Convex Cocompactness in Mod(S) via Quasiconvexity in RAAGs

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Thms (Farb, Mosher; Hamenstädt; Kent, Leininger)

For finitely generated $G < \text{Mod}(S)$, TFAE:

- $G$ acts cocompactly on its “weak hull”, is $\delta$-hyperbolic, . . .
- Orbits of $G$ are quasiconvex in $\text{Teich}(S)$
- Orbit maps of $G$ into $\mathcal{C}(S)$ are quasi-isometric embeddings.
Convex cocompactness in mapping class groups

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\[
\begin{array}{cccccc}
1 & \rightarrow & \pi_1(S) & \rightarrow & E_G & \rightarrow & G & \rightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \pi_1(S) & \rightarrow & \text{Mod}(\hat{S}) & \rightarrow & \text{Mod}(S) & \rightarrow & 1
\end{array}
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\end{align*}
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Thms (Farb-Mosher, Hamenstädt)

$E_G$ is word hyperbolic if and only if $G$ is convex cocompact.

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Which subgroups of $\text{Mod}(S)$ are convex cocompact?

purely pseudo-Anosov subgroups

Examples of convex cocompact?

Q: Is \{ convex cocompact \} same as \{ f.g. all-pA \}?
Which subgroups of \( \text{Mod}(S) \) are convex cocompact?

purely pseudo-Anosov subgroups

(virtually) free

examples

convex cocompact

Q:

Is \{ convex cocompact \} same as \{ f.g. all-pA (v. free) \}?
RAAGs in mapping class groups

Definition

\[ A_\Gamma = \langle v_i \mid \text{vertices of } \Gamma \mid [v_i, v_j] = id \text{ iff } (v_i, v_j) \text{ is an edge of } \Gamma \rangle \]

Thms (Koberda, Clay-Leininger-M, Crisp-Paris/-Weiss/-Farb)

Many ways to embed \( A_\Gamma \) in some \( \text{Mod}(S) \).
RAAGs in mapping class groups

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**Thm (Clay-Leininger-M)**

For partially pA \( \{f_1, \ldots, f_n\} \) supported on connected, non-nested \( X_i \) with disjointess recorded in the graph \( \Gamma \), for large enough \( p_i \),

\[ A_\Gamma \to \langle f_1^{p_1}, \ldots, f_n^{p_n} \rangle < ModS \]

is a quasi-isometric embedding.
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**Thm (Clay-Leininger-M)**

*For partially pA \( \{f_1, \ldots, f_n\} \) supported on connected, non-nested \( X_i \) with disjointess recorded in the graph \( \Gamma \), for large enough \( p_i \),

\[ A_\Gamma \to \langle f_1^{p_1}, \ldots, f_n^{p_n} \rangle < \text{ModS} \]

is an admissible* embedding.

*meaning \( A_\Gamma \hookrightarrow \text{Mod}(S) \):

(i) Comes with large subsurface curve complex projections, and

(ii) Word partial order matches subsurface partial order

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Thm (M-Taylor)

If $A_\Gamma < \text{Mod}(S)$ is admissible and $G < A_\Gamma < \text{Mod}(S)$ is convex cocompact, then $G$ is (word) quasiconvex in $A_\Gamma$.

Thm (M-Taylor)

Suppose $A_\Gamma < \text{Mod}(S)$ is admissible and $G < A_\Gamma$ is fin. gen. and $K$-quasiconvex. There exists $L = L(K, |\Gamma|)$ such that if $w \in G$ with $0 < |w| < L$ are pseudo-Anosov, then $G$ is convex cocompact (thus all-pseudo-Anosov, thus free).

Corollary

All-pA $G < A_\Gamma < \text{Mod}(S)$ is convex cocompact in $\text{Mod}(S)$ if and only if it is word quasiconvex in $A_\Gamma$. 
The Cayley graph of $A_\Gamma$ completes to a CAT(0) cube complex $\tilde{S}_\Gamma$

**Thm (Haglund 2008)**

*For $G < A_\Gamma$, TFAE:*

- Exists (non-empty) convex subcomplex $C \subset \tilde{S}_\Gamma$ which is $G$-invariant and cocompact.
- $G$ (word) quasiconvex in $A_\Gamma$ (vertex orbits $G \cdot v$ are combinatorially $q$-convex in $\tilde{S}_\Gamma$.)
Convex cocompactness in RAAGs

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Interesting examples

\[ n = g - 1 \quad \rho^n = \text{id} \]

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\[ Y_0 \supset g_0 \]

Thm (M-Taylor)
For any \( k \), \( \langle h_1, h_2, \ldots, h_k \rangle \sim = F_k \) is convex cocompact

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\[ h = (\rho f_0 g_0)^n \]

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\[ h = (\rho f_0 g_0)^n = f_1 g_1 f_2 g_2 \cdots f_n g_n \in \langle f_i, g_i \rangle \quad \text{trans}(h) \sim 1/g \]
Interesting examples

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**Thm (M-Taylor)**

For any \( k \), \( \langle h_1, h_2, \ldots, h_k \rangle \cong F_k \) is convex cocompact
Q: Construct a non-cyclic convex cocompact subgroup containing pseudo-Anosovs with $1/g^2$ translation length in curve complex.
Further questions

Q:

*Construct a non-cyclic convex cocompact subgroup containing pseudo-Anosovs with $1/g^2$ translation length in curve complex.*

Q:

*Does $G$ all-pseudo-Anosov imply $G$ convex cocompact in $\text{Mod}(S)$?*
Further questions

Q:
Construct a non-cyclic convex cocompact subgroup containing pseudo-Anosovs with $1/g^2$ translation length in curve complex.

Q:
Does $G$ all-pseudo-Anosov imply $G$ convex cocompact in $\text{Mod}(S)$?

Q:
Does $G < A_\Gamma$ all-loxodromic imply $G$ (word) quasiconvex in $A_\Gamma$?

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Fun pictures

\[ \pi_1 \left( \begin{array}{c} f_0 \\ f_1 \\ f_2 \end{array} \right) \]

= \left( \begin{array}{c} A \end{array} \right)

right-angled Artin group

\[ \left( \begin{array}{c} f_0 \\ g_0 \\ f_1 \\ g_1 \\ f_2 \end{array} \right) \]

mapping class subgroup

\[ \text{pseudo-Anosov} \]
Consequences of convex cocompactness in Mod(S)

Requirements for word hyperbolicity:

1. No subgroups \( BS(p, q) = \langle a, b | a^{-1}b^p a = b^q \rangle \)
2. Has finite \( K(G, 1) \) if torsion-free (in general, type \( FP_\infty \)).

Q: (Gromov, Farb-Mosher)

If \( G \) with finite \( K(G, 1) \) has no BS subgroups, is it hyperbolic?

Example (which might not exist)

If \( G \) is all-pA, then \( E_G \) has finite \( K(G, 1) \) and no BS subgroups.
Recall if \( G \) fails to be convex cocompact, it also fails hyperbolicity.

Q:

Does there exist free, non-quasiconvex \( G < A_\Gamma \) and admissible embedding \( A_\Gamma < \text{Mod}(S) \) such that \( G \) is all-pA?