THE BOX PRODUCT PROBLEM TWENTY-FIVE YEARS LATER

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ABSTRACT:

The central box product problem states: Given the box topology, is the product of countably many copies of the real line a normal space? In the first part of this paper, we survey recent results we believe related to the central problem. In the second part of this paper we present interesting additional open problems about box products related to the central problem.

In what follows, a space each of whose finite products is Hurewicz is called a power Hurewicz space.

In the third part of this paper we present new consistency results related to the central problem: We expand Lawrence's 1988 result that $\mathfrak{b} = \mathfrak{d}$ and $\mathfrak{d} = \mathfrak{c}$ imply the box product of countable many copies of the space of the rationals, to all power Hurewicz metric spaces (under $\mathfrak{d} = \mathfrak{c}$) and all power Hurewicz first countable spaces (under $\mathfrak{d} = \mathfrak{c}$). We remove "zero-dimensional" and weaken to power Hurewicz the hypothesis of Winger's 1994 Theorem that $\mathfrak{d} = \aleph_1$ implies the box product of countably many zero-dimensional σ-compact spaces of weight at most $\aleph_1$ is paracompact. We prove $\mathfrak{b} = \mathfrak{d} \Rightarrow$ If X is a σ-compact separable linear ordered space, then $\square^\omega X$ is paracompact.

Finally, we recognize that box products have had interesting applications, so we include many references to articles about box products written since The Handbook of Set-Theoretic Topology.

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