of exponential order.
** These theorems require $f(t)$ be piecewise continuous and of exponential order.