A Recipe for “Short-word” Pseudo-Anosovs

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Mapping class group of $S$:

$$\text{Mod}(S) = \{ f : S \to S \mid f \text{ o.p. diffeo. } \} / \{ f \sim \text{id} \}$$
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Nielsen-Thurston classification:

$f \in \text{Mod}(S)$ is either
Definitions

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- finite-order
- pseudo-Anosov
Definitions

Mapping class group of $S$:
\[ \text{Mod}(S) = \{ f : S \to S \mid f \text{ o.p. diffeo. } \} / \{ f \sim id \} \]

Nielsen-Thurston classification:
\[ f \in \text{Mod}(S) \text{ is either} \]
- finite-order
- pseudo-Anosov
- kind of pseudo-Anosov
Reducible mapping classes
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\[ \mathcal{A}(f) = \bigcup_i A_i \]
If \( a \) is pA on \( A \),
If $a$ is pA on $A$, and $b$ is pA on $B$, 
If $a$ is pA on $A$, and $b$ is pA on $B$, is $ab$ pseudo-Anosov on the whole surface?
If $a$ is pA on $A$, and $b$ is pA on $B$, is $ab$ pseudo-Anosov on the whole surface? Not always, but ...
“Short-word” question

Question (Fujiwara)

Is there an upper bound, depending only on $S$, for distance from the identity to the nearest pseudo-Anosov, in any Cayley graph of any subgroup of $\text{Mod}(S)$?
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**Question (Fujiwara)**

Is there an upper bound, depending only on $S$, for distance from the identity to the nearest pseudo-Anosov, in any Cayley graph of any subgroup of $\text{Mod}(S)$?

**Theorem (Yes)**

There exists a constant $K = K(S)$ with the property that, for any subset $\Sigma \subset \text{Mod}(S)$, there exists $f \in \langle \Sigma \rangle$ such that $|f|_{\Sigma} < K$ and $f$ is pseudo-Anosov.
“Short-word” question

Question (Fujiwara)
Is there an upper bound, depending only on $S$, for distance from the identity to the nearest pseudo-Anosov, in any Cayley graph of any subgroup of $\text{Mod}(S)$?

Theorem (Yes, and more)
There exists a constant $K = K(S)$ with the property that, for any subset $\Sigma \subset \text{Mod}(S)$, there exists $f \in \langle \Sigma \rangle$ such that $|f|_\Sigma < K$ and $A(g) \subset A(f)$ for all $g \in \langle \Sigma \rangle$. 
Proposition

There exists \( Q = Q(S) \) s.t. if pure reducibles \( a \) and \( b \) are pA on domains \( A \) and \( B \) resp., and \( A \cup B \) fills \( S \), then for any \( n, m \geq Q \),

- \( \langle a^n, b^m \rangle \cong \mathbb{F}_2 \)
- Elements of \( \langle a^n, b^m \rangle \) are pA except those conjugate to powers of \( a \) or \( b \).
- F.g. all-pA sbgps of \( \langle a^n, b^m \rangle \) are convex cocompact.
Proposition proof

In the curve complex $C(S)$ of $S$:

Show: if $w$ not conjugate to $a^k$ or $b^k$, $\langle w \rangle$ q.i.-embeds in $C(S)$. 
Proposition proof

In the curve complex $\mathcal{C}(S)$ of $S$:

$$a \ast b \ast a \ast b \ast a \ast (\beta) \quad a \ast b \ast a \ast b \ast (\alpha) \quad a \ast b \ast a \ast (\beta) \quad a \ast b \ast a \ast b \ast a \ast (\beta)$$

Show: if $w$ not conjugate to $a^k$ or $b^k$, $\langle w \rangle$ q.i.-embeds in $\mathcal{C}(S)$. 
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Show: if $w$ not conjugate to $a^k$ or $b^k$, $\langle w \rangle$ q.i.-embeds in $\mathcal{C}(S)$. Use Masur-Minsky theorems.
Proposition proof

In the curve complex $\mathcal{C}(S)$ of $S$:

Show: if $w$ not conjugate to $a^k$ or $b^k$, $\langle w \rangle$ q.i.-embeds in $\mathcal{C}(S)$. Use Masur-Minsky theorems.
Theorem proof (idea)

Pair of pAs

“Special case”

“Nested” case: 1 pA, 1 reducible

\[ a_1^P b_1^P a_1^{-P} \cdot b_1^P \text{ is pA on largest possible subsurface} \]
Extra: counterexample to naive question

\[ a = x^k y^k \]
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\[ a = x^k y^k \]
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\[ b = y^{-k} z^k \]
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\[ ab = x^k y^k \cdot y^{-k} z^k = x^k z^k \]