Journal of Knot Theory and Its Ramifications
Vol. 15, No. 3 (2006) 279–287

THE A-POLYNOMIAL n-TUPLE OF A LINK AND HYPERBOLIC 3-MANIFOLDS WITH NON-INTEGRAL TRACES

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Accepted 10 February 2005

ABSTRACT

We define A-polynomial n-tuple for a link of n-components and apply them to construct hyperbolic link manifolds with non-integral traces.

Keywords: Links; character variety; A-polynomial; non-integral traces.

Mathematics Subject Classification 2000: 57M25

By a link manifold we mean a connected, orientable, compact 3-manifold whose boundary consists of tori. If a link manifold has only one boundary component, it is also called a knot manifold. A link manifold is said to be hyperbolic if its interior has a complete hyperbolic structure of finite volume. A link manifold is said to be large if it contains a closed, embedded, orientable, incompressible, and non-boundary-parallel surface.

Let M be a compact 3-manifold. If for some representation ρ of $\pi_1(M)$ into $SL_2(\mathbb{C})$, $\{tr(\rho(\gamma)); \gamma \in \pi_1(M)\}$ is not contained in the set of algebraic integers, then by Bass [1], M contains a properly embedded, orientable, incompressible and non-boundary-parallel surface (we shall call such a surface a properly embedded essential surface). If in particular M is a hyperbolic link manifold or a closed hyperbolic manifold and if a discrete faithful representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ has non-integral traces, we say that M has non-integral traces. In such a case M has a closed embedded essential surface.

In [9] a method for constructing examples of closed hyperbolic 3-manifolds with non-integral traces was introduced. The construction makes use of the A-polynomial. More concretely, one starts with a hyperbolic knot manifold M. If r is a boundary slope of M associated to a boundary slope of the Newton polygon of the A-polynomial of M and if $M(r)$ (the Dehn filling of M with slope r) is a hyperbolic 3-manifold, then very likely $M(r)$ has non-integral traces.

In this paper we show that a similar method also works for constructing hyperbolic link manifolds with non-integral traces. Namely we start with a hyperbolic link manifold with at least $n \geq 2$ boundary tori $T_1, \ldots, T_n$. By [5] each fixed T_j contains at least two distinct boundary slopes which bound embedded essential surfaces in M disjoint from all other boundary components of M. If $r_j \subset T_j$ is such a boundary slope and if $M(T_j, r_j)$ (the Dehn filling of M along the torus T_j with the slope r_j) is a hyperbolic link manifold (or a hyperbolic knot manifold if $n = 2$), then very likely $M(T_j, r_j)$ has non-integral traces. To find such boundary slopes we make use of the A-polynomial n-tuple for a link manifold with n boundary tori.

We now define the A-polynomial n-tuple for an n-component link, which is a natural generalization of the A-polynomial for a knot introduced in [3]. As in [4], for any compact manifold M, $R(M) = \{\rho; \rho : \pi_1(M) \to SL_2(\mathbb{C})$ a homomorphism$\}$ denotes the set of $SL_2(\mathbb{C})$ representations of $\pi_1(M)$, $X(M) = \{\chi_\rho; \rho \in R(M)\}$ denotes the corresponding set of characters. Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be a link of n components in a closed 3-manifold W. Let M be the link manifold which is the exterior of L in W. Let $T_1, \ldots, T_n$ be the boundary tori of M corresponding to $K_1, \ldots, K_n$ respectively. Let μ_j be an oriented essential simple closed curve in T_j which is a meridian of the knot K_j. Fix another essential simple closed curve λ_j in T_j such that the geometric intersection number of μ_j and λ_j is 1. Orient λ_j using the right hand rule (so that the cross product of the orientations of μ_j and λ_j has the direction pointing inward the manifold M). Then $\mathcal{B}_j = \{\mu_j, \lambda_j\}$ is a basis of $H_1(T_j; \mathbb{Z}) \cong \pi_1(T_j)$.

Let $i_j^* : X(M) \to X(T_j)$ be the regular map induced by the inclusion induced homomorphism $i_j^{\#} : \pi_1(T_j) \to \pi_1(M)$. Note that in the knot manifold case, i.e. when $j = n = 1$, $i_1^*(X(M))$ is always at most one dimensional in $X(T_1) = X(\partial M)$. But when $n \geq 2$, it is possible for $i_j^*(X(M))$ to be two dimensional. In order to associated a two-variable polynomial to $i_j^*(X(M))$, we need to algebraically cut out a 1-dimensional subset from $i_j^*(X(M))$. This cutting should be canonical and retain as much interesting information as possible about the manifold M. Motivated by [5], we propose the following cutting procedure. For each element $\gamma \in \pi_1(M)$, let f_γ be the regular function on the character variety $X(M)$ defined by $f_\gamma(\chi_\rho) = (\chi_\rho(\gamma))^2 - 4$. For each fixed j, let $Y_j(M)$ be the subvariety of $X(M)$ defined by $f_{\mu_k} = 0$, $f_{\lambda_k} = 0$ for all $k \neq j$, excluding those algebraic components consisting of only reducible characters. Let V_j be the Zariski closure of $i_j^*(Y_j(M))$ in $X(T_j)$.

Lemma 1. *The dimension of V_j as a subvariety of $X(T_j)$ is at most one.*

Proof. Suppose otherwise that V_j is two-dimensional. Then for any primitive element α_j of $\pi_1(T_j)$, f_{α_j} cannot be a constant function on V_j. Therefore the subvariety U_j of V_j defined by $f_{\alpha_j} = c$ is at least one-dimensional for some choice of constant c. Further if β_j is a primitive element of $\pi_1(T_j)$ different from $\alpha_j^{\pm 1}$, then f_{β_j} is non-constant on U_j. The subvariety $(i_j^*)^{-1}(U_j) \cap Y_j(M)$ of $Y_j(M)$ is at least one dimensional and thus contains an irreducible curve C. By construction,

f_{μ_k} and f_{λ_k} are both constantly equal to zero on C for each $k \neq j$. Also f_{α_j} is a constant function on C but f_{β_j} is not. These conditions imply, by [4], that there is a properly embedded essential surface F in M associate to some ideal point of the curve C, as defined in [4], such that F is disjoint from T_k for all $k \neq j$ and has boundary slope on T_j represented by α_j. But α_j is an arbitrary primitive element of $\pi_1(T_j)$. Therefore there are infinitely many boundary slopes on T_j bounding embedded essential surfaces in M disjoint from all other boundary components T_k. But this contradicts a result of [8] which asserts the finiteness of such boundary slopes. $\square$

For the fixed component T_j of ∂M and the given basis $\mathcal{B}_j = \{\mu_j, \lambda_j\}$ on T_j, let $q_j : R(T_j) \to X(T_j)$ be the canonical quotient map, let Θ_j be the set of diagonal representations of $\pi_1(T_j)$, i.e.

$$\Theta_j = \{\rho \in R(T_j) \mid \rho(\mu_j), \rho(\lambda_j) \text{ are diagonal matrices}\}.$$

Then Θ_j is a subvariety of $R(T_j)$ and $q_j|_{\Theta_j} : \Theta_j \to X(T_j)$ is a degree 2 surjective map. We may identify Θ_j with $\mathbb{C}^* \times \mathbb{C}^*$ through the eigenvalue map $E_j : \Theta_j \to \mathbb{C}^* \times \mathbb{C}^*$, which sends $\rho \in \Theta_j$ to $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$ if $\rho(\mu_j) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$ and $\rho(\lambda_j) = \left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)$.

Let V_j^1 be the set of one-dimensional components of V_j, let Z_j be the algebraic curve $(q_j|_{\Theta_j})^{-1}(V_j^1)$ in Θ_j, and let D_j be the Zariski closure of $E_j(Z_j)$ in $\mathbb{C} \times \mathbb{C}$. Define $A_j(u, v)$ to be the defining polynomial of the plane curve D_j with no repeated factors, normalized so that it is in $\mathbb{Z}[u, v]$, which is well defined up to sign (the common factors of the coefficients of $A_j(u, v)$ are ± 1). When V_j^1 is an empty set, we define $A_j(u, v)$ to be the constant one and call it trivial.

The reason that the polynomial $A_j(u, v)$ has integer coefficients is similar to that given in [3, 2.3], noticing that all the equations $f_{\mu_k} = 0$, $f_{\lambda_k} = 0$ are defined over $\mathbb{Q}$, besides those algebraic sets and regular maps involved in the definition of the polynomial.

In this way we get a polynomial $A_j(u, v)$ for each $j = 1, \ldots, n$. The ordered polynomial n-tuple $[A_1(u, v), \ldots, A_n(u, v)]$ is called the A-polynomial n-tuple of the triple $(W, L, \mathcal{B})$, where $\mathcal{B} = [\mathcal{B}_1, \ldots, \mathcal{B}_n]$ is the n-tuple of bases. To express explicitly this association, we may write the polynomial n-tuple as $[A_1(u, v), \ldots, A_n(u, v)]_{(W,L,\mathcal{B})}$.

Note that for the basis $\mathcal{B}_j = \{\mu_j, \lambda_j\}$, if we change the orientation of μ_j, then we need to change the orientation of λ_j as well, by our requirement on a basis (the right hand rule). But $A_j(u, v)$ is invariant under such change. If W is a homology 3-sphere, we always choose $\mathcal{B}_j = (\mu_j, \lambda_j)$ to be the standard meridian-longitude basis, considering K_j as a knot in W. Under this convention, we may drop $\mathcal{B}$ from the notation of the A-polynomial n-tuple and consider the A-polynomial n-tuple $[A_1(u, v), \ldots, A_n(u, v)]_{(W,L)}$ as an invariant of the link L in W. If W is the 3-sphere, we simply write $[A_1(u, v), \ldots, A_n(u, v)]_L$ for $[A_1(u, v), \ldots, A_n(u, v)]_{(S^3,L)}$.

Remark 2. The C-polynomial for a knot was defined in [11]. It can be similarly generalized to a C-polynomial n-tuple for a link of n components.

Here are some key properties of an A-polynomial n-tuple. Recall the Newton polygon of a two-variable polynomial $\sum a_{ij}u^iv^j$ is the convex hull in $\mathbb{R}^2$ of the set $\{(i,j); a_{ij} \neq 0\}$. The slope of a side of the Newton polygon is called a boundary slope of the polygon.

Theorem 3. *Let M be a link manifold with n boundary tori $T_1, \ldots, T_n$.*

(1) *The boundary slopes of the Newton polygon of $A_j(u,v)$ are boundary slopes on T_j bounding properly embedded essential surfaces in M disjoint from all other tori T_k, $k \neq j$.*
(2) *When M is hyperbolic, the A-polynomial $A_j(u,v)$ contains an irreducible factor whose Newton polygon has at least two distinct boundary slopes, for each $j = 1, \ldots, n$. In particular, $A_j(u,v)$ is nontrivial, for each $j = 1, \ldots, n$.*

Proof. The proof that boundary slopes of the Newton polygon of $A_j(u,v)$ correspond to boundary slopes on T_j is similar to the proof of [3, Theorem 3.4]. The proof for the assertion that these boundary slopes of T_j bound properly embedded essential surfaces disjoint from T_k for all $k \neq j$ is similar to that of Lemma 1. We omit the details.

Part (2) follows from [5]. In fact, let X_0 be a component of $X(M)$ which contains the character of a discrete faithful representation of $\pi_1(M)$. For any fixed j, it was shown in [5] that the subvariety of X_0 defined by the equations $f_{\mu_k} = 0$, $f_{\lambda_k} = 0$, for all $k \neq j$, has a 1-dimension component W_0 such that W_0 contains the discrete and faithful character and that on W_0, the function f_{α_j} is non-constant for any nontrivial element α_j of $\pi_1(T_j)$. Therefore $i_j^*(W_0)$ is one dimensional and thus the set V_j^1 in the definition of $A_j(u,v)$ is non-empty. Moreover there are at least two distinct boundary slopes associated to the factor of $A_j(u,v)$ corresponding to the component W_0. This follows from [5] as well. Again we omit the details. □

Remark 4. If the Newton polygon of an irreducible factor $a(u,v)$ of $A_j(u,v)$ contains at least two distinct boundary slopes, then it can be used to define a norm on $H_1(T_j;\mathbb{R})$. If the Newton polygon of an irreducible factor $a(u,v)$ of $A_j(u,v)$ contains only one boundary slope, then it can be used to define a semi-norm on $H_1(T_j;\mathbb{R})$ and the corresponding slope on T_j can be named as a semi-norm slope associated to $a(u,v)$. This can be done as in [2] where knot manifolds were considered.

Now we proceed to construct concrete examples of hyperbolic knot manifolds with non-integral traces, using the method indicated earlier. Let $L = K_1 \cup K_2$ be the 2-bridge link in S^3 shown in Fig. 1(a). Let M be the link manifold which is the exterior of L in S^3, and let T_1, T_2 be the two torus boundary components of M corresponding to K_1 and K_2 respectively. Then M is a small link manifold [7]

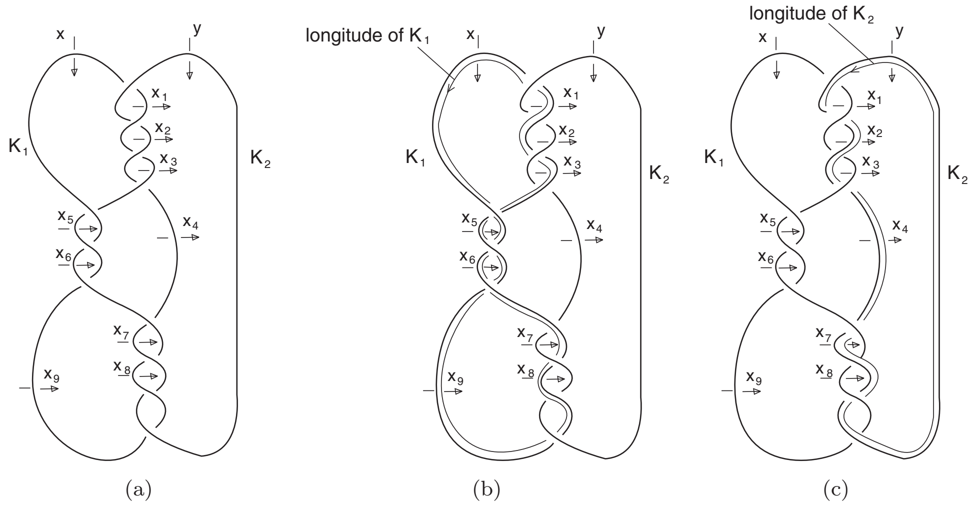

Fig. 1. The link projection, its Wirtinger generators, the two longitudes.

(here small means that M contains no closed essential surfaces) and hyperbolic (this assertion can be checked using Week's Snappea program, and can also be proved directly). Each torus T_j is given the standard meridian-longitude basis (μ_j, λ_j) obtained by considering K_j as a knot in S^3. We show

Theorem 5. *Let M be the exterior of the* 2*-bridge link shown in Fig.* 1(*a*)*. Then each of the three slopes on T_1 represented by $\mu_1^2\lambda_1$, $\mu_1^{-2}\lambda_1$ and $\mu_1^7\lambda_1$ respectively is a boundary slope bounding an essential surface disjoint from T_2, and filling M along T_1 with each of these slopes yields a hyperbolic knot manifold with non-integral traces.*

Proof. We first calculate the A-polynomial 2-tuple of L with respect to the standard bases. From the link projection in Fig. 1(a), we read off a presentation for the fundamental group of M (a Wirtinger presentation):

$$\begin{aligned}\pi_1(M) = \langle x, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, y \mid\ & yx_1y^{-1} = x^{-1},\ x_1x_2x_1^{-1} = y,\\ & x_2x_3x_2^{-1} = x_1,\ x_3x_4x_3^{-1} = x_2,\ xx_3x^{-1} = x_5,\ x_5xx_5^{-1} = x_6,\\ & x_6x_5x_6^{-1} = x_9,\ x_6x_4x_6^{-1} = x_7,\ x_7x_6x_7^{-1} = x_8,\ x_8x_7x_8^{-1} = y\rangle.\end{aligned}$$

This presentation can be simplified to one with two generators, the x and y, and one relation which is

$$\begin{aligned}& y^{-1}xy^{-1}xy^{-1}x^{-1}yx^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}yx^{-1}y^{-1}xy^{-1}xyx^{-1}yx^{-1}\\ &\quad yxy^{-1}xy^{-1}x^{-1}yx^{-1}yxy^{-1}xy^{-1}xyx^{-1}yx^{-1}y^{-1}xy^{-1}xyx^{-1}yx^{-1}yxy^{-1}xy^{-1}x^{-1}\\ &\quad yx^{-1}yxy^{-1}xy^{-1}xyx^{-1}yx^{-1}y^{-1}xy^{-1}xy^{-1}x^{-1}yx^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}yx^{-1}y^{-1}\\ &\quad xy^{-1}xyx^{-1}yx^{-1} = 1.\end{aligned}$$

We can also read off the expression of the longitude λ_j of the component K_j in terms of x and y. From Fig. 1(b) we get

$$\begin{aligned}\lambda_1 &= x^{-1}x_5^{-1}x_6^{-1}x_7^{-1}y^{-1}x_6x_5xx_2^{-1}y^{-1}\\ &= y^{-1}xy^{-1}xyx^{-1}yx^{-1}y^{-1}xy^{-1}xy^{-1}x^{-1}yx^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}yx^{-1}y^{-1}\\ &\quad xy^{-1}xyx^{-1}yx^{-1}y^{-1}xy^{-1}xy^{-1}x^{-1}yx^{-1}yxy^{-1}xy^{-1}x^{-1},\end{aligned}$$

and from Fig. 1(c) we get

$$\begin{aligned}\lambda_2 &= x_1x_3x_6^{-1}x_8^{-1}\\ &= y^{-1}x^{-1}yx^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}yx^{-1}y^{-1}xy^{-1}xyx^{-1}yx^{-1}y^{-1}\\ &\quad xy^{-1}xy^{-1}x^{-1}yx^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}yx^{-1}y^{-1}xy^{-1}xyx^{-1}yx^{-1}.\end{aligned}$$

Now we are going to find a 1-dimensional subvariety R_1 of $R(M)$ such that the image of this subvariety under the canonical map $q : R(M)\rightarrow X(M)$ is the subvariety of $X(M)$ defined by $f_{\mu_2} = 0$, with no components consisting of only reducible characters. Note that x is our choice of the meridian μ_1 for K_1, and y is the meridian μ_2 for K_2. One can easily see that every irreducible representation ρ of $\pi_1(M)$ with the trace of $\rho(y) = \pm 2$ can be conjugated so that its image is of the form $\rho(x) = \left(\begin{smallmatrix} u & 1 \\ 0 & u^{-1}\end{smallmatrix}\right)$ and $\rho(y) = \left(\begin{smallmatrix} \pm 1 & 0 \\ t & \pm 1\end{smallmatrix}\right)$. So to find R_1, let $X = \left(\begin{smallmatrix} u & 1 \\ 0 & u^{-1}\end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} \pm 1 & 0 \\ t & \pm 1\end{smallmatrix}\right)$ and substitute X and Y into the single relation of $\pi_1(M)$. We thus get the subvariety which consists of three 1-dimensional components defined by the following three two-variable polynomials respectively (we use the Maple program for the calculation):

$$\begin{aligned}P_1(u,t) &= -t^2u + 2tu^2 - 8t^2u^3 + 6t^3u^2 - u^3 - u^5 + 4tu^4 + 13t^3u^4 - 13t^4u^3\\ &\quad - 8t^2u^5 + 12u^4t^5 + 6t^5u^2 - 4t^4u + t^3 - 4t^6u^3 - t^2u^7 + 6t^3u^6 - 13t^4u^5\\ &\quad - 4t^4u^7 + 6t^5u^6 - 4t^6u^5 + t^7u^4 + t^3u^8 + 2u^6t,\\ P_2(u,t) &= 2t^2u^3 + 3t^3u^2 + 2u^4t + t^3u^4 - 12t^4u^3 - 5t^2u^5 + u^5 - 37t^4u^7 + 65t^5u^6\\ &\quad - 73t^6u^5 + 75t^7u^4 + 8t^3u^8 + 65t^5u^8 - 108t^6u^7 + 147t^7u^6 - 115t^8u^5\\ &\quad - 5t^2u^9 - 27t^4u^9 - 8u^7t^2 + 8u^6t^3 - 27u^5t^4 + 28u^4t^5 + t^4u - 22t^6u^3\\ &\quad - u^6t + 21t^7u^2 - 7t^6u + t^5 - t^{12}u^7 + 35t^9u^4 - 35t^8u^3 - 21t^{10}u^5 + 7t^{11}u^6\\ &\quad + u^{13}t^4 + 3u^{12}t^3 + 94t^9u^6 - 40t^{10}u^7 + 7t^{11}u^8 + t^5u^{14} - 73t^6u^9\\ &\quad + 147t^7u^8 + 28t^5u^{10} - 166t^8u^7 - u^8t + t^3u^{10} - 12t^4u^{11} + 2u^{11}t^2\\ &\quad - 22t^6u^{11} + 75t^7u^{10} + 21t^7u^{12} - 35t^8u^{11} - 7t^6u^{13} - 115t^8u^9 + 35t^9u^{10}\\ &\quad + 94t^9u^8 - 21t^{10}u^9 + 2u^{10}t + u^9,\\ P_3(u,t) &= t^4u^2 + t^2 - 2t^3u + 2t^2u^2 + u^2 - 2t^3u^3 + t^2u^4.\end{aligned}$$

We use Maple to check that each of the three polynomials is irreducible over $\mathbb{C}$. Therefore each of the three curves C_1, C_2, C_3 defined by the three polynomials is an irreducible subvariety of $R(M)$, on which the function f_{μ_2} is constantly equal to 0.

(Note that points in C_1, C_2, C_3, with $u = 0$ are not contained in $R(M)$. But there are only finitely many such points. So strictly speaking, only an open dense subset of C_1, C_2, C_3 are contained in $R(M)$, which are irreducible quasi-affine curves. In other words, C_1, C_2, C_3 can be considered as irreducible curves in $R(M)$ only up to birational isomorphisms. This subtlety will not affect the calculation of the A-polynomial 2-tuple.) Obviously f_{μ_1} is not constant on each of the three curves. Using Maple, one can also easily check that f_{λ_1} is not constant on the first two curves C_1 or C_2, but is constant on the third curve C_3, and that f_{λ_2} is constantly equal to zero on all the three curves. Hence we have actually $q(R_1) = Y_1(M)$, where R_1 is the union of C_1, C_2 and C_3 (again this is defined only up to birational isomorphisms). It also follows that one of C_1 and C_2 contains a point corresponding to a discrete faithful representation of $\pi_1(M)$.

To find $A_1(u,v)$, let $L[1,1]$ be the upper-left entry of the corresponding matrix for λ_1 (calculated using Maple). Calculating the resultant of $L[1,1]-v$ with $p_1(u,t)$ and $p_2(u,t)$ respectively, we get the corresponding factors of the A-polynomial $A_1(u,v)$ associated to the boundary component T_1 of M:

$$\begin{aligned}
a_1(u,v) = {} & -u^6 + (-3u^4 + 7u^6 + 3u^8)v + (-3u^2 + 14u^4 - 34u^6 + 96u^8 - 222u^{10} \\
& + 224u^{12} - 136u^{14} + 49u^{16} - 10u^{18} + u^{20})v^2 + (-1 + 7u^2 - 39u^4 \\
& + 168u^6 - 436u^8 + 638u^{10} - 502u^{12} + 308u^{14} - 157u^{16} + 64u^{18} \\
& - 17u^{20} + 2u^{22})v^3 + (-2u^2 + 17u^4 - 64u^6 + 157u^8 - 308u^{10} + 502u^{12} \\
& - 638u^{14} + 436u^{16} - 168u^{18} + 39u^{20} - 7u^{22} + u^{24})v^4 + (-u^4 + 10u^6 \\
& - 49u^8 + 136u^{10} - 224u^{12} + 222u^{14} - 96u^{16} + 34u^{18} - 14u^{20} + 3u^{22})v^5 \\
& + (-3u^{16} - 7u^{18} + 3u^{20})v^6 + u^{18}v^7, \\
a_2(u,v) = {} & u^6 + (3u^4 - 8u^6 + 29u^8 - 16u^{10} + 4u^{12})v + (3u^2 - 21u^4 + 95u^6 - 177u^8 \\
& + 354u^{10} - 304u^{12} + 156u^{14} - 49u^{16} + 10u^{18} - u^{20})v^2 + (1 - 18u^2 \\
& + 111u^4 - 370u^6 + 918u^8 - 1164u^{10} + 1442u^{12} - 1100u^{14} + 530u^{16} \\
& - 154u^{18} + 26u^{20} - 2u^{22})v^3 + (-5 + 55u^2 - 302u^4 + 971u^6 - 2019u^8 \\
& + 3582u^{10} - 3604u^{12} + 3066u^{14} - 1807u^{16} + 704u^{18} - 166u^{20} + 21u^{22} \\
& - u^{24})v^4 + (10 - 102u^2 + 499u^4 - 1059u^6 + 3419u^8 - 4692u^{10} \\
& + 5946u^{12} - 4676u^{14} + 3024u^{16} - 1398u^{18} + 417u^{20} - 70u^{22} + 5u^{24})v^5 \\
& + (-10 + 113u^2 - 566u^4 + 1739u^6 - 3868u^8 + 6538u^{10} - 6968u^{12} \\
& + 6538u^{14} - 3868u^{16} + 1739u^{18} - 566u^{20} + 113u^{22} - 10u^{24})v^6 \\
& + (5 - 70u^2 + 417u^4 - 1398u^6 + 3024u^8 - 4676u^{10} + 5946u^{12} - 4692u^{14} \\
& + 3419u^{16} - 1590u^{18} + 499u^{20} - 102u^{22} + 10u^{24})v^7 + (-1 + 21u^2 \\
& - 166u^4 + 704u^6 - 1807u^8 + 3066u^{10} - 3604u^{12} + 3582u^{14} - 2019u^{16} \\
& + 971u^{18} - 302u^{20} + 55u^{22} - 5u^{24})v^8 + (-2u^2 + 26u^4 - 154u^6 + 530u^8
\end{aligned}$$

$$- 1100u^{10} + 1442u^{12} - 1164u^{14} + 918u^{16} - 370u^{18} + 111u^{20}$$
$$- 18u^{22} + u^{24})v^9 + (-u^4 + 10u^6 - 49u^8 + 156u^{10} - 304u^{12} + 354u^{14}$$
$$- 177u^{16} + 95u^{18} - 21u^{20} + 3u^{22})v^{10} + (4u^{12} - 16u^{14} + 29u^{16} - 8u^{18}$$
$$+ 3u^{20})v^{11} + u^{18}v^{12}.$$

Corresponding to the curve C_3, we have the factor $a_3(u,v) = v - 1$ of $A_1(u,v)$.

So $A_1(u,v) = a_1(u,v)a_2(u,v)a_3(u,v)$.

A similar calculation shows that $A_2(u,v)$ is equal to $A_1(u,v)$ for this link.

The Newton polygons of $a_1(u,v)$ and $a_2(u,v)$ have the same set of boundary slopes, which are $\mu_1^2\lambda_1$, $\mu_1^{-2}\lambda_1$ and $\mu_1^7\lambda_1$. Each of the three corresponding boundary slopes on T_1 bounds an essential surface in M disjoint from T_2 by Theorem 3. (So both $a_1(u,v)$ and $a_2(u,v)$ provide norms on $H_1(T_1,\mathbb{R})$, while $a_3(u,v)$ provides a semi-norm with λ_1 as its associated slope.)

We can check using the Snappea program that the filling of M along T_1 with each of the three boundary slopes yields a hyperbolic knot manifold. One can also prove directly that at least one of the fillings with slopes $\mu_1^{-2}\lambda_1$ or $\mu_1^7\lambda_1$ is hyperbolic, applying [6].

We shall only show that Dehn filling of M along T_1 with the slope $r = \mu_1^{-2}\lambda_1$ yields a hyperbolic knot manifold with non-integral traces; the same assertion for the other two slopes can be proved similarly. Note that any discrete faithful representation ρ of $M((T_1,r))$ is contained, up to conjugacy, in C_1 or C_2. For $\rho(\mu_2)$ has to be a parabolic element and $\rho(\lambda_1)$ has to be a hyperbolic element. Now from $\rho(r) = I$, we get an additional equation, $v - u^2 = 0$, for the eigenvalues u and v of $\rho(x)$ and $\rho(y)$ respectively. Together with the polynomial equation $a_1(u,v) = 0$ or $a_2(u,v) = 0$, we see that the eigenvalue u of $\rho(x)$ has to satisfy the equation

$$8u^{12} + 28u^{10} + 100u^8 + 111u^6 + 100u^4 + 28u^2 + 8 = 0$$

(if ρ is contained in C_1) or the equation

$$8u^{36} - 52u^{34} + 300u^{32} - 977u^{30} + 2864u^{28} - 5714u^{26} + 10776u^{24} - 14879u^{22}$$
$$+ 19844u^{20} - 20244u^{18} + 19844u^{16} - 14879u^{14} + 10776u^{12} - 5714u^{10} + 2864u^8$$
$$- 977u^6 + 300u^4 - 52u^2 + 8 = 0$$

(if ρ is contained in C_2). Both equations are irreducible over $\mathbb{Z}$. Therefore u is not an algebraic unit, in either case. Thus $tr(\rho(x)) = u + u^{-1}$ cannot be an algebraic integer. Hence ρ has non-integral traces. The proof of the theorem is now complete. $\square$

Remark 6. For any two bridge link in S^3, its A-polynomial 2-tuple is calculable with the method given in the proof of Theorem 5. So presumably one could construct an abundance of examples of hyperbolic knot manifolds with non-integral traces.

Acknowledgment

Partially supported by NSF grant DMS 0204428.

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