# A Recipe for “Short-word” Pseudo-Anosovs


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# Definitions

Mapping class group of $S$:

$$\mathrm{Mod}(S) = \{f : S \to S | f \text{ o.p. diffeo. } \}/\{f \sim id\}$$

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Nielsen-Thurston classification:

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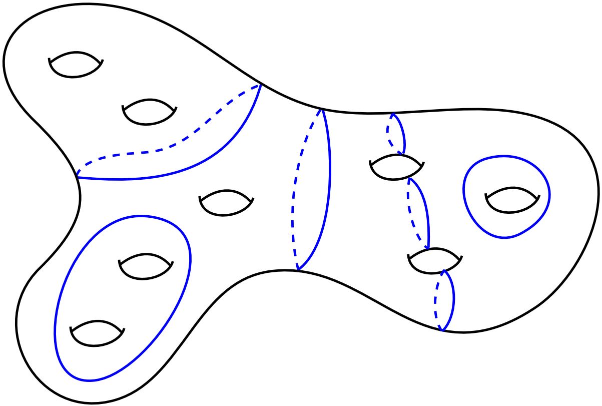
- finite-order
- pseudo-Anosov
- kind of pseudo-Anosov

# Reducible mapping classes



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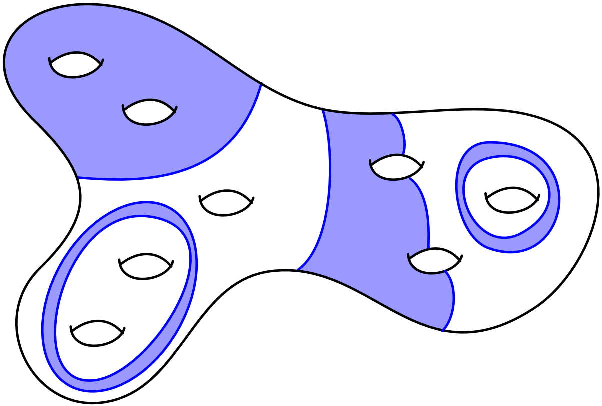

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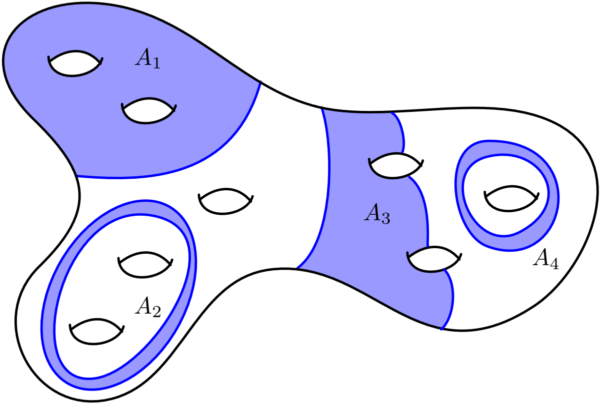
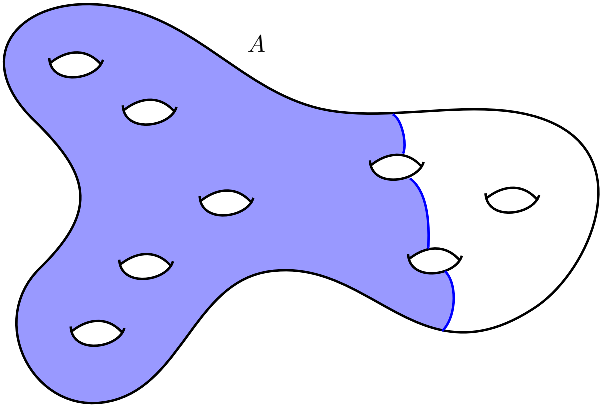



$$\mathcal{A}(f) = \bigcup_i A_i$$

# Naive question

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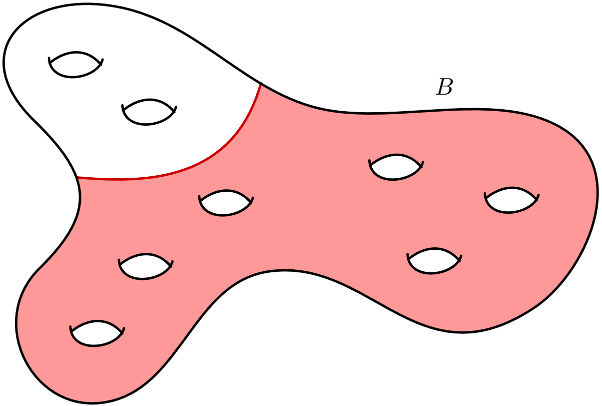


If $a$ is pA on $A$, and $b$ is pA on $B$,

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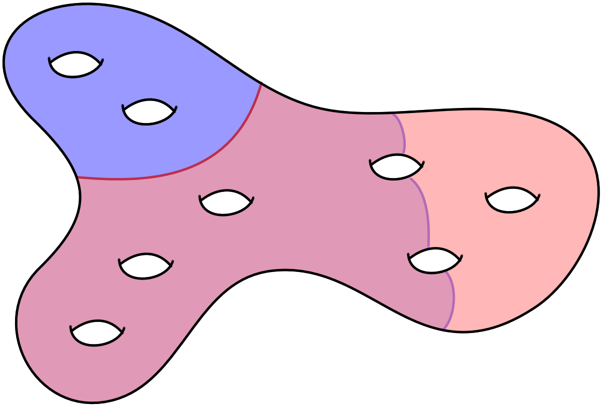

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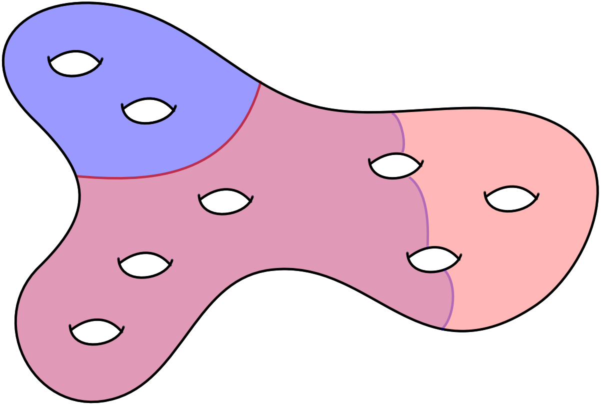

If $a$ is pA on $A$, and $b$ is pA on $B$,
is $ab$ pseudo-Anosov on the whole surface?
Not always, but ...

# "Short-word" question

**Question (Fujiwara)**

Is there an upper bound, depending only on $S$, for distance from the identity to the nearest pseudo-Anosov, in any Cayley graph of any subgroup of $\mathrm{Mod}(S)$?

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**Theorem (Yes)**

*There exists a constant $K = K(S)$ with the property that, for any subset $\Sigma \subset \mathrm{Mod}(S)$, there exists $f \in \langle \Sigma \rangle$ such that $|f|_\Sigma < K$ and $f$ is pseudo-Anosov.*

# "Short-word" question

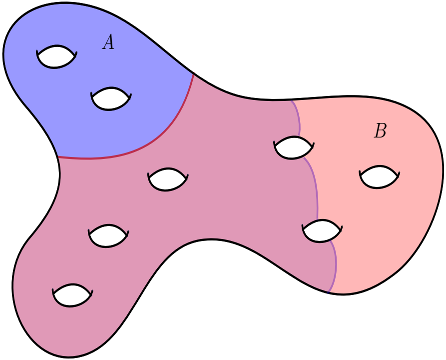
## Question (Fujiwara)

Is there an upper bound, depending only on $S$, for distance from the identity to the nearest pseudo-Anosov, in any Cayley graph of any subgroup of $\mathrm{Mod}(S)$?

## Theorem (Yes, and more)

*There exists a constant $K = K(S)$ with the property that, for any subset $\Sigma \subset \mathrm{Mod}(S)$, there exists $f \in \langle \Sigma \rangle$ such that $|f|_\Sigma < K$ and $\mathcal{A}(g) \subset \mathcal{A}(f)$ for all $g \in \langle \Sigma \rangle$.*

# Special-case construction



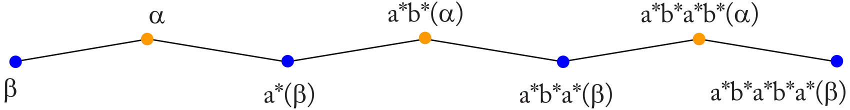

**Proposition**

There exists $Q = Q(S)$ s.t. if pure reducibles $a$ and $b$ are pA on domains $A$ and $B$ resp., and $A \cup B$ fills $S$, then for any $n, m \geq Q$,

- $\langle a^n, b^m \rangle \cong \mathbb{F}_2$
- Elements of $\langle a^n, b^m \rangle$ are pA except those conjugate to powers of $a$ or $b$.
- F.g. all-pA sbgps of $\langle a^n, b^m \rangle$ are convex cocompact.

# Proposition proof

In the curve complex $\mathcal{C}(S)$ of $S$:




Show: if $w$ not conjugate to $a^k$ or $b^k$, $\langle w \rangle$ q.i.-embeds in $\mathcal{C}(S)$.

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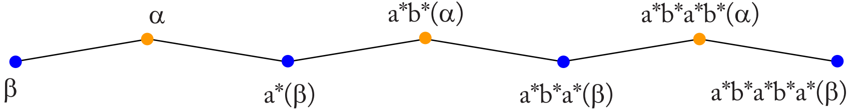


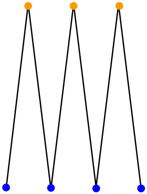

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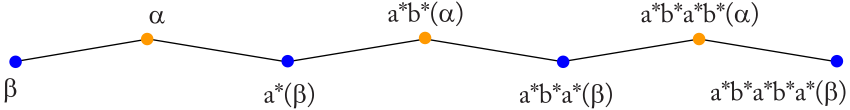


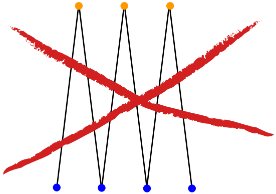

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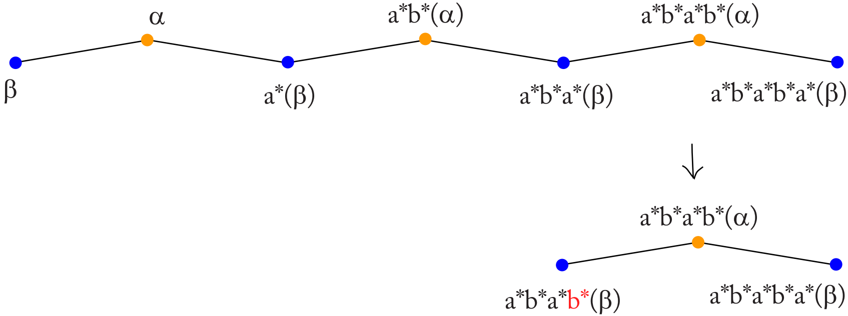


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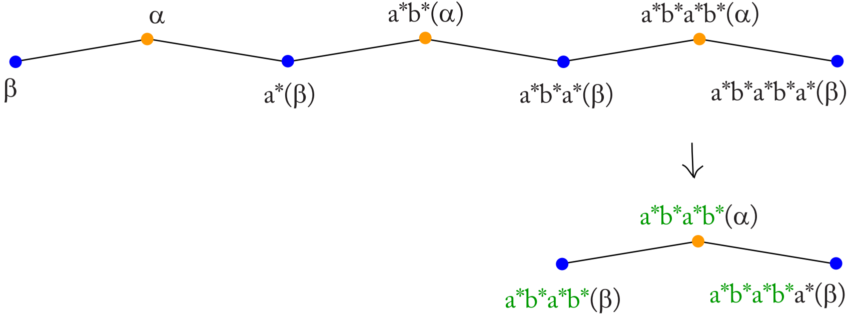


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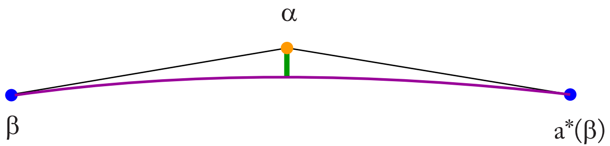


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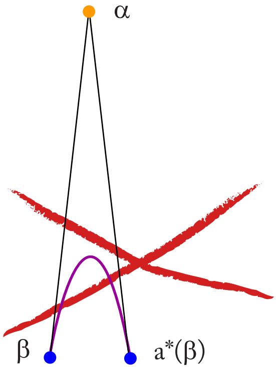


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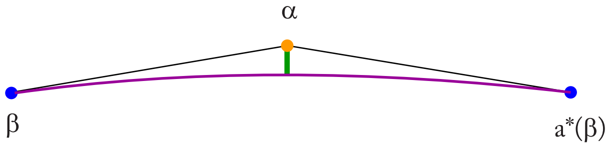


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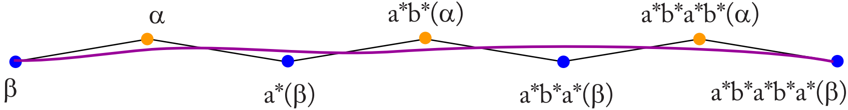


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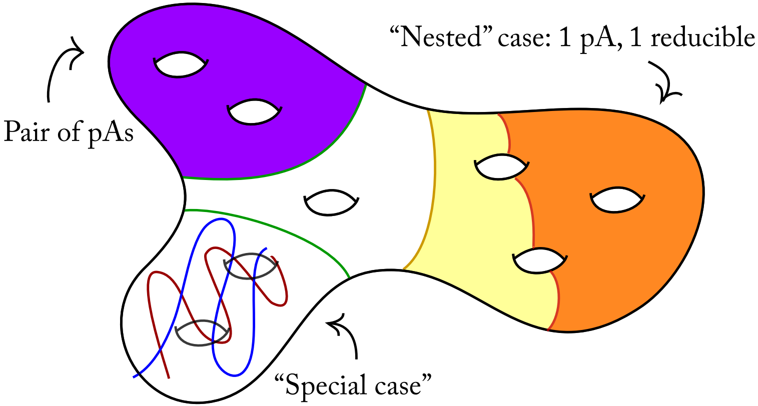
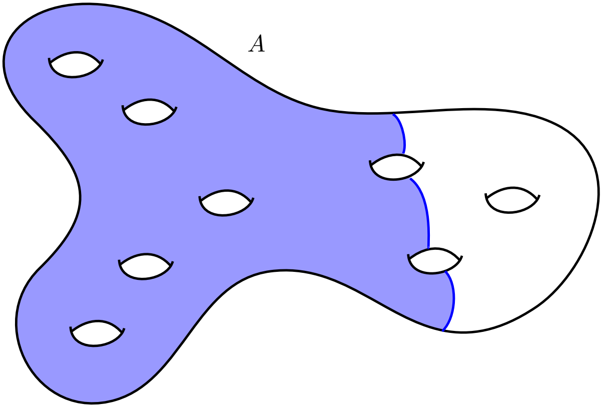
# Theorem proof (idea)

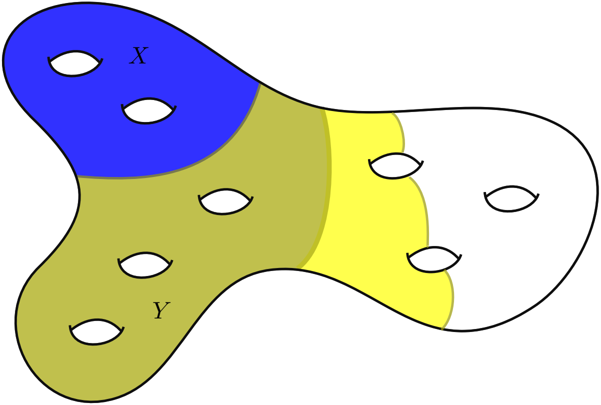



$a_1^P b_1^P a_1^{-P} \cdot b_1^P$ is pA on largest possible subsurface




$$a = x^k y^k$$

# Extra: counterexample to naive question

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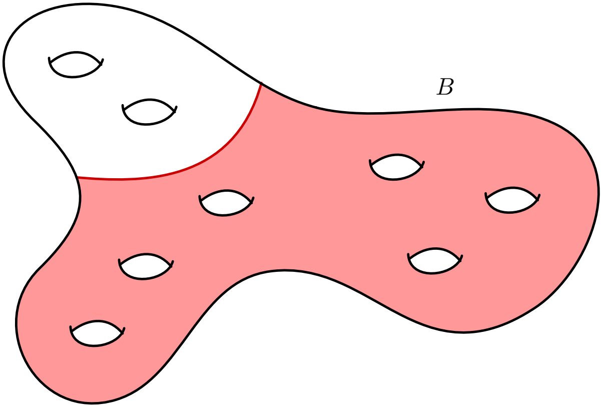


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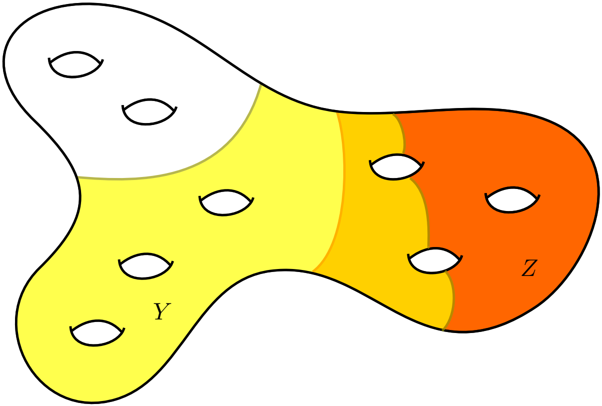


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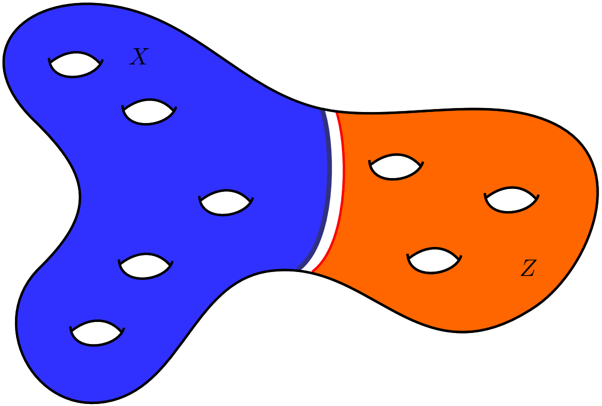


$$ab = x^k y^k \cdot y^{-k} z^k = x^k z^k$$