FIRST QUALIFYING EXAM
August 24, 2000

There are two sections, Algebra and Analysis. Solve **FOUR** *problems from each section. Do* **NOT** *hand in more than four problems for a section; if you do, only the first four problems will be counted. Do each problem on a separate sheet. Show all work. Be sure to write your person number clearly on* **EACH** *sheet. Remember, your goal is to convince the grader that you know what you are doing.*

Section I. Algebra. Do any four problems.

1. Show that a finite integral domain is a field.

2. Consider the conjecture: If a field F is infinite, then every nontrivial subfield of F is infinite. Prove or give a counterexample with proof.

3. Let $\varphi\colon \mathbf{Z}\oplus\mathbf{Z} \to \mathbf{Z}$ be the group homomorphism defined by $\varphi(m,n) = 4m-7n$. Let H be the subgroup of Z generated by the element 5. Find two elements that generate $\varphi^{-1}(H)$. (Part of this problem is to prove that the elements actually generate $\varphi^{-1}(H)$.)

4. Prove Eisenstein's Criteria. That is, show that if

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

is a polynomial over the ring $\mathbf{Z}$ of integers, and if p is a prime such that

$$\begin{gathered} p \text{ divides } a_i, \text{ for } 0 \le i < n \\ p \text{ does not divide } a_n \\ p^2 \text{ does not divide } a_0 \end{gathered}$$

then $P(x)$ is irreducible over $\mathbf{Z}$.

5. Let V be the vector space of all 2×2 matrices with real coefficients. Define an inner product $\langle\, ,\, \rangle$ on V by

$$\langle A, B\rangle = \operatorname{trace}\left(A\, B^t\right).$$

Find an orthonormal basis for the orthogonal complement in V of the 2×2 identity matrix.

6. Let V and W be finite dimensional vector spaces over $\mathbf{R}$. Let M be a non-zero subspace of V, and let N be a subspace of W. If $T(M) \subset N$ for all linear transformations $T\colon V \to W$, prove that $N = W$.

7. Let M be the vector space of upper triangular 2×2 matrices, with basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let T be the linear transformation on M defined by

$$TX = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} X \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Find the matrix $A = (a_{ij})$ of T. That is, find the matrix A that satisfies $TE_j = \sum_{i=1}^{3} a_{ij} E_i$.

8. A square matrix N is nilpotent if there is a positive integer m such that $N^m = 0$.

(a) Find a scalar λ and a nilpotent matrix N such that

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N.$$

(b) Let $P(x)$ be a polynomial with real coefficients. Use part a) to show that

$$P\left(\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\right) = \begin{pmatrix} P(2) + P'(2) & P'(2) \\ -P'(2) & P(2) - P'(2) \end{pmatrix}.$$

Section II. Analysis. Do any four problems.

9. Let $\sum_{n=0}^{\infty} a_n (x-a)^n$ be an infinite series with radius of convergence $R > 0$. Show that if $0 < r < R$, then the series converges uniformly for $|x - a| \le r$.

10. Let $f\colon [0,1] \to \mathbf{R}$ be continuous. Show that

$$\lim_{j \to \infty} \int_0^1 \cos\left(2\pi x + \frac{f(x)}{j}\right) dx = 0.$$

11. A set X in $\mathbf{R^n}$ is connected if there do not exist open sets A and B in $\mathbf{R^n}$ such that all the following hold:

(1) $X \cap A \neq \emptyset$
(2) $X \cap B \neq \emptyset$
(3) $X \subset (A \cup B)$
(4) $X \cap A \cap B = \emptyset$.

Let X and Y be connected subsets of $\mathbf{R^N}$ such that $X \cap Y \neq \emptyset$. Use the definition above to show that $X \cup Y$ is connected.

12. For each continuously differentiable simple closed curve $\mathcal{C}$ (oriented counter-clockwise) in $\mathbf{R^2}$, define $\tau(\mathcal{C})$ by

$$\tau(\mathcal{C}) = \int_{\mathcal{C}} 2y^3 dx + \left(3x - 2x^3\right) dy.$$

Use Green's Theorem to find the continuously differentiable simple closed curve $\mathcal{C}$ for which $\tau(\mathcal{C})$ is maximal.

13. Let $f\colon \mathbf{R^3} \to \mathbf{R}$ be a C^1 function such that $f(0,0,0) = 0$. Let $a, b,$ and c be real constants such that the derivative of f at $(0,0,0)$ is (a, b, c).

i. Give the minimal conditions on $a, b,$ and c that ensure there exists a neighborhood N of $(0,0)$ in $\mathbf{R^2}$ and a C^1 function $g\colon N \to \mathbf{R}$ such that $g(0,0) = 0$ and $f(x, y, g(x, y)) = 0$ for all (x, y) in N.

ii. Assume that suitable conditions on $a, b,$ and c are given for part i. Find the derivative of g at $(0,0)$.

14. Let S be a compact subset of $\mathbf{R^n}$. Prove S is closed and bounded.