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EVERY NONTRIVIAL KNOT IN S^3 HAS NONTRIVIAL A-POLYNOMIAL

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ABSTRACT. We show that every nontrivial knot in the 3-sphere has a nontrivial A-polynomial.

In Theorem 1 of [3], Kronheimer and Mrowka give a proof of the following remarkable theorem, thereby establishing the truth of the Property P conjecture.

Theorem 0.1 (Kronheimer-Mrowka)**.** *Let K be any nontrivial knot in S^3 and let $M(r)$ be the manifold obtained by Dehn surgery on K with slope r with respect to the standard meridian-longitude coordinates of K. If $|r| \leq 2$, then there is an irreducible homomorphism from $\pi_1(M(r))$ to $SU(2)$.*

The purpose of this note is to describe another consequence of Theorem 0.1, answering a question which has been around for about ten years. We show

Theorem 0.2. *Every nontrivial knot K in S^3 has nontrivial A-polynomial.*

The A-polynomial was introduced in [1]. We recall its definition for a knot in S^3.

For a compact manifold W, we use $R(W)$ and $X(W)$ to denote the $SL_2(\mathbb{C})$ representation variety and character variety of W respectively, and $q : R(W) \to X(W)$ to denote the quotient map sending a representation ρ to its character χ_ρ (see [2] for detailed definitions). Note that q is a regular map between the two varieties defined over the rationals.

Let K be a knot in S^3, M its exterior, and $\{\mu, \lambda\}$ the standard meridian-longitude basis for $\pi_1(\partial M)$. Let $i^* : X(M) \to X(\partial M)$ be the restriction map, also regular, induced by the homomorphism $i_* : \pi_1(\partial M) \to \pi_1(M)$, and let Λ be the set of diagonal representations of $\pi_1(\partial M)$, i.e.

$$\Lambda = \{\rho \in R(\partial M) \mid \rho(\mu), \rho(\lambda) \text{ are both diagonal matrices}\}.$$

Then Λ is a subvariety of $R(\partial M)$ and $q|_\Lambda : \Lambda \to X(\partial M)$ is a degree 2, surjective, regular map.

We may identify Λ with $\mathbb{C}^* \times \mathbb{C}^*$ through the eigenvalue map $E : \Lambda \to \mathbb{C}^* \times \mathbb{C}^*$ which sends $\rho \in \Lambda$ to $(x, y) \in \mathbb{C}^* \times \mathbb{C}^*$ if $\rho(\mu) = \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$ and $\rho(\lambda) = \left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)$. Let

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$X^*(M)$ be the set of components of $X(W_K)$ each of which has a 1-dimensional image in $X(\partial M)$ under i^* and then define

- V to be the Zariski closure of $i^*(X^*(M))$ in $X(\partial M)$;
- Z to be the algebraic curve $(q|_\Lambda)^{-1}(V)$ in Λ;
- D to be the Zariski closure of $E(Z)$ in $\mathbb{C}^2$.

It can be verified that each of $X^*(M), V, Z$, and D is defined over the rationals.

The A-polynomial of K is the defining polynomial $A_K(x,y)$ of the plane curve D determined up to sign by the requirements that it has no repeated factors, it lies in $\mathbb{Z}[x,y]$, and the greatest common divisor of its coefficients is 1. For every knot K in S^3, $X(M)$ has a unique component Y_0 consisting of reducible characters. The image of Y_0 under i^* is 1-dimensional and contributes the factor $y-1$ to $A_K(x,y)$. Thus the A-polynomial of the trivial knot is $y-1$. For a knot K in S^3, its A-polynomial is said to be nontrivial if $A_K(x,y) \neq y-1$. (See [1] for more details.)

We now proceed to the proof of Theorem 0.2. We take K to be a nontrivial knot in S^3 with exterior M and we let $M(r)$ denote the manifold obtained by Dehn surgery on K with slope r. By Theorem 0.1, for every integer $n \neq 0$, the fundamental group of the surgered manifold $M(1/n)$ has an irreducible representation ρ_n into $SU_2(\mathbb{C}) \subset SL_2(\mathbb{C})$. We shall consider ρ_n as a representation of $\pi_1(M)$ through the composition with the quotient homomorphism $\pi_1(M) \to \pi_1(M(1/n))$. Thus $\rho_n(\mu\lambda^n) = I$ and so the irreducibility of ρ_n implies that

$$\rho_n(\mu), \rho_n(\lambda) \neq \pm I \text{ for each } n \neq 0. \tag{1}$$

Moreover, by a result of Thurston (see [2], Proposition 3.2.1), any algebraic component of $X(M)$ which contains the character of ρ_n is at least 1-dimensional.

Claim. *There is a component X_0 of $X(M)$ containing some χ_{ρ_n} whose restriction to $X(\partial M)$ under i^* is 1-dimensional.*

Assuming the claim, we can quickly complete the proof of Theorem 0.2. For suppose that X_0 contributes a factor $(y-1)$ to $A_K(x,y)$. Then every representation in $q^{-1}(X_1)$ sends the longitude λ to an element of $SL_2(\mathbb{C})$ of trace 2. In particular, if n is chosen so that $\chi_{\rho_n} \in X_0$ we have $\rho_n(\lambda) = I$ or is a parabolic element of $SL_2(\mathbb{C})$. But the first possibility is prohibited by (1) while the second is prohibited by the fact that $SU_2(\mathbb{C})$ contains no parabolic elements. Thus X_0 contributes a factor to the A-polynomial different from $(y-1)$. In particular, $A_K(x,y)$ is nontrivial, so the theorem holds.

Proof of the Claim. We shall suppose that each component of $X(M)$ containing some χ_{ρ_n} restricts to a point in $X(\partial M)$ in order to arrive at a contradiction.

We begin by selecting a component X_1 of $X(M)$ which contains χ_{ρ_n} for at least two distinct n, say n_1, n_1'. (This is possible since $X(M)$ has only finitely many algebraic components.) Let $R_1 = q^{-1}(X_1)$.

Recall that every element $\gamma \in \pi_1(M)$ defines a regular function $\tau_\gamma : X(M) \to \mathbb{C}$ given by $\tau_\gamma(\chi_\rho) = trace(\rho(\gamma))$. Our assumption that $i_*(X_1)$ is a point is equivalent to the fact that for every element $\beta \in \pi_1(\partial M) \subset \pi_1(M)$, the function $\tau_\beta|X_1$ is constant.

Suppose that $\rho \in R_1$ and $\rho(\pi_1(\partial M))$ contains a parabolic element. Then the commutativity of $\pi_1(\partial M)$ shows that every element of $\rho(\pi_1(\partial M))$ is either parabolic or $\pm I$. Hence $\tau_\mu(\chi_{\rho_{n_1}}) = \tau_\mu(\chi_\rho) = \pm 2$, which is impossible as it implies that $\rho_{n_1}(\mu)$ ($\in SU_2(\mathbb{C})$) is $\pm I$ (cf. (1)). Thus for each $\rho \in R_1$, $\rho(\pi_1(\partial M))$ consists of diagonalisable elements. Since $i_*(X_1)$ is a point, it follows that for any such ρ we have that $\rho|\pi_1(\partial M)$ is conjugate in $SL_2(\mathbb{C})$ to $\rho_{n_1}|\pi_1(\partial M)$ and therefore $\rho(\mu\lambda^n) = I$ for each $\rho \in R_1$ and n such that $\chi_{\rho_n} \in X_1$. For such a ρ we therefore have $I = \rho(\mu\lambda^{n_1})\rho(\mu\lambda^{n_1'})^{-1} = \rho(\lambda^{n_1 - n_1'})$. Thus $\rho(\lambda)$ is of a fixed finite order $d_1 \geq 3$ (cf. (1)) and so for $\rho \in R_1$, $\rho(\mu\lambda^n) = I$ if and only if $n \in S_1 := \{n_1 + d_1 k; k \in \mathbb{Z}\}$. Note that

$$d_1\mathbb{Z} \subset \mathbb{Z} \setminus S_1 \tag{2}$$

as otherwise $n_1 \equiv 0 \pmod{d_1}$ and therefore $I = \rho_{n_1}(\mu\lambda^{n_1}) = \rho_{n_1}(\mu)$, which is absurd.

Now repeat the argument to produce a component X_2 of $X(M)$ satisfying the following conditions:

- there are at least two integers $n_2, n_2' \in d_1\mathbb{Z} \setminus \{0\}$ such that X_2 contains the characters of $\rho_{n_2}, \rho_{n_2'}$;
- $i^*(X_2)$ is a point in $X(\partial M)$;
- there is an integer $d_2 \geq 3$ such that for any $\rho \in R_2 = q^{-1}(X_2)$ we have $\rho(\mu\lambda^n) = I$ if and only if n belongs to the set $S_2 = \{n_2 + d_2 k; k \in \mathbb{Z}\}$;
- $d_1 d_2 \mathbb{Z} \subset \mathbb{Z} \setminus (S_1 \cup S_2)$.

The first of these conditions combines with (2) to show that $X_2 \neq X_1$.

Proceeding inductively, one can find, for each integer $j \geq 1$, a component X_j of $X(M)$ satisfying the following conditions:

- there are at least two integers $n_j, n_j' \in d_1 d_2 \dots d_{j-1}\mathbb{Z} \setminus \{0\}$ such that X_j contains the characters of $\rho_{n_j}, \rho_{n_j'}$;
- $i^*(X_j)$ is a point in $X(\partial M)$;
- there is an integer $d_j \geq 3$ such that for each $\rho \in R_j = q^{-1}(X_j)$ we have $\rho(\mu\lambda^n) = I$ if and only if n belongs to the set $S_j = \{n_j + d_j k; k \in \mathbb{Z}\}$;
- $d_1 d_2 \cdots d_j \mathbb{Z} \subset \mathbb{Z} \setminus (S_1 \cup S_2 \cup \cdots \cup S_j)$.

It is easy to see that these conditions imply that $X_i \neq X_j$ for $i \neq j$, which is clearly impossible as $X(M)$ has only finitely many components. Thus there must be a component X_0 of $X(M)$ containing some χ_{ρ_n} such that $i^*(X_0)$ is 1-dimensional. This completes the proof of the claim and therefore of Theorem 0.2.

Remark 0.3. Theorem 0.2 has been obtained independently by Nathan Dunfield and Stavros Garoufalidis.

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