Dynamics on the Irrationals·

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§0. Introduction. §1. A homeomorph of P. §2. A new system on P. §3. More homeomorphs of P. §4. Constructing points in Π. §5. Recurrent points in [Π;σ]. §6. Systems with all points recurrent. §7. Multiple Recurrence.

§1. Introduction.

We present here work which is, in part, expository with proofs,[1] exercises (2.4, 2.6, 4.3, and 7.6), and, in part, contains new results (3.1, 5.4, amd 7.7) so we ought to begin with some background: Dynamics travels a line of history from as far back as Newton, as a notion for his laws of motion, especially, as concerns the law of gravity, for which he developed the Calculus. Prior to the twentieth century, a dynamical system meant a motion whose parameters are functions of time and satsfy a system of differential equations. Eighteenth and Nineteenth century analysts used various analytical manipulations (including infinte series) to cause the differential equations to reveal information. However, 100 years ago Poincaré, using a proofs fortelling modern topology, shifted our attention from particular solutions to the relationships between all possible solutions and, in some cases, he used his methods to prove the existence of periodic solutions. In 1927, G.D. Birkhoff's work signifcantly justified Poincare's global approach by proving that in any dynamical system on a compact space has a solution stable in the sense of Poisson. In our terminolgy it is stated as each system on a compact space has a recurrent point.

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[1] Some of the material represents joint work with Jan Pelant of the Czech Academy of Sciences and represents a simplified version of results in **[15]**.

In the area of multiple systems, the first result came in with a little heralded paper by P. Erdös and A. H. Stone [7b]. However, H. Furstenburg, B. Weiss, and others did, in the late 1960's for multiple systems, what Birkhoff did for simple systems.

The primary results we exhibit here are:

3.1. Suppose [X,f] is a system on a separable complete metrizable space (X,d). Then there is a system [P,g] and a homomorphism $h:[P,g] \to [X,f]$.

5.2. Suppose $x \quad \Pi$. x is recurrent in $[\Pi;\sigma]$ iff there is a street S such that $x = \alpha S$.

5.3. If $\{\lambda B_z : z \quad N\}$ is finite, then x is almost-periodic in $[\Pi;\sigma]$.

5.4. There is an almost periodic-point x such that if $(F_z : z \quad N)$ is a street with $x = \alpha(F_z : n \quad N)$. Then $\{\lambda F_z : z \quad I\}$ is infinite.

6.3. There is a homeomorphism $f: P \to P$ such that [P;f] is a minimal system, and no point of P is almost-periodic in [P;f].

6.4. There is a continuous function $f : P \to P$ such that each point in P is recurrent in [P;f], but [P;f] has no minimal sets.

7.7. There is a continuous function $f: P \to P$ such that [P,f] is minimal, but no point of P is multiply recurrent in $[P;\{f,f^2\}]$.

7.9. There are commuting continuous functions $f,g : P \to P$ such that [P,{f,g}] is minimal, $p \quad P$, $\mathbf{OC}_f(p)$ **È** $\mathbf{OC}_g(p) = \{p\}$.

0.1. In general, the objects considered here are pairs [X;f], called a discrete dynamical system, where X is a metric space and $f:X \to X$ is a continuous function.[2] Our particular attention will focus upon the case when X is the topological space P of all irrationals in the real line. All the definitions below are standard (see **[1]**, **[6]**, **[10]**, **[11]**, and **[16]**).

0.2. Fix a system [X;f], let $f^1 = f$, and for each $n \quad N$ (= the set of positive integers) let $f^{n+1} = ff^n$. The *orbit* of a point $x \quad X$ is the set $\{f^n(x) : n \quad N\}$, denoted by $\mathbf{Orb}_f(x)$ (or $\mathbf{Orb}(x)$ when no confusion results). The *orbit closure* of a point $x \quad X$ is the set $cl(\mathbf{Orb}(x))$, denoted by $\mathbf{OC}_f(x)$ (or $\mathbf{OC}(x)$ when no confusion results).

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[2] It is common, but not necessary, to require f to be a homeomorphism, and I assure you **all** the material here have analogous resultsand harder proofs, in the homeomorphism case **[15]**.

0.3. Suppose x X. In [X;f],

1). x is said to be *fixed* provided f(x) = x.

2). x is said to be *periodic* provided m N such that $f^m(x) = x$. (of period m N, if m is the first such integer).

3). A point x X is said to be *almost-periodic* (also known as *uniformly recurrent* **[8]**) in [X;f] provided that for each neighborhood U of x, {n N : $f^n(x)$ U} ≠ is relatively dense in N; i.e., provided k = k(U) N such that m N, {n N : km ≤ n < (k+1)m, $f^n(x)$ U} ≠ .

4). A point x X is said to be *recurrent* in [X;f] provided x **OC**(x) (or equivalently, when X is a metric space, there is an increasing sequence $<k_n>$ in N such that $\lim_{n \to \infty} f^{k_n}(x) = x$).

5). A set M **ç** X is said to be *minimal* in [X;f] provided it is a minimal element in the partially ordered, by **ç**, set of all non-empty closed sets A **ç** X such that f(A) **ç** A, or equivalently, x M, **OC**(x) = M.

There are more interesting points we could study here (see exercise 2.6 and **[1]**, **[2]**, **[6]**, **[8]**, **[9]**, **[11]**, **[14]**, **[16]**, and **[18]**).

0.4. If X is minimal in [X;f], then [X;f] is said to be a *minimal system*. A system [X;f] is said to be a *transitive system*. when there is a point x X such that X = **OC**(x).

0.5. In dynamics, the basic property preserving properties between two systems are called homomorphisms. Specifically, suppose [X,f] and [Y,g] are systems. h is a *homomorphism* from [Y,g] to [X,f] (write h :[Y,g] → [X,f]) if h :Y → X is a continuous surjection satisfying gh = hf. It is very easy to show a homomorphism takes fixed (periodic, almost-periodic, recurrent) points to fixed (respectively, periodic, almost-periodic, recurrent) points, and that the composition of two homomorphisms is a homomorphism.

0.6. In the final section of this paper, we will consider *multiple systems* [X; f], where f is a commuting family of continuous functions $f: X \to X$; i.e., f,g f , fg = gf). Given a multiple system [X; f], let < f > = $\{f_1 f_2 \cdots f_k : \langle f_1, f_2, \ldots, f_k \rangle$ is a finite sequence in f }.

0.7. For x X, we set $\mathbf{OC}_f(x) = cl\{fx : f \ \langle f \rangle\}$. A *minimal* set in a multiple system [X; f] is minimal with respect to the condition: All non-empty closed sets A ç X such that f f , fA ç A.

0.8. x X is said to be *jointly recurrent (almost-periodic)* in the multiple system [X; f] provided f f , x is recurrent (almost-periodic) in [X;f]. In the case f is finite, x is said to be *multiply recurrent* **[8]** in [X; f] provided that for each neighborhood U of x n N such that f f , $f^n x$ U **[8]** (this is usually defined for when f is finite; the infinite case is considered in **[2]** where it is called *uniform multiple recurrence*).

0.9. Let P denote the subspace of irrationals on the real line, and let C denote the Cantor set. These objects are often considered bizarre, however, they occupy a place fundamental to dynamics, both in theory and example. Our primary goal in this paper is to present examples on P. The chief method is by the way of constructing "simple" systems on objects topologically the same (i.e., homeomorphic to) as P.

10. Notation. C, N, and P are all as above, Q, R, and Z denote, respectively, the space of rationals, the reals, and the integers. We also use ω to denote the non-negative integers. *card*(X) denotes the cardinality of a set X.

Given a space X and a set I, we use Logic's notation ^{I}X to denote all functions from I to X. $\Pi^{I}X$ denotes the Tychonov product of I many copies of X. Π will be used to denote the Baire space of all sequences <x(n)> of positive integers with the metric $d(x,y) = 2^{-(n-1)}$, if n is the least integer such that $x_n \neq y_n$.

cl and int denote, respectively, the closure and interior operators in a space X. C, N, P, Q, R, and Z all possess natural linear orders and we use [a,b] and (a,b), respectively, to denote closed and open intervals.

A *zero-dimensional* space is a space with a base of *clopen* (≡ simultaneously open and closed) sets. A space X is *separable* provided it has a countable dense set; i.e., there is a countable subset of X whose closure is X. A space is *completely metrizable* provided it is homeomorphic to a complete metric space. A space is *nowhere-locally compact* provided no non-empty open set has comnpact closure.

The *domain* and *range* of a function f are denoted, repectively, by dom and rng. **HENCEFORTH**, we use fx instead of f(x) for the image of a point unless some confusion would result.

§1. A homeomorph of P.

1.1. The usual topology on P. A basic nhbd of a point x has form $(x-\varepsilon,x+\varepsilon)$ (in P of course), where $\varepsilon > 0$. In R there is a rational between any pair of irrationals. Hence, we have alternate basic nhbds of form $(a,b)\cap P$, where a and b are rationals. When p is a rational $(p,\infty)\cap P = [p,\infty)\cap P$. Thus, $(p,\infty)\cap P$ is both open and closed in P. Similarly, $(-\infty,p)\cap P$ is both open and closed in P. As each point in P has a nhbd base of clopen sets in P, P is a zero-dimensional space.

1.2. The usual topology on Π. When we identify sequences of integers with elements of ${}^{N}N$, the set of all functions from N to itself. The topology on Π is the same as the Tychonov product topology on $\Pi^{N}N$, the Tychonov product of countably many copies of the positive integers, when the integers are given the (discrete) metric $d(n,m) = \begin{cases} 0 & \text{if } n = m \\ 1 & \text{if } n \neq m \end{cases}$. Please observe that the topology of Π is given by the complete(!) metric $d(x,y) = 0$ if $x = y$, otherwise $d(x,y) = 2^{-k}$, where k is the first integer n such that $x(n) \neq y(n)$.

When F ç N is finite, and when $B:F \to N$ is a function, $\{y\ \Pi:\ n\ F,\ y(n) = B(n)\}$ is an open set in Π; further, the set of all such sets forms a base for the topology of Π. Therefore, a basic nhbd of $x\ \Pi$ has the form $\{y\ \Pi:\ n \geq m,\ y|[1,m] = x|[1,m]\}$, where m varies in N.

Usually, topology is applied to subjects such as Algebra, Analysis, or Number Theory. The next result is seventy years old and reverses this process. It was one of the most beuatiful results I saw in my graduate topology course at Lehigh University in 1965. Some of its most notable corollaries are that P is homeomorphic to $P \times P$ (the double irrationals in the plane), and that P has a group operation which makes it a topological group.

1.3. Theorem [7]. P and Π are homeomorphic.

Sketch of Proof. Consider continued fractions; that is objects of the form $\cfrac{1}{x(1)+\cfrac{1}{x(2)+\cfrac{1}{x(3)+\ddots}}}$.[3, 4] They define a unique real using the limit of the fractions $\frac{1}{x(1)}$, $\cfrac{1}{x(1)+\cfrac{1}{x(2)}}$, $\cfrac{1}{x(1)+\cfrac{1}{x(2)+\cfrac{1}{x(3)}}}$, … . The function $f(x) = \cfrac{1}{x(1)+\cfrac{1}{x(2)+\cfrac{1}{x(3)+\ddots}}}$ defines a homeomorphism between Π and (0,1)È P. It is easy to see that P and (0,1)È P are homeomorphic. //

[3] There is are several variances on this idea, some allow a non-negative integer x(0) to be added to the above continued fraction. This defines a homeomorphism from (0,∞)È P to P, but adds considerable formulaic considerations to a few of the results in this paper.

[4] Note that when n, $x(n) = 1$, the resulting continued fraction yields $\sqrt{2} - 1$. While the Fibbonacci sequence, starting with the second term yields the golden rule −1. Using finite induction it is easy to prove that the continued fraction is a root of a quadratic equation iff the sequence <x(m)> is constant after some m.

When F ç N is finite iff it is a compact subset of N. Thus, Tychonov Product Theorem shows that when F is finite, $\Pi^N F$ is a compact subset of $\Pi^N N$. When F is finite with at least two points, it can shown that $\Pi^N F$ is homoemorphic to the Cantor middle thirds set C.

§2. A system on P**.** Applying 1.3, we define a function from P to itself by defining it on Π! $\sigma : \Pi \to \Pi$ is defined by $(\sigma x)(n) = x(n+1)$ (so for example, the sequence $\sigma\langle 1,2,3,4, \ldots\rangle = \langle 2,3,4,5, \ldots\rangle$).

For m N, $\sigma^{-1}(y|[1,m]) = \{x \ \Pi : \ n \ [2,m+1], x(n) = y(n-1)\}$, which, according to 1.2, is open. Therefore, σ is a continuous function called the *shift map* (on Π). [5] In this paper, all examples in this paper will concern the system $[\Pi;\sigma]$, and its subsystems.

The first study of the dynamics of (a variant of) $[\Pi;\sigma]$ appears in **[15]**, but when F ç N is finite, and in particular when F has just two elements e.g., 0-1 sequences), the systems $[\Pi^N F;\sigma]$ have been studied for more than 40 years (see **[8]**, **[12]** and **[13]** for more). They are called *symbolic dynamical systems* or *symbolic cascades.*

The next lemma follows directly from the definitions.

2.1. Lemma. For x Π, the following are true in $[\Pi;\sigma]$:

1). x is a fixed point iff n N, $x(n) = x(1)$.

2). x is a periodic point iff m N such that n N, $x(n+m) = x(n)$ iff m N such that k N, $x|[km+1,(k+1)m] = x|[1,m]$ as sequences.

3). x is almost-periodic iff u N, $k = k_u$ N such that m N, $\{n \ N : km \le n < (k+1)m, \sigma^n x|[1,u] = x|[1,u]\} \neq$ iff u N, $k = k_u$ N such that m N, $\{n \ N : km \le n < km+k, x|[n+1,n+u] = x|[1,u]\} \neq$ as sequences.

4). x is recurrent in iff u N, $a \ge u$ such that $\sigma^a x|[1,u] = x|[1,u]$ iff u N, $a \ge u$ such that $x|[a+1,a+u] = x|[1,u]$ as sequences. **//**

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[5] The reader should know there is another notion of "shift" on P, it adds 1 to each irrational and has no interesting dynamics.

Most fundamental to our constructions is the unique element in Π, denoted by α, given by

2.2. $\alpha(n) = k+1$ if k is the largest integer such that 2^k divides n.

Then α is the sequence
<1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,6 ...>.

The next lemma has a straight-forward induction proof.

2.3. Lemma. Suppose m,k N. Then the following hold:
1). If $n < 2^m$, then $\alpha(k2^m+n) = \alpha(n)$.
2). If $k \le i < j \le k+m$, then $\min\{\alpha(i), \alpha(j)\} \le 1+\log_2 m$. **//**

2.4. EXERCISE. Define points x,y Π. $x(n) = n$ n N; $y(n) = k$ if $n = 2^k$ and 1 otherwise.
1). Prove that x and y are not recurrent.
2). Prove that **OC**(x) is countably infinite.
3). Prove that **OC**(y) contains the fixed point <1,1,1,...>.
4). Prove or disprove **OC**(y) is countably infinite.

2.5. Example. There is an almost-periodic point in [Π;σ] which is not periodic.
\\. The point is α. For u N, let $k = 2^i$, where $i = \min\{j : u \le 2^j\}$. Then 2.1(3) and 2.3(1) prove α is almost periodic. Notice that 2.1(2) shows that any periodic point in [Π;σ] has finite range. However, $\alpha(2^m) = m+1$, and so α is not periodic. **//**

2.6. EXERCISE. Define x Π by $x(1) = 1$ and if $2^{n-1} < m \le 2^n$, then $z(m)=2-z(m-2^{n-1})$, so z = <1,2,2,1,2,1,1,2,2,1,1,2,1,2,2,1,...>.
1). Prove that z is almost-periodic.
2). Construct a point p **OC**(α)**Orb**(α) (we shall see in 4.5 that **OC**(α)**Orb**(α) is uncountable). Hint. Consider y = <2,1,4,1,2,1,3,1,2,1,6,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,8,...>.
3). A point x Π is said to be *non-wandering* in [Π;σ] provided n,m N, y Π with $x|[1,m] = y|[1,m] = y|[n+1,n+m]$ considered as sequences. Prove that each point of [Π;σ] is non-wandering.

The following result is well-known (with the same proof) in the case of the systems $[\Pi^N F;\sigma]$ when F ç N is finite.

2.7. Proposition. $[\Pi;\sigma]$ is a transitive system.
\\. Given n N, the set s_n of all sequences of length n is the same as the set of all n-tuples consisting of elements of N. In most elementary Anaylsis classes it is proved that s_n is countable. Since countable unions of countable sets is countable, the set s of all finite sequences in N is countable. Let $\{S_n : n \; N\}$ list the elements of s. Define a point x Π, recursively, by starting with S_1 and adjoining the sequence S_2 at its end. If we have the first m sequences adjoined in this manner, then adjoin S_{m+1} at the end. x will be the point constructed in this fashion. Given any point y Π and u N, then y|[1,u] can be considered as a finite sequence. Thus, we may find a N such that x|[a+1,a+u] = y|[1,u]. Clearly, y **OC**(x). Therefore, $[\Pi;\sigma]$ is transitive. **//**

2.8. Example. There is a recurrent point in $[\Pi;\sigma]$ which is not almost-periodic.
Proof. The point we use is the point x in the proof of 2.7. Notice that there are arbitrarily long constant finite sequences of the form 1+x(1). So for any k N, we may find a,b N such that b > k and n [a,a+b], $x(n) \neq x(1)$. Therefore, x is not almost-periodic. **//**

§3. More homeomorphs of P**.** We will build new dynamical systems on P using two major tools: The first is a theorem, 3.2, considerably expanding 1.3 and some "technology" expanding the essential idea in the proof of 2.7.

The first theorem of this section is completely new and shows why P is so important to Topological Dynamics. It says, "Any odd behavior of a discrete dynamical system on a separable complete metrizable space should be reflected by a system on P."

3.1. Theorem. Suppose $[X,f]$ is a system on a separable complete metrizable space (X,d). Then there is a system $[P,g]$ and a homomorphism $h:[P,g] \to [X,f]$.[13]

\\. n N, and r $^{[1,n]}N$, we will define, recursively, an open set G_r in X, and, if $n>1$, and an r^* $^{[1,n-1]}N$ all subject to the following six conditions:

1). $\{G_r : r \; ^{[1,1]}N\}$ is an open cover of X.

2). If $n>1$ and if $r \; ^{[1,n-1]}N$, then $G_r = Ë\{G_s : s \; ^{[1,n]}N,\ r = s|[1,n-1]\}$.

3). If $n>1$, then the diameter $\delta(G_r)$ of G_r, is at most $\frac{\delta(G_{r|[1,n-1]})}{2}$.

4). If $n>1$ and if $r \; ^{[1,n]}N$, then $cl(G_r)$ ç $G_{r|[1,n-1]}$.

5). If $n>1$ and if $r \; ^{[1,n]}N$, then $f(G_r)$ ç G_{r*}.

6). If $n>2$ and if $r \; ^{[1,n]}N$, then $G_{r|[1,n-1]*} = G_{r*|[1,n-2]}$.

Since X is Lindelöf, there is a countable open cover r of X by sets of diameter 1. Allowing repeats (in the case r is finite), let $\{R_m : m \; N\}$ index r. $r \; ^{[1,1]}N$, define $G_r = R_{r(1)}$. Thus, for $n=1$, the conditions (1)-(6) are satisfied.

Suppose $m>1$ and $n<m$, $r \; ^{[1,n]}N$, we have defined G_r, and if $n>1$, we have defined G_{r*} to satisfy the conditions (1)-(6). Fix $r \; ^{[1,m-1]}N$. $x \; G_r$, (2) finds $t(x) \; ^{[1,m-1]}N$ such that $f(x) \; G_{t(x)}$. So choose an open nhbd U_x ç G_r of x such that $f(U_x)$ ç $G_{t(x)}$. In the case, $m>2$, then, by recursion and (5), $f(G_r)$ ç G_{r*}. Again using (2), $G_{r*} = Ë\{G_s : s \; ^{[1,m-1]}N,\ r^* = s|[1,m-1]\}$. So if $n>2$, we may choose $t(x)$ such that $r^* = t(x)|[1,m-2]$. Now choose an open ball $B(x,\varepsilon_x)$ centered at x with $B(x,\varepsilon_x)$ ç U_x. As X is separable metric, each subspace of X is Lindelöf. Thus, we may choose a countable subset $\{V_k : k \; N\}$ of $\{B(x,\frac{\varepsilon_x}{2}) : x \; G_r\}$ covering G_r. $t \; ^{[1,m]}N$, with $r = t|[1,m-1]$, let $G_t = V_{t(m)}$. Clearly, the conditions (1)-(6) are satisfied.

[13] The analgous result for compact metric spaces and the Cantor set is well-known and can be proved similarly

We may now assume all the construction of the sets G_r and functions r^* are defined n N, and r ${}^{[1,n]}$N, r^* ${}^{[1,n-1]}$N. Suppose π Π. According to (2) and (3), $\{cl(G_{\pi|[1,n]}) : n \ \ N\}$ is a descending family of closed sets with diameters converging to 0. Since X is complete, there is a unique element x_π $\bigcap_{n \ N} cl(G_{\pi|[1,n]})$. Define a function $h : \Pi \to X$ by $h(\pi) = x_\pi$. Notice that given x X, (1) and (2) inductively define π Π such that x $\bigcap_{n \ N} G_{\pi|[1,n]}$. Thus, h is surjective.

We show h is continuous. Suppose π Π and $h(\pi) = x$ X. By (4), $\bigcap_{n \ N} cl(G_{\pi|[1,n]}) = \bigcap_{n \ N} G_{\pi|[1,n]}$. So (3) shows $\{G_{\pi|[1,n]} : n \ \ N\}$ is a nhbd base at x. Hence, $h(\theta) = x_\theta$ $G_{\pi|[1,n]}$ implies m > n such that $G_{\theta|[1,m]} \subseteq G_{\pi|[1,n]}$. Thus, $h(\{\sigma \ \ \Pi : \sigma|[1,n] = \theta|[1,n]\}) \subseteq G_{\pi|[1,n]}$. Therefore, h is continuous.

Now consider π Π. According to (3), (4), (5), and (6), there is a unique π^* Π such that n N, $\pi^*|[1,n] = (\pi|[1,n])^*$. So we may define a function $g : \Pi \to \Pi$ by $g(\pi) = \pi^*$. Let θ Π have $\theta^*|[1,m] = \pi^*|[1,m]$. Clearly, $g(\{\sigma \ \ \Pi : \sigma|[1,m+1] = \theta|[1,m+1]\}) \subseteq \{\tau \ \ \Pi : \tau|[1,m] = \pi^*|[1,m]\})$. Therefore, h is continuous.

Fix π Π. Then $fh(\pi) = f(x_\pi)$ $\bigcap_{n \ N} G_{\pi^*|[1,n]} = \{x_{\pi^*}\}$. However, $hg(\pi) = x_{\pi^*}$. Therefore, h is a homomorphism. //

3.2. Corollary [17]. A space X is homeomorphic to P iff it is a separable zero-dimensional completely metrizable nowhere-locally compact space.

\\. Only if. Recall Q is the set of rational numbers. Fix p P, and let Q= {p+q : q Q}. Between each pair of reals is a member of Q, so given r,s P, q Q such that q is between r−p and s−p. But p+q is between r and s; thus, Q witnesses that P, and any homeomorphs, is separable. As any open set of P contains a sequence converging to a missing rational, each compact set in P has empty interior; hence, P is nowhere-locally compact. P is not complete, but N and hence, Π is - see any standard topology text; e.g., **[7]**. Thus, P, and any homeomorphs, is completely metrizable.

If. Assume X is a separable zero-dimensional nowhere locally compact complete metric space. Then X has a base c consisting of clopen sets. Given C r , C is complete since it is closed in X. C is not compact since it is open and all compact sets in X have empty interior. However, C is Lindelöf; hence, it is the union a countably infinite family of pairwise disjoint clopen subset s (C) ç c of sets with diameter at most half the diameter of C. Thus, within the proof of 3.1, we can require an extra condition:

7). If r,s $^{[1,n]}$N, r ≠ s, and if r = s|[1,dom (r)−1] = s|[1,dom (s)−1], then G_s È G_t = .

Thus, the function h : Π → X is an injection. Given r $^{[1,n]}$N. Then (7) also implies h({π Π : π|[1,n] = r}) = G_r . Therefore, h is a homeomorphism. //

3.3. Lemma. A closed subset A of Π is compact iff it is bounded in the pointwise product partial order.

Proof. If. Suppose A is a bounded closed subset A of Π; i.e, there is f Π such that A ç K = $\Pi_{n \quad N}$[1,f(n)]. As K is the product of finite sets, K is compact. Since A is a closed subset of K, it is compact.

Only if. Suppose K is a compact set in Π. For m N, K projects on to the mth-coordinate as a compact, and hence finite subset of N with maximum f(n). As K ç $\Pi_{n \quad N}$[1,f(n)], A is bounded. //

Here is the tool we spoke about at the beginning of this section.

3.4. Theorem. A non-empty closed subspace X of Π is homeomorphic to P iff each non-empty open set of X is unbounded in the pointwise product partial order.

\\. Since, from 3.2, X is nowhere-locally compact, 3.3 immediately implies the properties above. Conversely, suppose X ç Π has the properties above. Clearly, each subspace of Π is zero-dimensional separable metric, so X is. 3.3 shows X is nowhere locally compact iff each of its non-empty open subsets are unbounded. Therefore, 3.2 applies to prove the result. //

§4. Constructing points in Π. Here we introduce our second tool used to build and discuss points in Π, we need some special notions about finite sequences in N. A *block* will be a finite function with domain dom B an (a possibly empty) interval in N, and range rng B a subset of N. All blocks will be assumed to have as domain an initial segment of N unlesss otherwise stated. If the block B has (non-) empty domain, we write B = (B ≠). The length λ B = *card* (dom B). If B ≠ is a block, if a ≤ b N, and if dom B = [a,b], then by a *tail* (a *head*) of B, we mean any block of the form B|[c,b] (respectively, B|[a,c]), where a ≤ c ≤ b.

4.1. A partial order on blocks. Suppose A and B are blocks. We will say A is a *copy* of B and write A ≡ B provided the following two conditions are satisfied:

4.1(1). λA = λB and

4.1(2). œ z N such that n dom A, A(n) = B(n+z).

So ≡ is an equivalence relation on the set of all blocks. We will also need a partial order on the equivalence classes: Let us agree that A is *in* B and write A ≤ B provided there is an interval I ç dom B such that A ≡ B|I.

We can expand these ideas to points in Π. For x. Π, we say A is a block in x and we write A ≤ x provided there is an interval I ç N such that A ≡ x|I. Suppose B is a block and I ç N is an interval. We say that x|I is a *maximal tail* of B in x provided x|I is a tail of B and for each interval J◊ I, x|J ≤ B.

4.2. A finitary operation on classes of blocks: Now suppose I and J are finite intervals in N, f is a function with dom f = I and rng f = J. Further, suppose $(B_z : z \ \ J)$ is a sequence of blocks. Let us agree that $f(B_z : z \ \ I)$ or $(B_{fz} : z \ \ J)$, will denote the unique (up to the equivalence ≡) block obtained by allowing z, min I ≤ z < max I, B_{fz} to be immediately followed by $B_{f(z+1)}$. When f : [a,b] → [a,b] is the identity function, we also let $B_a \ \ B_{a+1} \ \cdots \ B_b$ denote $f(B_z : z \ \ [1,n])$.

4.3. EXERCISE. Suppose $n \;\; N$ and $(B_z : z < 2^n)$ is a family of blocks. Prove that $\lambda\alpha(B_z : z < 2^n) = \sum_{k=1}^{n} 2^{n-k}\lambda B_k$.

4.4. Lemma. For $x \;\; \Pi$, **OC**(x) is homeomorphic to P provided that for each block A in x, there is a $m \;\; N$ such that

$$L = \{\text{rng}\, C : \lambda C = m \text{ and } A \;\; C \text{ is a block in } x\} \text{ is infinite.}$$

\\. Given $a < b \;\; N$ let $G_{a,b} = \{y \;\; \mathbf{Ob}(x) : y|[1,b-a] \equiv x|[a,b]\}$. Then $G_{a,b}$ is open in **OC**(x) and each open set in **OC**(x) contains a $G_{a,b}$ for suitable $a < b \;\; N$.

Let $c = \{\text{blocks } C : \lambda C = m \text{ and } A \;\; C \text{ is a block in } x\}$. Since L is infinite, there is a first $k \leq m$ such that $\{C(k) : C \;\; c\}$ is unbounded. As $\;\; C \;\; c, \lambda C = m$, $\{C|[1,k) : C \;\; c\}$ is finite. Without loss of generality, we may assume that *card* $(\{C|[1,k) : C \;\; c\}) = 1$ and $\;\; C \;\; c$, $C = C|[1,k]$. Suppose $C \;\; c$, Let $A = x|[u,v]$ and $w = v+k$. Then $G_{u,w}$ **ç** $G_{u,v}$. But if $A = x|[a,b]$, then $G_{u,v} = G_{a,b}$ and $\{y(b\text{-}a+k) : y \;\; G_{a,b}\}$ is unbounded. Hence, $G_{a,b}$ is unbounded. According to 3.4, **OC**(x) is homeomorphic to P. **//**

4.5. Corollary. For $x \;\; \Pi$, **OC**(α) is homeomorphic to P.

\\. According to 2.3(1), $\;\; m \;\; N$, $\alpha|[1,2^m) \equiv \alpha|[2^m, 2^{m+1})$. Since $\alpha(2^{m+1}) = m$, the hypothesis of 4.4 is satisfied. **//**

A *street* is a sequence $S = (B_z : z \;\; N)$ of blocks indexed by N and such that $\;\; z \;\; N$, $B_z \neq \;\;$. An infinitary operation on streets: Suppose $x \;\; \Pi$. and $S = (B_z : z \;\; N)$ is a street. Then

4.6. xS denotes the unique element y of Π such that if $m \;\; N$, if $f = x|[1,m)$, and if $s = \sum_{z=1}^{m-1} \lambda B_{fz}$, then $y|[1,s] = f(B_z : z \;\; [1,m))$.

4.7. Lemma. Suppose $S = (B_z : z \; N)$ is a street, $f \; {}^N N$ is an increasing function, and $k \; N$ has

$$I = \{z \; N : k \leq \lambda B_z , B_z(k) \geq f(z)\}$$

infinite. Then **OC**(αS) is homeomorphic to P.

\\. Suppose A is a block in $x = \alpha S$. Let $i \; I$ be such that $A \leq \alpha(B_w : w \; [1,2^i))$, say $A = x|[a,b]$ for $1 \leq a \leq b < s = \sum_{w \leq 2^i} \lambda B_{\alpha(w)}$.

$z \; I$ with $z \geq i$, let $C_z = x|[s-b,s] \; B_z|[1,k]$. From 2.3, $\{\text{rng}\, C_z : z \; I, \; A \; C_z$ is a block in $x\}$ is infinite. For $m = k+s-b$, the hypothesis of 4.4 is satisfied. Therefore, **OC**(αS) is homeomorphic to P. **//**

§5. Recurrent points in [Π;σ].

5.1. Lemma. Suppose φ is recurrent in [Π;σ] and S is a street. Then φS is recurrent in [Π;σ].

\\. Let $x = \varphi S$. Fix $m \; N$ and let $p > m$ such that $x|[1,m]$ is a head of $\varphi(B_z : z \leq p)$. As φ is recurrent, 2.1(4) finds $k > p$ such that $\varphi|[k+1,k+p] \equiv \varphi|[1,p]$. Hence, $\varphi(B_z : k+1 \leq z \leq k+p) \equiv \varphi(B_z : z \leq p)$. Therefore, $x|[1,m]$ is a head of $\varphi(B_z : k+1 \leq z \leq k+p)$. From 2.1(4), x is recurrent. **//**

5.2. Theorem. Suppose x Π. x is recurrent in [Π;σ] iff there is a street S such that $x = \alpha S$.

\\. The "if" is a consequence of 5.1 once we see α is recurrent in [Π;σ]. But 2.3(1) shows α is almost-periodic!

Only if. Conversely, suppose x is recurrent in [Π;σ]. By induction, we construct a street $(B_z : z \; N)$. Let $B_1 = x|\{1\}$, and suppose $m \; N$ is such that $z \leq m$, the following hold:

1). B_z is defined (and hence, $z \; [1,2^m)$, $B_{\alpha z}$ is defined).

2). $x|[1,s] \equiv f(B_z : z \quad [1,2^m]))$, where $s = \sum_{z=1}^{2^m-1} \lambda B_{\alpha z}$, and

$f = \alpha|[1,2^m])$.

Since x is recurrent, we may find a smallest $b > s$ such that $x|[1,s] \equiv x|[b+1,b+s]$. Now define $B_{m+1} = x|[s+1,b]$. As it is clear that we can proceed, in the above fashion, defining B_z and exhausting x, condition (2) shows $x = \alpha(B_z : z \quad N)$. **//**

Unlike 5.2, for recurrence, **I know of no interesting characterization for almost periodic points in** $[\Pi;\sigma]$. However, the 5.3 and 5.4 below exhibit what is known at present.

5.3. Theorem. Suppose $S = (B_z : z \quad N)$ is a street.

1). If $\{\lambda B_z : z \quad N\}$ is finite and if φ is almost-periodic in $[\Pi;\sigma]$, then φS is almost-periodic in $[\Pi;\sigma]$.

2). If $x = \varphi S$ is almost-periodic in $[\Pi;\sigma]$ and if $\quad m \quad N$ such that $Z = \{z \quad N : \alpha(B_w : w \quad [1,2^{m+1})) \text{ is not a block in } B_z\}$ is infinite, then $L = \{\lambda B_z : z \quad I\}$ is finite.

\\. 1). Let $x = \varphi S$ and suppose $u \quad N$ is an upper bound for $\{\lambda B_z : z \quad N\}$. Fix $m \quad N$ and let $p > m$ such that $x|[1,m]$ is a head of $\varphi(B_z : z \leq p)$. As φ is almost-periodic, 2.1(3) finds $k = k_p \quad N$ such that

$$\quad q \quad N,\ \{n \quad [kq,kq+k) : \varphi|[n+1,n+p] \equiv \varphi|[1,p]\} \neq \quad .$$

Hence,

$$\quad q \quad N,\ \{n \quad [kq,(k+1)q) : \varphi(B_z : n+1 \leq z \leq n+p) \equiv \varphi(B_z : z \leq p)\} \neq \quad .$$

Therefore, $\quad q \quad N,\ \{n \quad [kuq,(ku+1)q) : \varphi|[n+1,n+p] \equiv \varphi|[1,p]\} \neq \quad$, and, according to 2.1(3), x is almost-periodic in $[\Pi;\sigma]$.

2). Suppose L is infinite and $k \quad N$. We may choose $z \quad Z$ such that $\lambda B_z \geq k + \lambda\alpha(B_w : w \quad [1,2^{m+1}))$. But then B_z is a block in x of length greater than k failing to contain a copy of $\alpha(B_w : w \quad [1,2^{m+1}))$. From 2.1(3), x is not almost-periodic. **//**

5.4. Example. An almost periodic-point x such that if $(S_z : z \quad N)$ is a street with $x = \alpha(S_z : n \quad N)$. Then $\{\lambda S_z : z \quad I\}$ is infinite.

11. Define $B_1 = \langle 1 \rangle$, and for $n \geq 2$, define

1). $B_n = \langle n \rangle \ \ \alpha(B_k : k < 2^n) \ \ \langle n \rangle$..

Let $\beta = \alpha(B_n : n \ \ N)$. Then the first few terms of β are $\langle$1,2,1,2,1,3,1,2,1,2,
1,3,1,2,1,2,1,4,1,2,1,2,1,3,1,2,1,2,1,3,1,2,1,2,1,4,1,2,1,2,1,3,1,2,1,2,1,3,1,2,1,2,1,5, ...$\rangle$.

Clearly, 2.2, 4.3, and an elementary finite induction argument shows the next two statements

2). If $n > 1$, then $\lambda B_n = 2{\cdot}3^{n-1}{+}1$ and $\lambda\alpha(B_k : k < 2^n) = 2{\cdot}3^{n-2}{-}1$.
3). $\ \ n,m \ \ N$, $\alpha(B_k : k < 2^n) \leq \beta|[m{+}1, m{+}4{\cdot}3^{n-1}{-}1]$.

Acording to 2.1(3), (3) shows β is almost periodic.
Let $t \ \ N$ and let $(S_z : z \ \ N)$ be a street with $\beta = \alpha(S_z : z \ \ N)$ and $\ \ z \ \ N$, $\lambda S_z \leq t$. Choose $m \ \ N$ to be the first integer with $t \leq 2{\cdot}3^{m-1}$. Choose $r \ \ N$ to be the first integer with $m \ \ \text{rng}\, S_r$. Let $R_1 = \alpha(S_z : z < 2^{r-1})$, and $A_1 = \alpha(B_n : n < 2^{m-1})$. Since $m \ \ \text{rng}\, R_1$, we have
4). R_1 is a head of A_1 which is a head of β.

So (4) shows

5). $A_1 \ \ \langle m \rangle$ is a head of $R_1 \ \ S_r$ which is a head of β.

From (1), $A_1 \ \ \langle m \rangle \ \ A_1$ and $A_1 \ \ \langle m \rangle \ \ A_1 \ \ \langle m \rangle$ are heads of β. As $\lambda S_r \leq t$, (2) shows $\langle m \rangle$ appears in S_r but once. Using 2.3, 4.6, and (4), we have

6). $R_1 \ \ S_r \ \ R_1$ is a head of $A_1 \ \ \langle m \rangle \ \ A_1$.

From (6) there is a second integer $s \ \ N$ with $m \ \ S_s$. Let $R_2 = \alpha(S_z : z < s)$, and $A_2 = A_1 \ \ \langle m \rangle \ \ A_1 \ \ \langle m \rangle \ \ A_1$. Clearly,

7). $A_1 \ \ \langle m \rangle \ \ A_1 \ \ \langle m \rangle$ is a head of $R_1 \ \ S_r \ \ R_1 \ \ S_s$ which is a head of β.

Now (5) and (7) show A_1 <m> A_1 <m> A_1 <m> is a subsequence of $T = R_1\ S_r\ R_1\ S_s\ R_1\ S_r\ R_1$. Since $z \le s$, $\lambda S_z \le t$ and $\mathrm{rng}\,S_z$ is bounded by m, $\mathrm{rng}\,S_z$ is bounded by m. Since T is a head of β, T must be a head of A_2. Thus, <m,m,m> is a subsequence of which contradicts (1). //

5.5. Lemma. Suppose that $S = (B_z : z\ \ N)$ is a street, y **OC**(αS), and suppose $\lambda\ \ N$ such that for each block F with $\lambda F = \lambda$, $\{z\ \ N : F$ is a head or tail of $B_z\}$ is finite. If y is not recurrent in $[\Pi;\sigma]$ or if **OC**(y) $\ne$ **OC**(αS), then h Π such that z N, $y|[1,z] \le B_{hz}$.

\\. Let $x = \alpha S$. For simplicity, we will use α when we restrict its domain. Suppose m N, we define h(m).

Since y **OC**(x), m N, k N $y|[1,m] = \sigma^k x|[1,m]$, so we may choose functions $a,b : N \to N$ such that

1). $a(m) \le b(m)$, and

2). m N, $b(m) - a(m)$ is minimal with respect to $(y|[1,m]) \le \alpha(B_z : a(m) \le z \le b(m))$.

Now, m N, find c(m) N and (possibly empty) blocks H(m) and T(m) such that

3). H(m) (y|[1,m]) is a head of $\alpha(B_z : a(m) \le z \le c(m))$,

4). T(m) is a tail of $\alpha(B_z : c(m) \le z \le b(m))$, and

5). $a(m) \le c(m) \le b(m)$ and H(m) (y|[1,m]) $T(m) \equiv \alpha(B_z : a(m) \le z \le b(m))$.

Notice that the conditions (2) to (5) implies

6). $\alpha(B_z : a(m) < z < b(m)) \le (y|[1,m])$.

From (6), **œ** $f,g : N \to N$ with m N, $(y|[f(m),g(m)]) \equiv B_{\alpha c(m)}$. According to the hypothesis of this lemma (concerning heads and tails of B_z), we have one of two possibilities:

7). either there is an infinite N **ç** N such that both f|N and g|N are monotone, or

8). there is an infinite N **ç** N such that (αc)|N is constant.

Of course if (7) holds, we are done - just set h(m)=αc(i_m), where the natural indexing of N is {i_m : m N}. So we assume (8) is true. From the hypothesis, we have just two possibilities for the functions c−a and b−c.

CASE1. For each infinite I **ç** N, both rng(c−a)|I and rng(b−c)|I are unbounded.

CASE2. There is an infinite I **ç** N, such that at least one of rng(c−a)|I and rng(b−c)|I is bounded, while the other is unbounded.

Assume CASE1 holds, and suppose k N is arbitrary. As y **OC**(x), p N such that x|[p+1, p+k]=y|[1,k]. Thus, we may find k_1 N such that

9). both x|[1,k] and y|[1,k] are in α(B_z : z ≤ k_1).

Let I={m N : 2^{k_1+1}+1 < c(m)−a(m)}. Since CASE1 holds, I is infinite. Again applying CASE1 shows J={m I : 2^{k_1+1}+1 < b(m)−c(m)} is infinite. Now choose j J. Using (6), 2.3(1) shows

10). α(B_z : z ≤ k_1) ≤ y|[1,j].

Applying (6) to (10), we see x|[1,k] and y|[1,k] are in y|[1,j]. It is now clear that q,r N such that σ^qy|[1,k]=x|[1,k], and σ^ry|[1,k]=y|[1,k]. As k N is arbitrary, we see, respectively, that **OC**(y)=**OC**(x) and so y is recurrent.

Assume CASE2 is true - say rng(c−a)|I is bounded. Then, without loss of generality, we may assume:

11). $(c-a)|I$ is constant, c_0 N such that m I $c(m) = a(m)+c_0$.

From (8), (11), and 2.3, j $[0,c_0]$ such that i $[0,c_0)\setminus\{j\}$,

$\alpha(c+i)|(I \,\grave{E}\, [k,\infty))$ is constant for $k = 1+\log_2(\sum_{z=1}^{c_0}\lambda B_z)$. From (3), $j = 0$.

So b_0 N such that m > k, m I, $y|[1,b_0] \equiv \alpha(B_z : a(m) < z \le c(m))$. Therefore, there is an increasing $d : I \to N$ and an infinite valued $e : I \to N$ such that $(y|[-d(m),a_0))$ is a tail of $B_{\alpha ce(m)}$ - a contradiction.

The proof, in case rng $(b-c)|I$ is bounded, is similar. **//**

5.6. Lemma. Suppose $S = (B_z : z \;\; N)$ is a street satisfying

#). for $n < m$ N, rng $B_n \,\grave{E}\,$ rng $B_m =$.

Then **OC**(αS) is minimal, and y **OC**(αS) iff y can be written as

$$(T \;\; \alpha(B_z : z < h(1))) \qquad {}_{n>1}(B_{\alpha(h(n))} \;\; \alpha(B_z : z \;\; [1,h(n)))),$$

where h Π is increasing and T is a maximal tail of $B_{\alpha(h(1))}$ in y.

\\. To see that [**OC**(αS);σ] is minimal, suppose x **OC**(αS). As the B_n's are pairwise-disjoint, the hypothesis of 5.5 is satisfied. So if **OC**(x) $\neq$ **OC**(αS), then 5.5 finds an h Π such that n N, $x|[1,n] \le B_{h(n)}$. Clearly, h is constant, say $h(1) = m$. Choose $n > 2^m$, then $n = \lambda x|[1,n] \le \lambda B_m \le 2^m$ - ridiculous. So **OC**(x) = **OC**(αS). Therefore, [**OC**(αS);σ] is minimal.

<u>IF</u> Just check (see exercise 2.6(2), where T = <2>) that $k \ge 0$ such that

$$\sigma^k(\alpha S) = (T \;\; \alpha(B_z : z < h(1))) \;\; (\quad {}_{1<n<m}(B_{\alpha(h(n))} \;\; \alpha(B_z : z \;\; [1,h(n))))) \quad B_{\alpha(h(m))} \;\; \alpha(B_z : z > m).$$

<u>Only if</u>. Suppose y **OC**(αS). Then a **Orb**(αS) and z,n N with $B_z(n) = a(1) = y(1)$. Choose the first h(1) N such that $\alpha(h(1)) = z$. Let $p = \lambda B_z-(n-1)+\lambda\alpha(B_k : k < h(1))$. (#) implies whenever x **Orb**(αS) and $x(1) = B_z(n)$, we have $x|[1,\lambda B_z-(n-1)] = B_z|[n,\lambda B_z]$. Since we may assume $a|[1,p] = y|[1,p]$, there is a tail T of B_z which is a head of y. WLOG we may assume T is a maximal tail of B_z in y. Further, (#) and 2.3(1) implies whenever x **Orb**(αS) and T is a head of x, then

$T \ \alpha(B_k : k < h(1))$ is a head of x. So $T \ \alpha(B_k : k < h(1)) = a|[1,p]$ is a head of y.

Suppose $m \ N$ and $n < m$ we have defined $h(n)$ so that $h|[1,m)$ is an increasing sequence and

$$*). \ H = (T \ \alpha(B_z : z < h(1))) \qquad {}_{1 < n < m}(B_{\alpha(h(n))} \ \ \alpha(B_z : z \ \ [1,h(n))))$$

is a head of y. Let $q = 1 + \lambda H$. Then b **Orb**(αS) with $b|[1,q] = y|[1,q]$. Find the first $u \ N$ with $<b(q)> \leq B_{\alpha(u)}$. As $<\alpha(h(m-1))> \ \alpha|[1,h(m-1))$ is a block in α, 2.2, (#), and (*) imply $u > h(m-1)$. Let $h(m) = u$. Since b **Orb**(αS), we can again apply (#) to show $b(q) = B_{\alpha(u)}(1)$. Now follow the methods of the previous paragraph to show

$$(T \ \alpha(B_z : z < h(1))) \qquad {}_{1 < n \leq m}(B_{\alpha(h(n))} \ \ \alpha(B_z : z \ \ [1,h(n))))$$

is a head of y. Thus, the construction of h is completed by recursion. y clearly satisfies the conclusion. **//**

§6. Systems on P with all points recurrent. We begin with two standard and easy to prove results in Topological Dynamics. Note 6.1 is usually stated for the compact case, but x is remains almost-perodic in $[\beta X, \beta f]$, where βX is the Stone-Čech compactification of X and βf is the extension of f to βX.

6.1. If x is almost-periodic in a system $[X;f]$, then **OC**(x) is minimal in $[X;f]$ and each point of **OC**(x) is almost-periodic.

6.2. If $[X;f]$ is a minimal system, then each point of $[X;f]$ is recurrent.

Our first example actually lead to the discovery of α. Please contrast it with 6.1.

6.3. Example. There is a continuous function $f: P \to P$ such that $[P;f]$ is a minimal system, and no point of P is almost-periodic in $[P;f]$.

\\. n N, let $[2^{n-1}, 2^{n-1}+n-1]$ be also considered a sequence B_n in its natural order. Let $x = \alpha(B_n : n \ N)$. According to 4.7, **OC**(x) is homemorphic to P. 5.6 shows [**OC**(x);σ] is minimal.

According to 6.1 it suffices to show that x is not almost-periodic. $U = \{y \ \Pi : y(1) = 1\}$ is a neighborhood of x. Of course, $\sigma^n x \ U$ iff $x(n+1) = 1$ iff $z \neq 1$ implies $x(n+1) \ \text{rng}\, B_z$. Fix m N. $z \geq m$, $\text{dom}\, B_z$ contains an interval I of length m such that 1 $\text{rng}\, B_z|I$. Therefore, 2.1(4) shows x cannot be almost-periodic. **//**

The next example is an expansion of the idea of proof of 5.4 and should be contrasted with 6.2.

6.4. Example. There is a continuous function $f : P \to P$ such that each point in P is recurrent in [P;f], but [P;f] has no minimal sets.
\\. n N, define
$I_n = \{2^{m-1}(2n-1) : m \ N\}$. During the remainder of this proof we will consider blocks as either functions or ordered sequences. n N, let B_1^n denote the one-element sequence <2n−1>, and define, recursively

1). m N, $n > 1$, $m \neq 0$,

$$B_m^n = \langle 2^{m-1}(2n-1)\rangle \ \alpha(B_z^{n+1} : z < m) \ \langle 2^{m-1}(2n-1)\rangle.$$

Here are samples: $B_1^1 = \langle 1\rangle$, $B_1^2 = \langle 3\rangle$, $B_2^1 = \langle 2,3,2\rangle$, $B_1^3 = \langle 5\rangle$, $B_2^2 = \langle 6,5,6\rangle$, $B_3^1 = \langle 4,3,6,5,6,3,4\rangle$, $B_1^4 = \langle 7\rangle$, $B_2^3 = \langle 10,7,10\rangle$, $B_3^2 = \langle 12,5,10,7,10,5,12\rangle$,
$B_4^1 = \langle 8,3,6,5,6,3,12,5,10,7,10,5,12,3,6,5,6,3,8\rangle$.

We claim the following is true:

2). n,m N, $\text{rng}\, B_m^n$ **ç** $\{2^{m-1}(2k-1) : n \leq k \leq n+m\}$.

Certainly (2) is true when $m = 1$. Fix n N and suppose (2) is true m, i, m N, $i < n$. If $z < m$, then 2.3(2) shows $\alpha z \le \log_2 m < m$. Hence, by induction, $\mathrm{rng}\, B^{n+1}_{\alpha z}$ ç Ë$\{2^{m-1}(2k-1) : n+1 \le k \le n+m\}$. Because $2^{m-1}(2n-1)$ is the only element of $\mathrm{rng}\, B^{n}_{m}$ not considered by the induction hypothesis, $\mathrm{rng}\, B^{n}_{m}$ ç $\{2^{m-1}(2k-1) : n \le k \le n+m\}$.

We claim the following is true:

3). $n \ge 0$, m, z N, $m \ne z$, no head or tail of B^{n}_{m} is a head or tail of B^{n}_{z}.

Suppose n, m, z N, $z < m$. If $k \ne z$, then $2^{z-1}(2k-1)$ I_k. So (2) shows, $2^{z-1}(2n-1)$ $\mathrm{rng}\, B^{n}_{m}$. As $\langle 2^{z-1}(2n-1)\rangle$ is both a head and a tail of B^{n}_{z}, (3) is true.

n N, define $x_n = \alpha(B^{n}_{z} : z$ N$)$. From 5.2, x_n is recurrent in $[\Pi;\sigma]$. Since m N, $m \le 2^{m-1}(2n-1) = B^{n}_{m}(1)$, 4.7 proves:

4). n N, **OC**(x_n) is homeomorphic to P.

n, p N, let $G_p = \{y$ $\Pi : y|(1,p) = \alpha(B^{n+1}_{z} : z < p)\}$. Then $\{G_p : p$ N$\}$ forms a nhbd base at x_{n+1}. But (1) shows that p N, m N such that $\alpha(B^{n+1}_{z} : z < p) \le B^{n}_{m} = x_n|I$ for some interval I of N. So
q N
with $\sigma^q x_n$ G_p ***È*****Ob**(x_n). Thus, x_{n+1} **OC**(x_n). Similarly, r N, $\sigma^r x_{n+1}$ **OC**(x_n). Further, since $x_n(1) = 2n-1$ $\{2^{m-1}(2k-1) : k > n\}$, (2) shows x_n **OC**(x_{n+1}). Hence, we have:

5). n N, **OC**(x_{n+1})√ **OC**(x_n).

Since $n > 1$, $n < 2n-1$. (2) implies that $n > 1$, $n < \min \mathrm{rng}\, x_n$. Hence, the following holds:

6). k N, ***È***$_{n > k}$ **OC**$(x_n) =$.

Now fix n N. Suppose that y **OC**(x_n) is not recurrent or satisfies **OC**(y) ≠ **OC**(x_n). According to (3), 5.5 finds an h Π such that z N, $y|[1,z] \le B_{hz}$. So m N, $y|[1,m+1] \le B^{n}_{h(\alpha(m))}$. Applying (1), $y|[1,z] \le \alpha(B^{n+1}_{h(\alpha(m))}$: $1 \le z < h(\alpha(m)))$. Hence, y **OC**(x_{n+1}). From (6):

7). y **OC**(x_0), y is stable, and $n \ge 0$ such that **OC**(y) = **OC**(x_n).

Finally notice that (5), (7), and 6.1 prove **OC**(x_0) contains no minimal sets. Since (4) shows **OC**(x_0) is homeomorphic to P, **OC**(x_0) satisfies our requirements. **//**

§7. Multiple Recurrence. In this section we are concerned with multiple systems - systems comprising of more than one map from a space to itself. One of the earliest results on multiple systems is due to P. Erdös and A. Stone **[7b]**:

7.1. x is recurrent (almost-periodic) in a system [X;f] iff x is jointly recurrent (almost-periodic) in the system [X;{f^n : n N}].

More recent is the Furstenberg-Weiss theorem **[8]** (improved in **[4]**):

7.2. If f is a finite family of commuting maps on a compact metric space X, then [X; f] has a multiply recurrent point.

7.3. Lemma. The following are true:
1). x is multiply recurrent in [Π,{σ^1, ... ,σ^n}] iff m N, $k = k_m$ N such that $p \le n$, $x|[kp+1,kp+m] \equiv x|[1,m]$
2). x is multiply recurrent in [Π,{σ^n : n N}] iff m N, $k = k_m$ N such that n N, $x|[kn+1,kn+m] \equiv x|[1,m]$.
3). If x is multiply recurrent in [Π,{σ^n : n N}], then x is almost-periodic in [Π;σ].
\\. (1) and (2) are immediate from the definition. Notice that (2) implies that for m N, we can choose $k > m$ in the conclusion; hence, n N, $x|[1,m] \le x|[nk+1,(n+1)k]$. So (3) holds. **//**

7.4. Theorem. Suppose $(B_z : z \;\; N)$ is a street and $<\lambda B_z : z \;\; N>$ is a constant sequence. If φ is multiply recurrent in $[\Pi,\{\sigma^1, \ldots ,\sigma^n\}]$ (in $[\Pi,\{\sigma^n : n \;\; N\}]$), then $\varphi(B_z : z \;\; N)$ is multiply recurrent in $[\Pi,\{\sigma^1, \ldots ,\sigma^n\}]$ (in $[\Pi,\{\sigma^n : n \;\; N\}]$).

\\. The proof is analogous to the proof of 5.3(1). **//**

7.5. Corollary. Suppose $(B_z : z \;\; N)$ is a street with $<\lambda B_z : z \;\; N>$ a constant sequence. Then $\alpha(B_z : z \;\; N)$ is multiply recurrent in $[\Pi,\{\sigma^n : n \;\; N\}]$.

\\. $\;\; m \;\; N$, $\alpha|[1,2^{m-1}-1] = \alpha|[n \cdot 2^m+1, n \cdot 2^m+2^{m-1}-1] = (\sigma^n)^{2m}\alpha|[1,2^{m-1}-1]$. So α is multiply recurrent in $[\Pi,\{\sigma^n : n \;\; N\}]$. Now use 7.2. **//**

To see that 7.5 cannot be reversed try (1) of the next exercise. (2) shows that almost-periodic points are not necessary to get points multiply recurrent in each $[\Pi,\{\sigma^1, \ldots ,\sigma^n\}]$.

7.6. EXERCISE. 1). Prove β, the point defined in 5.4, is multiply recurrent in $[\Pi,\{\sigma^n : n \;\; N\}]$.
2). Prove that the point x defined in 6.3 is, $\;\; n \;\; N$, a multiply recurrent point in $[P,\{f^1, \ldots ,f^n\}]$. Are all of the points in **OC**(x) multiply recurrent?

7.7. Example. There is a continuous function $f: P \to P$ such that $[P,f]$ is minimal and $\;\; y \;\; P$, y is not multiply recurrent in $[P;\{f,f^2\}]$.

\\. Let $B_1 = \lhd 1>$. Define $x = \alpha(B_z : z \;\; N)$, where B_z is the constant function z of recursively defined length $\sum_{k=1}^{z-1} 2^{z-k}\lambda B_k$. The first "few" terms of x are $\lhd 1,2,2,1,3,3,3,3,1,2,2,1,4,4,4,4,4,4,4,4,4,4,4,4,1,2,2,1,3,3,3,3,1,2,2,1, \ldots >$. Notice that $\lambda B_{\alpha(n)} = \lambda(B_z : \alpha(z) < n)$ (see exercise 4.3).

From 3.2 **OC**(x) is homeomorphic to P. 5.6 shows [**OC**$(x);\sigma$], and hence [**OC**$(x);\{\sigma,\sigma^2\}$], is minimal.

Suppose y **OC**(x). According to 5.6, y can be written as

$$((T \;\; \alpha(B_z : z < h(1))) \qquad {}_{n>1}(B_{\alpha(h(n))} \;\; \alpha(B_z : z \;\; [1,h(n))))),$$

where h Π is increasing and T is a maximal tail of $B_{\alpha(h(1))}$ in y. Let $t = \lambda T$ and suppose p N satisfies $T \equiv y|[p+1,p+t]$. Since the B_z's have disjoint ranges, $p > \lambda(T$ $\alpha(B_z : z < h(1))$ $B_{\alpha(h(2))})$. Suppose m N is the largest integer such that $p \geq \lambda H$, where H is the block

$((T$ $\alpha(B_z : z < h(1)))$ $($ $_{1<n<m}(B_{\alpha(h(n))}$ $\alpha(B_z : z$ $[1,h(n))))))$ $B_{\alpha(h(m))}$.

Then $\lambda B_{\alpha(h(m))} < p+1$. Since T and $B_{\alpha(h(m)+1)}$ have disjoint ranges,

$$p+1 < \sum_{k=1}^{m} 2^{m-k}\lambda B_{\alpha(n)} = 2\cdot\lambda B_{\alpha(h(m))}.$$ Thus,

$$\lambda(H)+\lambda B_{\alpha(h(m))} < 2p+1 \leq 3\cdot\lambda B_{\alpha(h(m))} = \lambda B_{\alpha(h(m)+1)} \leq \lambda B_{\alpha(h(m+1))}.$$

So $x(2p+1)$ $\operatorname{rng} B_{\alpha(h(m+1))}$. Since T and $B_{\alpha(h(m)+1)}$ have disjoint ranges, $\sigma(y(p)) = x(p+1) \neq x(2p+1) = \sigma^2(y(p))$. Therefore, y is not multiply recurrent in $[\mathbf{OC}(x);\{\sigma,\sigma^2\}]$. / /

7.8. CONJECTURES.

1. Suppose x is an almost-periodic point in $[\Pi,\sigma]$ and suppose f ç $\{\sigma^n : n$ $N\}$ is finite. Then $[\mathbf{OC}(x), f\,]$ has a multiply recurrent point.
2. There is a completely metrizable space X and an almost-periodic point x in the system $[X,f]$ such that each point of X fails to be multiply recurrent in $[X; f\,]$ for some finite f ç $\{f^n : n$ $N\}$.

7.9. Example. There are commuting homeomorphisms $f,g : P \to P$ such that $[P,\{f,g\}]$ is minimal, p P, $\mathbf{OC}_f(p)$ È $\mathbf{OC}_g(p) = \{p\}$.

Proof. Let P be $\mathbf{OC}(\alpha)$ in Π, where α is defined in 2.2. 4.4 shows P is homeomorphic to P. From 3.2, P^2 is homeomorphic to P. Let $f = \sigma\times\mathrm{id}$ and $g = \mathrm{id}\times\sigma$. Then f and g commute. From 2.3(a) and 6.2, $[P;\sigma]$ is minimal; hence, $[P^2;\{f,g\}]$ is a minimal multiple system. Clearly, (x,y) P^2, $\mathbf{OC}_f((x,y)) = P\times\{y\}$ and $\mathbf{OC}_g((x,y)) = \{x\}\times P$. So $\mathbf{OC}_f((x,y))$ È $\mathbf{OC}_g((x,y)) = \{(x,y)\}$. / /

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