MAGNETIC POINT VORTEX DYNAMICS IN THE PLANE

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Summary It is shown that an interpretation of the Hamiltonian dynamics of a one-dimensional chain of point magnets bears a striking resemblance to that of point vortices in a planar ideal fluid. This leads naturally to a consideration of planar generalizations of point magnet fields that exhibit vortex-like dynamical characteristics. A class of such generalizations – (quasi-) 2-dimensional approximations of Bose—Einstein condensate vortex dynamics – is analyzed using a combination of theory and simulations. Special attention is paid to integrability, bifurcations and perturbations of the Hamiltonian dynamics. Possible variations of the fields and extensions to approximate models of (topological) vortex dynamics in Maxwell—Chern—Simon—Higgs fields are also briefly considered.

INTRODUCTION

The dynamics of a 1-dimensional chain of small (point) magnets has several useful applications as shown in [5] and experiments and simulations by the authors indicate considerable energy harvesting potential. Naturally, this leads to an interest in higher dimensional generalizations, which catalyzed the work described in this paper on 2-dimensional extensions related to physical phenomena of current interest, especially those involving vortex-like dynamics. This emphasis is a result both of the resemblance of the equations of motion of to that of point vortices in fluid dynamics (see *e.g.* [1]) and certain similarities between these equations and those associated to vortex-like behavior in various electro-magnetic gauge fields, especially Bose—Einstein condensates (BEC) as illustrated in papers such as [2-4, 6].

This paper begins with a brief description of the Hamiltonian dynamics of a 1-dimensional chain of magnets; especially its similarity to the dynamics of point vortices in an ideal fluid. We also describe the Hamiltonian equations for point magnetic fields in general Euclidean spaces. Next, the vortex dynamics of quasi-two-dimensional BECs is recognized as a natural 2-dimensional generalization of the 1-dimensional point magnet chain. The remainder of the exposition is mainly devoted to an investigation of the BEC vortex dynamics, wherein we summarize one of the main results of our research so far. In the final section, we describe some of our conclusions and identify several possible generalizations and interesting problems related to vortex dynamics in other gauge fields.

POINT MAGNET AND RELATED DYNAMICS

We assume that the magnetic monopoles are continuously distributed along a line, but concentrated in finitely many minute intervals identified as *point magnets* at the interval midpoints. Integration of the $1/r^2$ forces over the intervals produces $1/r$ interaction forces among the point magnets. Let N point magnets of masses $m_1,\dots, m_N$ and strengths per unit mass $\gamma_1, \dots, \gamma_N$ lie along $\mathbb{R}$ at the points $x_1,\dots, x_N$. The Hamiltonian equations of motion with $1/r$ interaction forces are

(1)
$$\dot{x}_k = \{H_0, x_k\}, \quad \dot{y}_k = \{H_0, y_k\} \quad (1 \le k \le N),$$

where the momenta y_k, Hamiltonian function H_0 and (nonstandard) Poisson bracket $\{\bullet,\bullet\}$ are defined, respectively, as

(2)
$$y_k := \dot{x}_k,\ H_0 := (1/2)\sum_{k=1}^{N}\gamma_k y_k^2 - \sum_{1\le j<k\le N}\gamma_j\gamma_k \log\left|x_j - x_k\right| \text{and } \{f,g\} := \sum_{k=1}^{N}\gamma_k^{-1}\{f_{y_k}g_{x_k} - f_{x_k}g_{y_k}\}.$$

For point magnets in Euclidean n-space $\mathbb{R}^n$, the equations of motion are the same *mutatis mutandis* the notation.

If one omits the kinetic energy in H_0 in (2) for massless point vortices [3], it is essentially identical to that of point vortices in an ideal fluid and has three independent invariants in involution [1], whereas the original has just two.

BOSE—EINSTEIN CONDENSATE VORTEX DYNAMICS IN THE PLANE

Vortices occur in BEC dynamics in the plane as shown in the experimental observation in Fig. 1, which can be inferred from a well-known approximate point vortex model of the form

(3)
$$\gamma_k \dot{z}_k = -2i\gamma_k \partial_{\bar{z}_k}\Phi(z) - i\sum_{1\le j\ne k\le N}\gamma_j\gamma_k\left(\bar{z}_j - \bar{z}_k\right)^{-1} \quad (1\le k\le N),$$

where $z_k = x_k + iy_k \in \mathbb{C}$, the complex plane, and the leading term on the right-hand side is the component of the *precession velocity*. We note that the system (3) is symplectic, with Poisson bracket as in (2) and Hamiltonian function

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(4) $$H = \Phi(z) - \sum_{1 \le j < k \le N} \gamma_j \gamma_k \log\left|z_j - z_k\right|.$$

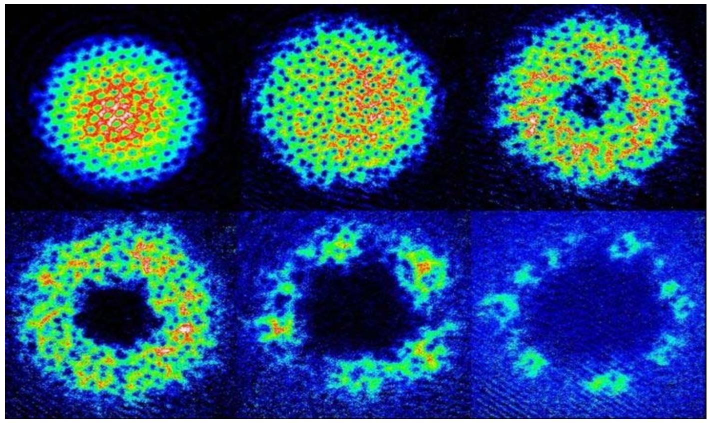

Fig. 1. Evolution of vortices in a BEC.

The Hamiltonian dynamical system corresponding to (3) is

(5) $$\dot{x}_k = \gamma_k^{-1} \partial_{y_k} H = \{H, x_k\},\ \dot{y}_k = -\gamma_k^{-1} \partial_{x_k} H = \{H, y_k\} \quad (1 \le k \le N).$$

As for the choice for the precession velocity, the function

(6) $$\Phi(z) := a \log\left[\prod_{k=1}^{N} \left(b - \left|z_k\right|^2\right)^{\gamma_k}\right],$$

where *a* and *b* are positive constants, is realistic and often chosen [4].

As an example of the type of results we have been able to prove, we offer the following:

Theorem. *The Hamiltonian system* (5) *has, depending on* Φ, *either one, two or three independent constants of motion (including H) in involution, in which case it is completely Liouville—Arnold (L-A) integrable for N* = 1, *N* =2 *or N* = 3 *point magnets, respectively. Additional point vortices in each of these cases can cause chaotic dynamics (proving that there are no additional independent invariants in involution).*

Indeed, if $\Phi = 0$, the Hamiltonian function is the same as for point vortices in a fluid, for which a system of at most three vortices is completely L-A integrable [1], when Φ is as in (6) it is not difficult to show that the system (5) is L-A integrable for $N = 2$, but not $N = 3$, and it is easy chose Φ so that *H* is the only independent integral of (5).

CONCLUDING REMARKS

We have shown how certain 2-dimensional generalizations of the dynamics of a 1-dimensional chain of point magnets includes well-known approximate 2-dimensional dynamical models of the evolution of BECs, and have begun to analyze these models. Additional investigation of these models is planned as part of our ongoing research. Moreover, inasmuch as the BEC models are derived from the 3-dimensional Gross—Pitaevskii partial differential equation, it is reasonable to assume that approximate planar point magnet models can be derived from gauge-field equations of more general types falling within the sphere of Maxwell—Chern—Simons—Higgs theory. We are also planning to delve into these possibilities.

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