FQE Problems for 9/22/03

1. Prove that an integral domain with finitely many elements is a field.

2. Let H be a normal subgroup of a finite group G. If G/H has an element of order n, prove that G has an element of order n.

3. A subgroup H of a group G is called a characteristic subgroup if $\varphi(H) \subset H$ for all automorphisms φ of G.

(a) Prove that each subgroup of a cyclic group is a characteristic subgroup.

(b) Find a finite Abelian group G and a subgroup H of G such that H is not a characteristic subgroup of G.

4. Is the polynomial $2x^{10} - 25x^3 + 10x^2 - 30$ irreducible over $\mathbf{Q}$? Why or why not?

5. Prove that M_2, the ring of 2×2 matrices with real entries, has exactly two ideals: M_2 and $\{0\}$.

6. There is a general theorem that groups of certain orders are cyclic. Without using the general theorem, prove the special case that every group of order 15 is cyclic.