FQE Problems for 10/27/03

1. Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^6 + 3y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

(a) Use the definition of directional derivative to show that f has a directional derivative in every direction at $(0,0)$.

(b) Show that f is not continuous at $(0,0)$.

2. Prove or disprove the following:

(a) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then so does $\sum_{n=1}^{\infty} {a_n}^2$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$, then $\sum_{n=1}^{\infty} b_n$ converges.

3. Does $\lim_{n\to\infty} x^n(1-x^n) = 0$ uniformly on the closed interval $0 \le x \le 1$? Justify your answer.

4. Let $f(x) = \sqrt{x}$ for $x \in A = [0,\infty)$.

(a) Prove that f is uniformly continuous on A.

(b) Does there exist a real constant K such that

$$|f(x) - f(y)| \le K|x-y|$$

for all x and y in A?

5. For each continuously differentiable simple closed curve $\mathcal{C}$ (oriented counter-clockwise) in $\mathbf{R}^2$, define $\tau(\mathcal{C})$ by

$$\tau(\mathcal{C}) = \int_{\mathcal{C}} 2y^3 dx + \left(3x - 2x^3\right) dy.$$

Use Green's Theorem to find the continuously differentiable simple closed curve $\mathcal{C}$ for which $\tau(\mathcal{C})$ is maximal.

6. Let $f\colon [a,b] \to \mathbf{R}$ be continuous. If $\int_a^b f(x)g(x)\,dx = 0$ for all continuous functions g on $[a,b]$ such that $g(a) = g(b) = 0$, prove that f is identically 0.

7. Determine the convergence or divergence of the sequence

$$a_n = \frac{1}{n+1} + \cdots + \frac{1}{2n}.$$

8. Let

$$F(u, v, x, y) := u^2 + v^3 + x^2 - 3y$$
$$G(u, v, x, y) := u^2 + v^4 + 3x + y^4.$$

a. Justify the existence of a unique solution $x = h(u, v), y = k(u, v)$ to the simultaneous equations $F(u, v, x, y) = 0, G(u, v, x, y) = 0$ on a neighborhood of the point $(u, v, x, y) := (1, 1, -1, 1)$, where h and k are continuously differentiable.

b. Let $x = h(u, v), y = k(u, v)$ be the solution from part a. Calculate $\frac{\partial h}{\partial u}(1, 1)$.

9. Let $f(x)$ be a real-valued function, defined and continuous for each x in $\mathbf{R}$. Let a be a fixed real number. Prove there is a point on the graph of $f(x)$ whose distance to $(a, 0)$ is a minimum.

10. Let $f(x)$ be continuous on $[0, 1]$ and let $n \geq 2$ be a fixed integer. If $f(0) = f(1)$, prove that there exists a number c in $[0, 1]$ such that $f(c) = f(c + \frac{1}{n})$. Hint: consider the function $g(x) := f(x + \frac{1}{n}) - f(x)$.

11. Let $f\colon [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Prove that if

$$\lim_{x \to a^+} f'(x) = A,$$

then the (one-sided) derivative $f'(a)$ exists and $f'(a) = A$.

12. Assume f is a real-valued continuous function on the rectangle $\{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$. Let

$$F(x) = \int_c^d f(x, y)dy.$$

Prove that F is a continuous function on $[a, b]$.