FQE Problems for 9/29/03

1. Let $\mathbf{Q}$ be the additive group of rational numbers. The additive group $\mathbf{Z}$ is a subgroup. Show that $\mathbf{Q}/\mathbf{Z}$ has infinite order, but each element of $\mathbf{Q}/\mathbf{Z}$ has finite order.

2. Let $G = \{(g_1, g_2) \in \mathbf{R^2} \mid g_1 \neq 0\}$ and let $e = (1, 0)$. Then G is a group under the multiplication $(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_1 h_2 + g_2)$, and e is the identity of G (you may assume all of this).

a. Let H be the subgroup of G defined by $H = \{(1, h) \mid h \in \mathbf{R}\}$. Show that H is normal in G.

b. Show that G/H is isomorphic to the multiplicative group $\mathbf{R}^*$ of non-zero real numbers. Hint: find a homomorphism of G onto $\mathbf{R}^*$ with kernel H.

3. Let n be a positive integer, let $G_n = \{(g_1, g_2) \in \mathbf{R^2} \mid \mathbf{g_1} \neq \mathbf{0}\}$, and let $e = (1, 0)$. Then G_n is a group under the multiplication $(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_1 h_2 + g_2 {h_1}^n)$, and e is the identity of G (you may assume all of this). Prove that for $n > 1$, the center $Z(G_n)$ of G_n is

$$Z(G_n) = \begin{cases} \{(1,0)\} & \text{for } n \text{ even} \\ \{(1,0), (-1,0)\} & \text{for } n \text{ odd.} \end{cases}$$

4. Let G be a group and let H be the subgroup generated by the squares of elements in G (so $h \in H$ if and only if h is of the form $h = {g_1}^2 {g_2}^2 \cdots {g_k}^2$ where k is some positive integer and $g_1, \ldots, g_k$ are in G).

(a) Show that H is normal in G.

(b) Show that the quotient group G/H is Abelian.

5. How many homomorphisms are there from the group $\mathbf{Z}/\langle \mathbf{20} \rangle$ to the group $\mathbf{Z}/\langle \mathbf{8} \rangle$? How many of these homomorphisms are onto?

6. Show that there is no non-trivial automorphism of the field $\mathbf{Q}$ of rational numbers.

7. Let F be a field and let $P(x) = a_0 + \ldots + a_n x^n$ be an irreducible polynomial of degree $n \geq 2$ in $F[x]$. Let $\alpha = x + \langle P(x) \rangle$, where $\langle P(x) \rangle$

denotes the ideal generated by $P(x)$. Express α^{-1} in $F[x]/\langle P(x)\rangle$ in terms of $\alpha^0, \ldots, \alpha^{n-1}$.