FQE Problems for 9/15/03

1. Let A be an $n \times n$ matrix with complex entries. If $A^m = 0$ for some positive integer m, prove that $A^n = 0$.

2. Let $T\colon \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation represented by the matrix

$$\begin{bmatrix} 0 & 1 & 2 & -2 \\ 1 & 0 & 2 & -3 \\ 1 & 1 & 4 & -5 \end{bmatrix}$$

(a) Find a basis for the kernel (null space) of T

(b) Find a basis for the range (image) of T.

3. Let A be a square matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ and corresponding eigenvectors $v_1, \ldots, v_m$. Prove that $v_1, \ldots, v_m$ are linearly independent.

4. Let V be an $(n-1)$-dimensional subspace of $\mathbf{R}^\mathbf{n}$. Prove that there exists x_0 in $\mathbf{R}^\mathbf{n}$ such that

$$V = \{x \in \mathbf{R}^\mathbf{n} \mid \mathbf{x} \cdot \mathbf{x_0} = \mathbf{0}\}$$

5. Let K be the vector space of $n \times n$ skew-symmetric matrices ($A^t = -A$) with real coefficients, and let S be the vector space of $n \times n$ symmetric matrices ($A^t = A$) with real coefficients. Define a linear transformation $T\colon K \to S$ as follows. If the (i,j)-th entry of A is a_{ij}, then the (i,j)-th entry of $T(A)$ is given by

$$T(A)_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -a_{ij} & \text{if } i > j \end{cases}$$

i. Is T one to one? Explain why or why not.

ii. Is T onto? Explain why or why not.

6. Let $T\colon \mathbf{R}^\mathbf{n} \to \mathbf{R}^\mathbf{m}$ be a linear transformation. Show that if T is onto (surjective), then $n \geq m$.

7. Prove that a 2×2 matrix A with positive entries is diagonalizable over $\mathbf{R}$.

8. (08/23/01) Let $\mathcal{P}_{\in}$ denote the vector space consisting of all real polynomials of the form $p(x) = a_0 + a_1 x + a_2 x^2$. For any two polynomials p and q in $\mathcal{P}_{\in}$, define a symmetric, bilinear form $\langle p, q \rangle$ by $\langle p, q \rangle = p(0)q(0) + p(\frac{1}{2})q(\frac{1}{2}) + p(1)q(1)$.

a. Show that $\langle p, q \rangle$ is an inner product on $\mathcal{P}_{\in}$. (Note: it is routine to check that $\langle p, q \rangle$ is symmetric and bilinear. You need not do this part of the argument.)

b. Find an orthonormal basis for the subspace of $\mathcal{P}_{\in}$ that is spanned by $p(x) = x$ and $q(x) = x^2$

9. Denote by P_n the vector space of all real polynomials of degree $\leq n$. It is well known that $\{1, x, x^2, \ldots, x^n\}$ is a basis for P_n.

(a) Must every basis for P_n contain a polynomial of degree n? Verify your assertion.

(b) Find a basis for P_n which consists entirely of polynomials of degree n.

10. Suppose A and B are linear transformations from $\mathbf{R^3}$ to $\mathbf{R^5}$, both of rank 3. Show that there are non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ such that $A\mathbf{x} = B\mathbf{y}$.

11. Let V be the vector space of real polynomials of degree ≤ 2. Define a linear transformation $L\colon V \to V$ by

$$L(P(x)) = (-3x + x^2)P''(x) + 3P'(x) + P(x) + 3xP(0),$$

where $P(x) = ax^2 + bx + c$.

a) Find the matrix representations of L and L^{-1} with respect to the basis $x^2, x, 1$ in V.

b) Find a basis for V consisting of eigenvectors for L.

12. Let V be a vector space over a field F and let $T\colon V \to V$ be a linear transformation which is *nilpotent* ($T^k = 0$ for some $k > 0$). Prove that $I + T$ is invertible.