Convex cocompactness in RAAGs and MCGs
or
The geometry of purely loxodromic subgroups of
right-angled Artin groups
or
Some interesting subgroups of MCGs and RAAGs

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$f : X \rightarrow Y$ is an $(K, L)$–quasi-isometric embedding if, $\forall p, q \in X$

$$\frac{1}{K} \cdot d_X(p, q) - L \leq d_Y(f(p), f(q)) \leq K \cdot d_X(p, q) + L$$
Coarse language

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\[ \frac{1}{K} \cdot d_X(p, q) - L \leq d_Y(f(p), f(q)) \leq K \cdot d_X(p, q) + L \]

Whether a subgroup \( H < G \) is qi-embedded \textbf{does not} depend on generating set
$Z \subset Y$ is a $K$–quasiconvex set if

$N_K(Z)$ contains all geodesics between points in $Z$

Whether a subgroup $H < G$ is quasiconvex does depend on generating set
Stability (Durham–Taylor)

Definition
A f.g. subgroup $H < G$ is stable if it is

(1) quasi-isometrically embedded, and

(2) any pair of $K$–quasigeodesics* between points in $H$ have Hausdorff distance bounded by $M(K)$.

*K–quasigeodesic: $(K, K)$–qi-embedding of an interval.
Stability (Durham–Taylor)

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A f.g. subgroup $H < G$ is *stable* if it is

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* $K$–*quasigeodesic*: $(K, K)$–qi-embedding of an interval.

Stable subgroups of $G$ are quasiconvex with respect to any word metric on $G$. 

![Diagram showing quasigeodesics and Hausdorff distance](image-url)
Convex cocompactness in mapping class groups

Definition (Farb–Mosher)

Finitely generated $G < MCG(S)$ is convex cocompact if its orbit $G \cdot X \subset \text{Teich}(S)$ is quasiconvex.
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Finitely generated $G < \text{MCG}(S)$ is convex cocompact if its orbit $G \cdot X \subset \text{Teich}(S)$ is quasiconvex.

Compare:

Finitely generated, discrete $G < \text{Isom}(\mathbb{H}^n)$ is convex cocompact iff its orbit $G \cdot p \subset \mathbb{H}^n$ is quasiconvex.
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Well-known theorems (Ivanov, Masur, Masur–Minsky)

$\text{MCG}(S) = \text{Isom}(\text{Teich}(S))$ \emph{and} $\text{Teich}(S)$ is \textbf{not} hyperbolic.

$\text{MCG}(S) = \text{Isom}(\text{Curve}(S))$ \emph{and} $\text{Curve}(S)$ is hyperbolic.
Convex cocompactness in mapping class groups

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Compare:

*Finitely generated, discrete \( G < \text{Isom}(\mathbb{H}^n) \) is convex cocompact iff its orbit \( G \cdot p \subset \mathbb{H}^n \) is quasiconvex.*

Well-known theorems (Ivanov, Masur, Masur–Minsky)

\[ MCG(S) = \text{Isom}(\text{Teich}(S)) \text{ and } \text{Teich}(S) \text{ is not hyperbolic.} \]
\[ MCG(S) = \text{Isom}(\text{Curve}(S)) \text{ and } \text{Curve}(S) \text{ is hyperbolic} \]

Definition

*Pseudo-Anosov* mapping classes are elements of \( MCG(S) \) with N-S dynamics along translation axis in \( \text{Teich}(S) \) (equiv., in \( \text{Curve}(S) \))
Goal: motivate theorem on right

Theorems (Kent–Leininger, Hamenstädt, Durham–Taylor)

TFAE for f.g. $G < \text{MCG}(S)$

(0) $G$ is convex cocompact
(1) The orbit map $G \hookrightarrow G \cdot v \subset \text{Curve}(S)$ is a q.i.-embedding
(2) $G$ is stable in $\text{MCG}(S)$.
   Also, these imply $G$ is purely pseudo-Anosov.

Theorem (Koberda–M.–Taylor)

TFAE for f.g. $G < A(\Gamma)$

(1) The orbit map $G \hookrightarrow G \cdot v \subset \text{Curve}(\Gamma)$ is a q.i.-embedding
(2) $G$ is stable in $A(\Gamma)$.
(3) $G$ is purely loxodromic.
Theorems (Farb-Mosher, Hamenstädt)

$E_G$ is word hyperbolic if and only if $G$ is convex cocompact.
MCG convex cocompactness & surface group extensions

\[ 1 \longrightarrow \pi_1(S) \longrightarrow E_G \longrightarrow G \longrightarrow 1 \]

\[ 1 \longrightarrow \pi_1(S) \longrightarrow \text{Mod}(\hat{S}) \longrightarrow \text{Mod}(S) \longrightarrow 1 \]

**Theorems (Farb-Mosher, Hamenstädt)**

*\(E_G\) is word hyperbolic if and only if \(G\) is convex cocompact.*

**Theorem (Thurston’s geometrization of mapping tori)**

\[ 1 \longrightarrow \pi_1(S) \longrightarrow \pi_1(M_\phi) \longrightarrow \langle \phi \rangle \longrightarrow 1 \]

*M_\phi is hyperbolizable if and only if \(\phi\) is pseudo-Anosov, i.e. iff \(\langle \phi \rangle\) is convex cocompact.*
MCG convex cocompactness & hyperbolic groups

Theorem (Thurston’s geometrization, proved by Perelman)

$M$ closed, aspherical 3–mfld admits a hyperbolic metric if and only if $\pi_1(M)$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$.

Question (Gromov)

If $H$ has finite $K(H,1)$ and no subgroups of the form $BS(p, q) = \langle a, b | a^{-1} b^p a = b^q \rangle$, is $H$ hyperbolic?
MCG convex cocompactness & hyperbolic groups

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Potential counterexample

f.g. purely pA \( G \) \( \rightarrow \) \( E_G \) has finite \( K(E_G,1) \) and no \( BS \) subgroups.

Question (Farb–Mosher)

Is f.g. purely pA \( G \) \( < \) \( MCG(S) \) necessarily convex cocompact?
Theorem (Thurston’s geometrization, proved by Perelman)

$M$ closed, aspherical 3–mfd admits a hyperbolic metric if and only if $\pi_1(M)$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$.

Question (Gromov)

If $H$ has finite $K(H, 1)$ and no subgroups of the form $BS(p, q) = \langle a, b | a^{-1} b^p a = b^q \rangle$, is $H$ hyperbolic?

Potential counterexample

f.g. purely pA $G \Rightarrow E_G$ has finite $K(E_G, 1)$ and no $BS$ subgroups.

Question (Farb–Mosher)

Is f.g. purely pA $G < \text{MCG}(S)$ necessarily convex cocompact? (No here means no to Gromov, since $E_G$ would not be hyperbolic).
Q: Does f.g. purely pA imply convex cocompact?

Yes answers in special cases

$H < G$ for $G$ from a certain family of $MCG(S)$–subgroups:

- $H < \text{Isom}(\mathbb{H}^2)$ for $\mathbb{H}^2 \subset \text{Teich}(S)$ (Veech groups);
- Leininger–Reid combinations of Veech groups (Leininger);
- $H < \pi_1(M_\phi)$ when $M_\phi$ is hyperbolic;
  - quasiconvex $H < E_G$ when $E_G$ is hyperbolic
    (Dowdall–Kent–Leininger, generalizing Kent–Leininger–Schleimer);
- $H < A(\Gamma) < MCG(S)$ for “admissible” $A(\Gamma)$
  (M.–Taylor, Koberda–M–Taylor)
RAAGs in MCGs

Definition

\[
A(\Gamma) = \langle v_i \in V(\Gamma) \mid [v_i, v_j] = id \text{ if } (v_i, v_j) \in E(\Gamma) \rangle
\]


Many ways to embed \( A(\Gamma) \) in some \( MCG(S) \).
RAAGs in MCGs

Definition

\[ A(\Gamma) = \langle v_i \in V(\Gamma) \mid [v_i, v_j] = id \text{ if } (v_i, v_j) \in E(\Gamma) \rangle \]

Theorem (Clay–Leininger–M)

For partially \( pA \) \{\( f_1, \ldots, f_n \)\} supported on connected, non-nested \( X_i \) with disjointess recorded in the graph \( \Gamma \), for large enough \( p_i \),

\[ A(\Gamma) \to \langle f_1^{p_1}, \ldots, f_n^{p_n} \rangle < MCG(S) \]

is a quasi-isometric embedding.
RAAGs in MCGs

Definition

\[ A(\Gamma) = \langle v_i \in V(\Gamma) \mid [v_i, v_j] = id \text{ if } (v_i, v_j) \in E(\Gamma) \rangle \]

Theorem (Clay–Leininger–M)

For partially \( pA \) \( \{f_1, \ldots, f_n\} \) supported on connected, non-nested \( X_i \) with disjointness recorded in the graph \( \Gamma \), for large enough \( p_i \),

\[ A(\Gamma) \to \langle f_1^{p_1}, \ldots, f_n^{p_n} \rangle < MCG(S) \]

is an admissible* embedding.

*meaning \( A(\Gamma) \hookrightarrow MCG(S) \):

(i) Comes with large subsurface curve complex projections, and
(ii) Word partial order matches subsurface partial order
Our special case

Suppose $A(\Gamma) < MCG(S)$ admissible.

**Theorem (M–Taylor)**

*F.g. purely pA $H < A_\Gamma < MCG(S)$ is convex cocompact in $MCG(S)$ if and only if $H$ is combinatorially quasiconvex* in $A(\Gamma)$.  

*word metric using standard vertex generators

**Easy fact**

$\phi \in A(\Gamma) < MCG(S)$ pseudo-Anosov $\implies \phi \in A(\Gamma)$ loxodromic.

**Corollary (Koberda–M–Taylor)**

$H < A(\Gamma) < MCG(S)$ is convex cocompact if and only if $H$ is f.g. purely pA.
Curve(S) and Curve(Γ)

Curve(S):
- Vertices $\leftrightarrow$ ess. simple closed curves on $S$ up to isotopy
- Edge $(\alpha, \beta) \iff \alpha, \beta$ are disjoint

Curve(Γ) aka extension graph $\Gamma^e$ of Γ, defined by Kim–Koberda:
- Realize $A(\Gamma) \hookrightarrow MCG(S)$ by
  - vertex generators $\rightarrow$ high-powered Dehn twists (Koberda)
- Vertices $\leftrightarrow$ base curves of $A(\Gamma)$–conjugates of vertex gens
- Edge $(\alpha, \beta) \iff \alpha, \beta$ are disjoint

Theorem (Kim–Koberda)
Curve(Γ) is hyperbolic (in fact, it is a quasi-tree).
Loxodromic elements

Definition
\( \phi \in A(\Gamma) \) is loxodromic if \( \phi \cdot v \subset \text{Curve}(\Gamma) \) is unbounded.

Note
\( \phi \in MCG(S) \) is pseudo-Anosov iff \( \phi \cdot v \subset \text{Curve}(S) \) is unbounded.

Theorems (Kim–Koberda, Servatius, Behrstock–Charney)

For \( \phi \in A(\Gamma) \), TFAE:
- \( \phi \) is loxodromic
- \( \phi \notin \mathbb{Z} \oplus \mathbb{Z} < A(\Gamma) \)
- \( \phi \) acts as a rank-1-isometry of \( \widetilde{S}(\Gamma) \), the CAT(0) cube complex whose 1–skeleton is \( \text{Cayley}(A(\Gamma), V(\Gamma)) \)
Theorem (Haglund)
For $H < A_{\Gamma}$, TFAE:

- Exists (non-empty) convex subcomplex $C \subset \tilde{S}(\Gamma)$ which is $H$-invariant and cocompact.
- $H$ combinatorially quasiconvex in $A(\Gamma)$, i.e. vertex orbit $H \cdot v$ quasiconvex in $\tilde{S}(\Gamma)_{(1)}$.

Proposition (K–M–T)
Non-loxodromic elements: join-words and star-words

Theorem (Servatius)

\[ \phi \text{ not loxodromic} \implies c\phi c^{-1} \in A(J) \text{ for a join } J \subset \Gamma. \]

Definition

A join \( J = \Gamma_1 \ast \Gamma_2 \).

join words conj. into \( A(J) \)
Non-loxodromic elements: join-words and star-words

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Non-loxodromic elements: join-words and star-words

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Definition

A join \( J = \Gamma_1 * \Gamma_2 \).

A star \( T = \Gamma_1 * v \):

join words conj. into \( A(J) \)

star words conj. into \( A(T) \)

\{purely loxo. (no join words)\} \subsetneq \{star-free (no star words)\}
If $G < A(\Gamma)$ is finitely generated and star-free, then

1. $G$ is a free group,
2. $G$ is quasi-isometrically embedded in $A(\Gamma)$, and
3. $G \cap A(\Lambda)$ is finitely generated, for any subgraph $\Lambda \subset \Gamma$. 

**Theorem (Koberda–M–Taylor)**

**Star-free RAAG subgroups**

- comb.
- qcvx
- purely loxo.
- star-free