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The AJ-Conjecture and cabled knots over torus knots

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ABSTRACT

We show that most cabled knots over torus knots in S^3 satisfy the AJ-conjecture, namely each (r, s)-cabled knot over each (p, q)-torus knot satisfies the AJ-conjecture if r is not a number between 0 and pqs.

Keywords: Colored Jones polynomials; A-polynomials; the AJ-conjecture; cabled knots.

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1. Introduction

For a knot K in S^3, let $J_{K,n}(t)$ denote the *n-colored Jones polynomial* of K with the zero framing, normalized so that for the unknot U,

$$J_{U,n}(t) = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

A remarkable result, proved in [4], asserts that for every knot K, $J_{K,n}(t)$ satisfies a nontrivial linear recurrence relation. By defining $J_{K,-n}(t) := -J_{K,n}(t)$ and $J_{K,0}(t) = 0$, one may treat $J_{K,n}(t)$ as a discrete function

$$J_{K,-}(t) : \mathbb{Z} \to \mathbb{Z}[t^{\pm 1}].$$

The quantum torus

$$\mathcal{T} = \mathbb{C}[t^{\pm 1}]\langle L^{\pm 1}, M^{\pm 1}\rangle/(LM - t^2 ML)$$

acts on the set of discrete functions $f : \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]$ by

$$(Mf)(n) := t^{2n} f(n), \quad (Lf)(n) := f(n+1).$$

Then linear recurrence relations of $J_{K,n}(t)$ correspond naturally to annihilators of $J_{K,n}(t)$ in $\mathcal{T}$. The latter set, which we denote by

$$\mathcal{A}_K := \{P \in \mathcal{T} \mid PJ_{K,n}(t) = 0\},$$

is obviously a left ideal of $\mathcal{T}$, called the *recurrence ideal* of K. The result of [4] cited above states that $\mathcal{A}_K$ is not the zero ideal for every knot K.

The ring $\mathcal{T}$ can be extended to a principal left ideal domain $\widetilde{\mathcal{T}}$ by adding inverses of polynomials in t and M; that is, $\widetilde{\mathcal{T}}$ is the set of Laurent polynomials in L with coefficients rational functions of t and M with a product defined by

$$f(t, M)L^a \cdot g(t, M)L^b = f(t, M)g(t, t^{2a}M)L^{a+b}.$$

The left ideal $\widetilde{\mathcal{A}}_K = \widetilde{\mathcal{T}}\mathcal{A}_K$ is then generated by some nonzero polynomial in $\widetilde{\mathcal{T}}$, and in particular, this generator can be chosen to be in $\mathcal{A}_K$ and be of the form

$$\alpha_K(t, M, L) = \sum_{i=0}^{d} P_i L^i,$$

with d minimal and with $P_0, \ldots, P_d \in \mathbb{Z}[t, M]$ being coprime in $\mathbb{Z}[t, M]$. This polynomial α_K is uniquely determined up to a sign and is called the (*normalized*) *recurrence polynomial* of K.

The A-polynomial was introduced in [1]. For a knot K in S^3, its A-polynomial $A_K(M, L) \in \mathbb{Z}[M, L]$ is a two-variable polynomial with no repeated factors and with relatively prime integer coefficients, which is uniquely associated to K up to a sign. Note that $A_K(M, L)$ always contains the factor $L - 1$.

The AJ-conjecture was raised in [2] which states that for every knot K, its recurrence polynomial $\alpha_K(t, M, L)$ evaluated at $t = -1$ is equal to the A-polynomial of K, up to a factor of a polynomial in M. The conjecture is obviously of fundamental importance as it predicts a strong connection between two important knot invariants derived from very different backgrounds. This is also a very difficult conjecture; so far only torus knots, some classes of 2-bridge knots and pretzel knots are known to satisfy the conjecture [2, 3, 5–7, 11, 12].

In this paper, we consider the AJ-conjecture for cabled knots over torus knots. Recall that the set of nontrivial torus knots $T(p, q)$ in S^3 can be indexed, in a standard way, by pairs of relatively prime integers (p, q) satisfying $|p| > q \geq 2$. Also recall that an (r, s)-cabled knot on a knot K in S^3 is the knot which can be embedded in the boundary torus of a regular neighborhood of K in S^3 as a curve of slope r/s with respect to the meridian/longitude coordinates of K satisfying $(r, s) = 1$, $s \geq 2$. Note that r can be any integer relatively prime to s. We have the following.

Theorem 1.1. *The* AJ-*conjecture holds for each* (r, s)-*cabled knot* C *over each* (p, q)-*torus knot* T *if* r *is not an integer between* 0 *and* pqs.

A cabling formula for A-polynomials of cabled knots in S^3 is given in [9]. In particular when C is the (r, s)-cabled knot over the torus knot $T(p, q)$ in S^3, its

A-polynomial $A_C(M,L)$ is given explicitly as in (1.1). For a pair of relatively prime integers (p,q) with $q \geq 2$, define $F_{(p,q)}(M,L), G_{(p,q)}(M,L) \in \mathbb{Z}[M,L]$ to be the associated polynomials in variables M and L by:

$$F_{(p,q)}(M,L) := \begin{cases} M^{2p}L+1 & \text{if } q=2,\ p>0, \\ L+M^{-2p} & \text{if } q=2,\ p<0, \\ M^{2pq}L^2-1 & \text{if } q>2,\ p>0, \\ L^2-M^{-2pq} & \text{if } q>2,\ p<0 \end{cases}$$

and

$$G_{(p,q)}(M,L) := \begin{cases} M^{pq}L-1 & \text{if } p>0, \\ L-M^{-pq} & \text{if } p<0. \end{cases}$$

Then

$$A_C(M,L) = \begin{cases} (L-1)F_{(r,s)}(M,L)F_{(p,q)}(M^{s^2},L) & \text{if } s \text{ is odd;} \\ (L-1)F_{(r,s)}(M,L)G_{(p,q)}(M^{s^2},L) & \text{if } s \text{ is even.} \end{cases} \tag{1.1}$$

A cabling formula for the n-colored Jones polynomial of the (r,s)-cabled knot C over a knot K is given in [8] (see also [13]) which in our normalized form is:

$$J_{C,n}(t) = t^{-rs(n^2-1)} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4rk(ks+1)} J_{K,2ks+1}(t). \tag{1.2}$$

In particular the n-colored Jones polynomial of the (p,q)-torus knot T (which is the (p,q)-cabled knot over the unknot U) is:

$$\begin{aligned} J_{T,n}(t) &= t^{-pq(n^2-1)} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4pk(kq+1)} J_{U,2kq+1}(t) \\ &= t^{-pq(n^2-1)} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4pk(kq+1)} \frac{t^{4kq+2}-t^{-4qk-2}}{t^2-t^{-2}}. \end{aligned} \tag{1.3}$$

We divide the proof of Theorem 1.1 into the following cases:

(1) s is odd and $q>2$;
(2) s is odd and $q=2$;
(3) $s>2$ is even;
(4) $s=2$.

In each case, we will find an annihilator of $J_{C,n}(t)$ by applying the formulas (1.3) and (1.2) (where taking the general knot K to be the (p,q)-torus knot T), and then proceed to prove that it is the recurrence polynomial $\alpha_C(t,M,L)$ of C when r is not an integer between 0 and pqs, making use of the degree formulas given in Sec. 2. Of course we will also compare $\alpha_C(-1,M,L)$ with $A_C(M,L)$ given in (1.1) to complete the verification of the AJ-conjecture for C. For convenience, we often get $\alpha_C(t,M,L)$ in the form $P=\sum_{i=0}^{d} P_iL^i \in \widetilde{\mathcal{A}}_C$, with d minimal and with $P_i \in \mathbb{Q}(t,M)$ and with $P(-1,M,L)\neq 0$. Such P only differs from α_C by a factor of a rational function $f(t,M)\in\mathbb{Q}(t,M)$ with $f(-1,M)\neq 0$ and thus is clearly as good as the normalized recurrence polynomial in verification for the AJ-conjecture. We often simply call such P the recurrence polynomial of C. Also notice from the formula (1.1) that changing the sign of r or p only changes the A-polynomial of C up to a power of M, so in checking that $P(-1,M,L)=A_C(M,L)$ up to a factor of a rational function in M we do not need to worry about the sign of r or p.

Further investigation of the AJ-conjecture for more general cabled knots, such as iterated torus knots and cabled knots over some hyperbolic knots, is being continued in [10]. In particular for some cabled knots over the figure 8 knot the AJ-conjecture has been verified to be true.

2. Degrees of $J_{T,n}(t)$ and $J_{C,n}(t)$

From now on in this paper, T denotes the (p,q)-torus knot and C the (r,s)-cabled knot over T, with the index convention given in Sec. 1.

For a polynomial $f(t)\in\mathbb{Z}[t^{\pm 1}]$, let $\ell[f]$ and $\hbar[f]$ denote the lowest degree and the highest degree of f in t respectively. Obviously for $f(t),g(t)\in\mathbb{Z}[t^{\pm1}]$, $\ell[fg]=\ell[f]+\ell[g]$ and $\hbar[fg]=\hbar[f]+\hbar[g]$.

Lemma 2.1. (1) *When* $p>q$,

$$\ell[J_{T,n}(t)] = -pqn^2+pq+\frac{1}{2}(1-(-1)^{n-1})(p-2)(q-2),$$
$$\hbar[J_{T,n}(t)] = 2(p+q-pq)|n|+2(pq-p-q).$$

(2) *When* $p<-q$,

$$\ell[J_{T,n}(t)] = 2(p-q-pq)|n|+2(pq-p+q),$$
$$\hbar[J_{T,n}(t)] = -pqn^2+pq+\frac{1}{2}(1-(-1)^{n-1})(p+2)(q-2).$$

Proof. The formula for $\ell[J_{T,n}(t)]$ in part (1) is proved in [12, Lemma 1.4]. The rest of the lemma can be proved similarly. □

Note that $r\neq pqs$ since r is relatively prime to s.

Lemma 2.2. (1) *When* $p>q$,

$$\ell[J_{C,n}(t)] = -pqs^2n^2+(2pqs^2-2pqs+2r-2rs)n+2rs-2r+2pqs-pqs^2 + \frac{1}{2}(1-(-1)^{(n-1)s})(p-2)(q-2), \quad \textit{if } r<pqs,$$

$$\ell[J_{C,n}(t)] = -rsn^2 + rs + \frac{1}{2}(1-(-1)^{(n-1)})(s-2)(r-pqs)$$
$$+\frac{1}{2}[1-(-1)^{(n-1)s}](p-2)(q-2), \quad \text{if } r > pqs,$$
$$\hbar[J_{C,n}(t)] = -rsn^2 + rs + \frac{1}{2}(1-(-1)^{n-1})(s-2)(r-2pq+2p+2q), \quad \text{if } r < 0.$$

(2) *When* $p < -q$,

$$\hbar[J_{C,n}(t)] = -pqs^2n^2 + (2pqs^2 - 2pqs + 2r - 2rs)n + 2rs - 2r + 2pqs - pqs^2$$
$$+\frac{1}{2}(1-(-1)^{(n-1)s})(p+2)(q-2), \quad \text{if } r > pqs,$$
$$\hbar[J_{C,n}(t)] = -rsn^2 + rs + \frac{1}{2}(1-(-1)^{(n-1)})(s-2)(r-pqs)$$
$$+\frac{1}{2}(1-(-1)^{(n-1)s})(p+2)(q-2), \quad \text{if } r < pqs,$$
$$\ell[J_{C,n}(t)] = -rsn^2 + rs + \frac{1}{2}(1-(-1)^{n-1})(s-2)(r-2pq+2p-2q), \quad \text{if } r > 0.$$

Proof. (1) From the formula (1.2) for $J_{C,n}(t)$ (replacing K there by T), we can see that

$$\ell[J_{C,n}(t)] = -rs(n^2-1) + \min\left\{\ell[J_{T,2sk+1}(t)] + 4rk(ks+1) \,\middle|\, -\frac{n-1}{2} \le k \le \frac{n-1}{2}\right\}.$$

By Lemma 2.1(1), we have

$$\ell[J_{T,2ks+1}(t)] + 4rk(ks+1)$$
$$= -pq(2ks+1)^2 + pq + \frac{1}{2}(1-(-1)^{2ks})(p-2)(q-2) + 4kr(ks+1)$$
$$= (4rs - 4pqs^2)k^2 + (4r - 4pqs)k + \frac{1}{2}(1-(-1)^{2ks})(p-2)(q-2).$$

When n is odd, k is integer valued and thus the alternating term vanishes, so the above expression is quadratic in k. When n is even, k is half-integer valued and the alternating term is either always equal to zero (when s is even) or is always equal to $(p-2)(q-2)$ (when s is odd), and thus the above expression is again quadratic in k. So if $r < pqs$, it is minimized at $k = \frac{n-1}{2}$, which yields the first formula in part (1), and if $r > pqs$, it is minimized at $k = 0$ when n is odd and at $k = -1/2$ when n is even, which yields the second formula in part (1).

Similarly to get the third formula in (1), we look at

$$\hbar[J_{C,n}(t)] = -rs(n^2-1) + \max\left\{\hbar[J_{T,2sk+1}(t)] + 4rk(ks+1) \,\middle|\, -\frac{n-1}{2} \le k \le \frac{n-1}{2}\right\}.$$

By Lemma 2.1(1), we have

$$\begin{aligned}
&\hbar[J_{T,2ks+1}(t)] + 4rk(ks+1) \\
&\quad = 2(p+q-pq)|2ks+1| + 2(pq-p-q) + 4kr(ks+1) \\
&\quad = \begin{cases} 4rsk^2 + (4ps+4qs-4pqs+4r)k & \text{for non-negative } k\text{'s}, \\ 4rsk^2 + (-4ps-4qs+4pqs+4r)k + 4(pq-p-q) & \text{for negative } k\text{'s}. \end{cases}
\end{aligned}$$

If $r < 0$, it is maximized at $k = 0$ when n is odd and at $k = -1/2$ when n is even, which yields the third formula in part (1).

Part (2) can be proved similarly. □

3. Case $s > 2$ is Odd and $q > 2$

3.1. *An annihilator P of $J_{C,n}(t)$*

Define

$$\delta_j = \frac{t^{2(p+q)(j+1)+2} + t^{-2(p+q)(j+1)+2} - t^{2(q-p)(j+1)-2} - t^{-2(q-p)(j+1)-2}}{t^2 - t^{-2}},$$

$$S_n = \sum_{k=1}^{s} t^{-4pqskn+2pqsn+4pqk^2-12pqsk+6pqs}\delta_{s(n+3)-1-2k}.$$

By [12, Lemma 1.1], we have

$$J_{T,n+2}(t) = t^{-4pq(n+1)}J_{T,n}(t) + t^{-2pq(n+1)}\delta_n. \tag{3.1}$$

Note that (3.1) is valid for every torus knot (although in [12], only positive p was considered). The following two lemmas also hold for general C and T (without restriction on s and q) and they shall also be applied in later sections.

Lemma 3.1.

$$\begin{aligned}
J_{C,n+2}(t) &= t^{-4rsn-4rs}J_{C,n}(t) + (t^{2(r-rs)n-2rs+2r-4pqs(n+1)} \\
&\quad - t^{2(-r-rs)n-2rs-2r})J_{T,s(n+1)-1}(t) + t^{2(r-rs)n-2rs+2r-2pqs(n+1)}\delta_{s(n+1)-1}.
\end{aligned}$$

Proof. We know by the cabling formula (1.2)

$$\begin{aligned}
J_{C,n+2}(t) &= t^{-rs((n+2)^2-1)} \sum_{k=-\frac{n+1}{2}}^{\frac{n+1}{2}} t^{4rk(ks+1)}J_{T,2ks+1}(t) \\
&= t^{-rs(n^2+4n+3)}\left(\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4rk(ks+1)}J_{T,2ks+1}(t) + t^{4r(\frac{n+1}{2})((\frac{n+1}{2})s+1)}\right. \\
&\quad \left. \times J_{T,s(n+1)+1}(t) + t^{4r(-\frac{n+1}{2})((-\frac{n+1}{2})s+1)}J_{T,-s(n+1)+1}(t)\right).
\end{aligned}$$

Noting that $J_{T,-s(n+1)+1}(t) = -J_{T,s(n+1)-1}(t)$, we have

$$J_{C,n+2}(t) = t^{-rs(n^2+4n+3)}\Bigg(t^{rs(n^2-1)}t^{-rs(n^2-1)}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4rk(ks+1)}J_{T,2ks+1}(t) + t^{(n+1)^2rs+2r(n+1)}J_{T,s(n+1)+1}(t) - t^{(n+1)^2rs-2r(n+1)}J_{T,s(n+1)-1}(t)\Bigg)$$
$$= t^{-4rsn-4rs}J_{C,n}(t) + t^{2(r-rs)n-2rs+2r}J_{T,s(n+1)+1}(t) - t^{2(-r-rs)n-2rs-2r}J_{T,s(n+1)-1}(t).$$

Since $J_{T,s(n+1)+1}(t)$ and $J_{T,s(n+1)-1}(t)$ are related by Eq. (3.1) as

$$J_{T,s(n+1)+1} = t^{-4pqs(n+1)}J_{T,s(n+1)-1} + t^{-2pqs(n+1)}\delta_{s(n+1)-1},$$

we have

$$J_{C,n+2}(t) = t^{-4rsn-4rs}J_{C,n}(t) + (t^{2(r-rs)n-2rs+2r-4pqs(n+1)} - t^{2(-r-rs)n-2rs-2r})J_{T,s(n+1)-1}(t) + t^{2(r-rs)n-2rs+2r-2pqs(n+1)}\delta_{s(n+1)-1}.$$ □

Lemma 3.2. *For all positive integers m, we have*

$$J_{T,n}(t) = t^{-4pqm(n+1)+4pqm(m+1)}J_{T,n-2m}(t) + \sum_{k=1}^{m} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq}\delta_{n-2k}$$

and in particular, with s any positive integer,

$$J_{T,s(n+3)-1}(t) = t^{-4pqs^2n-8pqs^2-4pqs}J_{T,s(n+1)-1}(t) + S_n.$$

Proof. We induct on m. The base case $m = 1$ follows directly from Eq. (3.1). Then assume the formula holds for some positive integer m. Applying equation (3.1) again yields

$$J_{T,n}(t) = t^{-4pqm(n+1)+4pqm(m+1)}J_{T,n-2m}(t) + \sum_{k=1}^{m} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq}\delta_{n-2k}$$
$$= t^{-4pqm(n+1)+4pqm(m+1)}(t^{-4pq(n-2m-1)}J_{T,n-2m-2}(t) + t^{-2pq(n-2m-1)}\delta_{n-2m-2}) + \sum_{k=1}^{m} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq}\delta_{n-2k}$$
$$= t^{-4pqm(n+1)+4pqm(m+1)-4pq(n-2m-2+1)}J_{T,n-2m-2}(t)$$

$$+ t^{-4pqm(n+1)+4pqm(m+1)-2pq(n-2m-1)}\delta_{n-2m-2} + \sum_{k=1}^{m} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq}\delta_{n-2k}.$$

If we compare the terms in the summation to the δ_{n-2m-2} term outside, we can easily see that this is precisely the term where $k = m+1$. So moving it inside, we have

$$\begin{aligned} J_{T,n}(t) &= t^{-4pq(m+1)(n+1)+4pq(m^2+m)+4pq(2m+2)}J_{T,n-2(m+1)}(t) \\ &\quad + \sum_{k=1}^{m+1} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq}\delta_{n-2k} \\ &= t^{-4pq(m+1)(n+1)+4pq(m+1)(m+2)}J_{T,n-2(m+1)}(t) \\ &\quad + \sum_{k=1}^{m+1} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq}\delta_{n-2k} \end{aligned}$$

as needed. Applying the formula at $s(n+3)-1$ gives the particular equation. □

We shall now find an annihilator for $J_{C,n}(t)$. By Lemma 3.1, replacing t^{2n} with M gives us

$$\begin{aligned} J_{C,n+2}(t) &= M^{-2rs}t^{-4rs}J_{C,n}(t) + (M^{r-rs-2pqs}t^{-2rs+2r-4pqs} \\ &\quad - M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t) + M^{r-rs-pqs}t^{-2rs+2r-2pqs}\delta_{s(n+1)-1}, \end{aligned}$$

and since $J_{C,n+2}(t) = L^2 J_{C,n}(t)$, we find

$$\begin{aligned} &(L^2 - M^{-2rs}t^{-4rs})J_{C,n}(t) \\ &\quad = (M^{r-rs-2pqs}t^{-2rs+2r-4pqs} \\ &\quad - M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t) + M^{r-rs-pqs}t^{-2rs+2r-2pqs}\delta_{s(n+1)-1}. \end{aligned}$$

In this equation, let

$$a(t,M) = M^{r-rs-2pqs}t^{-2rs+2r-4pqs} - M^{-r-rs}t^{-2rs-2r},$$

which is the coefficient of $J_{T,s(n+1)-1}(t)$, then obviously $a(t,M) \neq 0$, and we have

$$\begin{aligned} &a^{-1}(t,M)(L^2 - M^{-2rs}t^{-4rs})J_{C,n}(t) \\ &\quad = J_{T,s(n+1)-1}(t) + a^{-1}(t,M)M^{r-rs-pqs}t^{-2rs+2r-2pqs}\delta_{s(n+1)-1}. \end{aligned} \tag{3.2}$$

From Lemma 3.2, we have

$$(L^2 - t^{-8pqs^2-4pqs}M^{-2pqs^2})J_{T,s(n+1)-1}(t) = S_n.$$

So multiplying (3.2) from the left by $(L^2 - t^{-8pqs^2-4pqs}M^{-2pqs^2})$ gives

$$\begin{aligned}(L^2 - t^{-8pqs^2-4pqs}M^{-2pqs^2})&a^{-1}(t,M)(L^2 - M^{-2rs}t^{-4rs})J_{C,n}(t)\\ &= S_n + (L^2 - t^{-8pqs^2-4pqs}M^{-2pqs^2})a^{-1}(t,M)\\ &\quad\times M^{r-rs-pqs}t^{-2rs+2r-2pqs}\delta_{s(n+1)-1}\\ &= S_n + a^{-1}(t,t^4M)M^{r-rs-pqs}t^{-6rs+6r-6pqs}\delta_{s(n+3)-1}\\ &\quad - a^{-1}(t,M)M^{r-rs-pqs-2pqs^2}t^{-2rs+2r-8pqs^2-6pqs}\delta_{s(n+1)-1}. \qquad (3.3)\end{aligned}$$

Let $b(t,M)/(t^2 - t^{-2})$ denote the right-hand side of (3.3). Then $b(t,M)$ is a rational function in t and M. We claim that $b \neq 0$, which we show by checking $b(-1,M) := \lim_{t\to -1} b(t,M) \neq 0$. Recall

$$S_n = \sum_{k=1}^{s} M^{-2pqsk+pqs}t^{4pqk^2-12pqsk+6pqs}\delta_{s(n+3)-1-2k}.$$

So we have

$$\begin{aligned}\lim_{t\to -1}&(t^2 - t^{-2})S_n\\ &= \sum_{k=1}^{s} M^{-2pqsk+pqs}(M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})\\ &= (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})\frac{M^{pqs}(1-M^{-2pqs^2})}{M^{2pqs}-1}. \qquad (3.4)\end{aligned}$$

Also

$$\begin{aligned}\lim_{t\to -1}(t^2 - t^{-2})&(a^{-1}(t,t^4M)M^{r-rs-pqs}t^{6r-6rs-6pqs}\delta_{s(n+3)-1}\\ &\quad - a^{-1}(t,M)M^{r-rs-pqs-2pqs^2}t^{-2rs+2r-8pqs^2-6pqs}\delta_{s(n+1)-1})\\ &= a^{-1}(-1,M)(M^{r-rs-pqs} - M^{r-rs-pqs-2pqs^2})\\ &\quad\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})\\ &= \frac{1}{M^{r-rs-2pqs} - M^{-r-rs}}(M^{r-rs-pqs} - M^{r-rs-pqs-2pqs^2})\\ &\quad\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)}).\end{aligned}$$

Summing up the two limits above, we get

$$\begin{aligned}b(-1,&M)\\ &:= \lim_{t\to -1} b(t,M)\\ &= \left(\frac{M^{pqs}(1-M^{-2pqs^2})}{M^{2pqs}-1} + \frac{M^{r-rs-pqs} - M^{r-rs-pqs-2pqs^2}}{M^{r-rs-2pqs} - M^{-r-rs}}\right)\end{aligned}$$

$$\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})$$
$$= \frac{(-M^{pqs-r-rs} + M^{r-rs+pqs})(1 - M^{-2pqs^2})(M^{ps} - M^{-ps})(M^{qs} - M^{-qs})}{(M^{2pqs} - 1)(M^{r-rs-2pqs} - M^{-r-rs})},$$

which is not zero. So $b \neq 0$ and we conclude that our recurrence (3.3) is inhomogeneous. Therefore,

$$P(t, M, L) = (L - 1)b^{-1}(t, M)(L^2 - t^{-8pqs^2-4pqs}M^{-2pqs^2})$$
$$\times a^{-1}(t, M)(L^2 - M^{-2rs}t^{-4rs})$$

is an annihilator of $J_{C,n}(t)$ in $\widetilde{\mathcal{A}}_C$.

Up to this point all the results above in this section are valid for general C over T. From now on in this section, we put in the restriction that s is odd and $q > 2$. Once we prove that P is of minimal degree in L, it will follow that P is the recurrence polynomial of $J_{C,n}(t)$ up to normalization. We can check the AJ-conjecture by evaluating P at $t = -1$.

$$P(-1, M, L) = b^{-1}(-1, M)a^{-1}(-1, M)(L - 1)(L^2 - M^{-2pqs^2})(L^2 - M^{-2rs}),$$

which, up to a nonzero rational function in M, is equal to the A-polynomial of C.

3.2. *P is the recurrence polynomial of C*

We now wish to show that the operator P is the recurrence polynomial of C, up to normalization. It is enough to show that if an operator $Q = D_4L^4 + D_3L^3 + D_2L^2 + D_1L + D_0$ is an annihilator of $J_{C,n}(t)$ with $D_0, \ldots, D_4 \in \mathbb{Z}[t^{\pm1}, M^{\pm1}]$, then $Q = 0$.

Suppose $QJ_{C,n}(t) = 0$, that is,

$$D_4J_{C,n+4}(t) + D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t) = 0.$$

We wish to show that $D_i = 0$ for $i = 0, 1, 2, 3, 4$. Applying our Lemma 3.1, we have

$$0 = D_4J_{C,n+4}(t) + D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)$$
$$= D_4(M^{-2rs}t^{-12rs}J_{C,n+2}(t) + (M^{r-rs-2pqs}t^{-6rs+6r-12pqs}$$
$$- M^{-r-rs}t^{-6rs-6r})J_{T,s(n+3)-1}(t) + M^{r-rs-pqs}t^{-6rs+6r-6pqs}\delta_{s(n+3)-1})$$
$$+ D_3(M^{-2rs}t^{-8rs}J_{C,n+1}(t) + (M^{r-rs-2pqs}t^{-4rs+4r-8pqs}$$
$$- M^{-r-rs}t^{-4rs-4r})J_{T,s(n+2)-1}(t) + M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1})$$
$$+ D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)$$
$$= (D_4M^{-2rs}t^{-12rs} + D_2)(M^{-2rs}t^{-4rs}J_{C,n}(t) + (M^{r-rs-2pqs}t^{-2rs+2r-4pqs}$$
$$- M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t) + M^{r-rs-pqs}t^{2r-2rs-2pqs}\delta_{s(n+1)-1})$$
$$+ D_4((M^{r-rs-2pqs}t^{-6rs+6r-12pqs} - M^{-r-rs}t^{-6rs-6r})J_{T,s(n+3)-1}(t)$$
$$+ M^{r-rs-pqs}t^{6r-6rs-6pqs}\delta_{s(n+3)-1}) + D_3(M^{-2rs}t^{-8rs}J_{C,n+1}(t)$$

$$
\begin{aligned}
&+(M^{r-rs-2pqs}t^{-4rs+4r-8pqs} - M^{-r-rs}t^{-4rs-4r})J_{T,s(n+2)-1}(t)\\
&+M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1}) + D_1 J_{C,n+1}(t) + D_0 J_{C,n}(t)\\
={}&(D_0 + D_2 M^{-2rs}t^{-4rs} + D_4 M^{-4rs}t^{-16rs})J_{C,n}(t)\\
&+(D_1 + D_3 M^{-2rs}t^{-8rs})J_{C,n+1}(t)\\
&+D_4(M^{r-rs-2pqs}t^{-6rs+6r-12pqs} - M^{-r-rs}t^{-6rs-6r})J_{T,s(n+3)-1}(t)\\
&+D_3(M^{r-rs-2pqs}t^{-4rs+4r-8pqs} - M^{-r-rs}t^{-4rs-4r})J_{T,s(n+2)-1}(t)\\
&+(D_4 M^{-2rs}t^{-12rs} + D_2)(M^{r-rs-2pqs}t^{-2rs+2r-4pqs}\\
&-M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t)\\
&+(D_4 M^{-2rs}t^{-12rs} + D_2)M^{r-rs-pqs}t^{2r-2rs-2pqs}\delta_{s(n+1)-1}\\
&+D_3 M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1} + D_4 M^{r-rs-pqs}t^{6r-6rs-6pqs}\delta_{s(n+3)-1},
\end{aligned}
$$

and applying Lemma 3.2,

$$
\begin{aligned}
={}&(D_0 + D_2 M^{-2rs}t^{-4rs} + D_4 M^{-4rs}t^{-16rs})J_{C,n}(t)\\
&+(D_1 + D_3 M^{-2rs}t^{-8rs})J_{C,n+1}(t)\\
&+(D_4(M^{r-rs-2pqs}t^{-6rs+6r-12pqs} - M^{-r-rs}t^{-6rs-6r})M^{-2pqs^2}t^{-8pqs^2-4pqs}\\
&+(D_4 M^{-2rs}t^{-12rs} + D_2)(M^{r-rs-2pqs}t^{-2rs+2r-4pqs}\\
&-M^{-r-rs}t^{-2rs-2r}))J_{T,s(n+1)-1}(t) + D_3(M^{r-rs-2pqs}t^{-4rs+4r-8pqs}\\
&-M^{-r-rs}t^{-4rs-4r})J_{T,s(n+2)-1}(t) + (D_4 M^{-2rs}t^{-12rs}\\
&+D_2)M^{r-rs-pqs}t^{2r-2rs-2pqs}\delta_{s(n+1)-1} + D_3 M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1}\\
&+D_4 M^{r-rs-pqs}t^{6r-6rs-6pqs}\delta_{s(n+3)-1} + D_4(M^{r-rs-2pqs}t^{-6rs+6r-12pqs}\\
&-M^{-r-rs}t^{-6rs-6r})S_n\\
={}&D'_4 J_{C,n}(t) + D'_3 J_{C,n+1}(t) + D'_2 J_{T,s(n+1)-1}(t) + D'_1 J_{T,s(n+2)-1}(t) + D'_0.
\end{aligned}
$$

We claim that each $D'_i = 0$, and it then follows that each $D_i = 0$. Indeed, it follows easily from $D'_3 = D'_1 = 0$ that $D_3 = D_1 = 0$. For the rest, it is enough to show that the two linear equations defined by $D'_0 = 0$ and $D'_2 = 0$ are linearly independent (with D_2 and D_4 as variables). We can check that the determinant of the linear system is nonzero, and in particular, we multiply by $(t^2 - t^{-2})$ and then evaluate at $t = -1$ in order to use Eq. (3.4):

$$
\begin{aligned}
&(M^{r-rs-2pqs} - M^{-r-rs})\Bigg(M^{r-3rs-pqs} + M^{r-rs-pqs} + (M^{r-rs-2pqs} - M^{-r-rs})\\
&\qquad\times M^{pqs}\frac{(1-M^{-2pqs^2})}{M^{2pqs}-1}\Bigg)(M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})
\end{aligned}
$$

$$
\begin{aligned}
&- (M^{r-rs-2pqs-2pqs^2} - M^{-r-rs-2pqs^2} + M^{r-3rs-2pqs} - M^{-r-3rs})\\
&\times M^{r-rs-pqs}(M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})\\
= &\frac{1}{M^{2pqs}-1}(M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})(M^{r-2pqs} - M^{-r})\\
&\times (M^{-rs}(M^{r-rs-pqs}(M^{-2rs}+1)(M^{2pqs}-1)\\
&+ (M^{r-rs-pqs} - M^{-r-rs+pqs})(1-M^{-2pqs^2}))\\
&- M^{r-rs-pqs}(M^{-2pqs^2-rs} + M^{-3rs})(M^{2pqs}-1)),
\end{aligned}
$$

expanding some of the terms to observe cancelation,

$$
\begin{aligned}
= &\frac{1}{M^{2pqs}-1}(M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})(M^{r-2pqs} - M^{-r})\\
&\times (M^{r-4rs+pqs} - M^{r-4rs-pqs} + M^{r-2rs+pqs} - M^{r-2rs-pqs}\\
&+ M^{r-2rs-pqs} - M^{r-2rs-pqs-2pqs^2} - M^{-r-2rs+pqs} + M^{-r-2rs+pqs-2pqs^2}\\
&+ M^{r-2rs-pqs-2pqs^2} + M^{r-4rs-pqs} - M^{r-2rs+pqs-2pqs^2} - M^{r-4rs+pqs})\\
= &\frac{(M^{r-2pqs} - M^{-r})}{M^{2pqs}-1}(M^{ps} - M^{-ps})(M^{qs} - M^{-qs})(M^{-2pqs^2}-1)\\
&\times (M^{-r-2rs+pqs} - M^{r-2rs+pqs})
\end{aligned}
$$

which is indeed nonzero.

We now prove that if r is not an integer between 0 and pqs, we have $D'_i = 0$ for each $i = 0, 1, 2, 3, 4$ and thus our annihilator P is of minimal L-degree.

We say that a function $f : \mathbb{Z} \to \mathbb{Z}$ is a *quasi-polynomial* if there exist periodic functions $a_0, \ldots, a_d$ each with integral period such that

$$f(n) = \sum_{i=0}^{d} a_i(n) n^i,$$

and f is of degree d if $a_d \neq 0$. In particular, we say f is *quasi-quadratic* if f is a quasi-polynomial of degree 2.

Lemma 3.3. (1) *When $p > q$ and either $r < 0$ or $r > pqs$, we have $D'_i = 0$ for $i = 0, 1, 2, 3, 4$.*

(2) *When $p < -q$ and either $r > 0$ or $r < pqs$, we have $D'_i = 0$ for $i = 0, 1, 2, 3, 4$.*

Proof. (1) Suppose $p > q$, $r > pqs$, and some $D'_i \neq 0$. Then there must be another nonzero D'_j such that one of the following equalities hold:

$$
\begin{aligned}
\ell[D'_4 J_{C,n}(t)] &= \ell[D'_3 J_{C,n+1}(t)],\\
\ell[D'_4 J_{C,n}(t)] &= \ell[D'_2 J_{T,s(n+1)-1}(t)],\\
\ell[D'_4 J_{C,n}(t)] &= \ell[D'_1 J_{T,s(n+2)-1}(t)],\\
\ell[D'_3 J_{C,n+1}(t)] &= \ell[D'_2 J_{T,s(n+1)-1}(t)],
\end{aligned}
$$

$$\ell[D'_3 J_{C,n+1}(t)] = \ell[D'_1 J_{T,s(n+2)-1}(t)],$$
$$\ell[D'_2 J_{T,s(n+1)-1}(t)] = \ell[D'_1 J_{T,s(n+2)-1}(t)].$$

That is, two of the summands must share a lowest degree, and since $\ell[D'_0]$ is only linear in n for large enough n while $\ell[J_{C,n}(t)]$ and $\ell[J_{T,n}(t)]$ are quasi-quadratic by Lemmas 2.2 and 2.1, we can immediately dispose of the cases involving D'_0.

Subcase 3.1. $\ell[D'_4 J_{C,n}(t)] = \ell[D'_3 J_{C,n+1}(t)]$:

From the second formula of Lemma 2.2(1), we have

$$\begin{aligned}\ell[D'_4] - \ell[D'_3] &= \ell[J_{C,n+1}(t)] - \ell[J_{C,n}(t)]\\ &= -2rsn - rs - (-1)^n((s-2)(r-pqs) + (p-2)(q-2)),\end{aligned}$$

but for sufficiently large n, $\ell[D'_4] - \ell[D'_3]$ is a linear function in n, while the right-hand side is not a polynomial, so we have a contradiction.

Subcase 3.2. $\ell[D'_4 J_{C,n}(t)] = \ell[D'_2 J_{T,s(n+1)-1}(t)]$:

From Lemmas 2.2(1) and 2.1(1), we have

$$\begin{aligned}\ell[D'_4] - \ell[D'_2] &= \ell[J_{T,s(n+1)-1}(t)] - \ell[J_{C,n}(t)]\\ &= s(r-pqs)n^2 + (2pqs - 2pqs^2)n - pqs^2 + 2pqs - rs\\ &\quad - \frac{1}{2}(1-(-1)^{n-1})(s-2)(r-pqs),\end{aligned}$$

which is quasi-quadratic, while the left-hand side is at most linear, giving us another contradiction.

Subcase 3.3. $\ell[D'_4 J_{C,n}(t)] = \ell[D'_1 J_{T,s(n+2)-1}(t)]$:

Here we have

$$\begin{aligned}\ell[D'_4] - \ell[D'_1] &= \ell[J_{T,s(n+2)-1}(t)] - \ell[J_{C,n}(t)]\\ &= s(r-pqs)n^2 + (2pqs - 4pqs^2)n - pq(4s^2 - 4s) - rs\\ &\quad - \frac{1}{2}(1-(-1)^{n-1})(s-2)(r-pqs) + (-1)^n(p-2)(q-2)\end{aligned}$$

again giving us a quasi-quadratic function on the right and a linear function on the left, which is a contradiction.

Subcase 3.4. $\ell[D'_3 J_{C,n+1}(t)] = \ell[D'_2 J_{T,s(n+1)-1}(t)]$:

We have

$$\begin{aligned}\ell[D'_3] - \ell[D'_2] &= \ell[J_{T,s(n+1)-1}(t)] - \ell[J_{C,n+1}(t)]\\ &= s(r-pqs)n^2 + (2pqs + 2rs - 2pqs^2)n - pq(s^2 - 2s)\\ &\quad - \frac{1}{2}(1-(-1)^{n-1})(s-2)(r-pqs) + (-1)^n(p-2)(q-2)\end{aligned}$$

which is again quasi-quadratic on the right and linear on the left, again a contradiction.

Subcase 3.5. $\ell[D'_3 J_{C,n+1}(t)] = \ell[D'_1 J_{T,s(n+2)-1}(t)]$:

We have

$$\begin{aligned}\ell[D'_3] - \ell[D'_1] &= \ell[J_{T,s(n+2)-1}(t)] - \ell[J_{C,n+1}(t)] \\ &= (rs - pqs^2)n^2 + (2pqs + 2rs - 4pqs^2)n - pq(4s^2 - 4s) \\ &\quad - \frac{1}{2}(1-(-1)^n)(s-2)(r-pqs)\end{aligned}$$

giving us another contradiction.

Subcase 3.6. $\ell[D'_2 J_{T,s(n+1)-1}(t)] = \ell[D'_1 J_{T,s(n+2)-1}(t)]$:

This time we have

$$\begin{aligned}\ell[D'_2] - \ell[D'_1] &= \ell[J_{T,s(n+2)-1}(t)] - \ell[J_{T,s(n+1)-1}(t)] \\ &= -2pqs^2 n + 2pqs - 3pqs^2 + \frac{1}{2}(-1)^{(n+1)s}(1-(-1)^s)(p-2)(q-2) \\ &= -2pqs^2 n + 2pqs - 3pqs^2 - (-1)^n (p-2)(q-2)\end{aligned}$$

which is alternating on the right and eventually linear on the left, which is a contradiction. This exhausts the possibilities of the case $r > pqs$.

Now assume $p > q$ and $r < 0$. We first consider the highest degrees of the summands in the equation

$$0 = D'_4 J_{C,n}(t) + D'_3 J_{C,n+1}(t) + D'_2 J_{T,s(n+1)-1}(t) + D'_1 J_{T,s(n+2)-1}(t) + D'_0$$

for large positive n's. If $D'_3 \neq 0$ or $D'_4 \neq 0$, then both of them cannot be zero since by Lemma 2.2(1) $\hbar[J_{C,n}]$ and $\hbar[J_{C,n+1}]$ are each quasi-quadratic while by Lemma 2.1(1) $\hbar[J_{T,s(n+1)-1}(t)]$ and $\hbar[J_{T,s(n+2)-1}(t)]$ are each linear in n (for positive n's), and we must have the following

Subcase 3.7. $\hbar[D'_4 J_{C,n}(t)] = \hbar[D'_3 J_{C,n+1}(t)]$:

Then we have, by Lemma 2.2(1),

$$\begin{aligned}\hbar[D'_4] - \hbar[D'_3] &= \hbar[J_{C,n+1}(t)] - \hbar[J_{C,n}(t)] \\ &= -2rsn - rs - (-1)^n(s-2)(r - 2pq + 2p + 2q)\end{aligned}$$

which is a linear polynomial for large n on the left but is not a linear polynomial on the right, giving a contradiction.

So both D'_4 and D'_3 are zero. So we have $0 = D'_2 J_{T,s(n+1)-1}(t) + D'_1 J_{T,s(n+2)-1}(t) + D'_0$. We can then analyze the lowest degrees in a similar fashion as above; if one of the D'_i is not zero, we must have

Subcase 3.8. $\ell[D'_2 J_{T,s(n+1)-1}(t)] = \ell[D'_1 J_{T,s(n+2)-1}(t)]$:

This can be treated similarly as Subcase 3.6. We conclude that each $D_i' = 0$. This completes the proof of part (1).

Part (2) of the lemma can be proved similarly with the use of Lemmas 2.1(2) and 2.2(2). □

Remark 3.1. In the proof of Lemma 3.3 we used the condition that $s > 2$ odd and $q > 2$ in several subcases. Some of these subcases will disappear accordingly in later sections when we impose the condition s odd and $q = 2$ or $s > 2$ even or $s = 2$.

4. Case $s > 2$ is Odd and $q = 2$

4.1. *An annihilator P of $J_{C,n}(t)$*

Define:

$$U_n = \sum_{k=1}^{s} (-1)^{k-1} t^{2psn-4psnk+2pk^2-8psk+2pk+4ps} \frac{t^{4sn+8s-2-4k} - t^{-4sn-8s+2+4k}}{t^2 - t^{-2}}.$$

When $q = 2$, we have by [12, Lemma 1.5] the identity

$$J_{T,n+1}(t) = -t^{-(4n+2)p} J_{T,n}(t) + t^{-2pn} \frac{t^{4n+2} - t^{-4n-2}}{t^2 - t^{-2}}. \tag{4.1}$$

Note again that (4.1) is valid for negative p as well.

Lemma 4.1. *When $q = 2$, for all positive integers m, we have*

$$\begin{aligned} J_{T,n}(t) &= (-1)^m t^{(-4mn+2m^2)p} J_{T,n-m}(t) \\ &\quad + \sum_{k=1}^{m} (-1)^{k-1} t^{-(4k-2)pn+(2k^2-2k+2)p} \frac{t^{4n+2-4k} - t^{-4n-2+4k}}{t^2 - t^{-2}} \end{aligned}$$

and in particular, when s is odd,

$$J_{T,s(n+2)-1}(t) = -t^{-4ps^2n+4ps-6ps^2} J_{T,s(n+1)-1}(t) + U_n.$$

Proof. Apply the relation (4.1) m times. □

Note that the relation (3.2) is valid for general C over T. Specializing it at $q = 2$ and s odd, we have

$$\begin{aligned} &a^{-1}(t, M)(L^2 - M^{-2rs} t^{-4rs}) J_{C,n}(t) \\ &\quad = J_{T,s(n+1)-1}(t) + a^{-1}(t, M) M^{r-rs-2ps} t^{2r-2rs-4ps} \delta_{s(n+1)-1}. \end{aligned} \tag{4.2}$$

From Lemma 4.1, we get

$$(L + M^{-2ps^2} t^{4ps-6ps^2}) J_{T,s(n+1)-1}(t) = U_n.$$

Multiplying (4.2) from the left by $(L + M^{-2ps^2} t^{4ps-6ps^2})$ yields

$$\begin{aligned}(L + M^{-2ps^2} t^{4ps-6ps^2}) & a^{-1}(t,M)(L^2 - M^{-2rs}t^{-4rs})J_{C,n}(t) \\ &= U_n + (L + M^{-2ps^2}t^{4ps-6ps^2})a^{-1}(t,M)M^{r-rs-2ps}t^{2r-2rs-4ps}\delta_{s(n+1)-1} \\ &= U_n + a^{-1}(t,t^2M)M^{r-rs-2ps}t^{4r-4rs-8ps}\delta_{s(n+2)-1} \\ &\quad + a^{-1}(t,M)M^{r-rs-2ps-2ps^2}t^{2r-2rs-6ps^2}\delta_{s(n+1)-1}. \end{aligned} \tag{4.3}$$

As in Sec. 3, let $b(t,M)/(t^2 - t^{-2})$ denote the right-hand side of (4.3). Then $b(t,M)$ is a rational function in t and M. We now show that $b \neq 0$ by checking $b(-1,M) := \lim_{t\to-1} b(t,M) \neq 0$. Rewrite U_n as a function of t and M by changing t^{2n} to M:

$$U_n = \sum_{k=1}^{s}(-1)^{k-1}M^{ps-2psk}t^{2pk^2-8psk+2pk+4ps}\frac{M^{2s}t^{8s-2-4k} - M^{-2s}t^{-8s+2+4k}}{t^2 - t^{-2}}.$$

So we have

$$\begin{aligned}\lim_{t\to-1}(t^2 - t^{-2})U_n &= \sum_{k=1}^{s}(-1)^{k-1}M^{ps-2psk}(M^{2s} - M^{-2s}) \\ &= \frac{(M^{2s} - M^{-2s})M^{-ps}(1 + M^{-2ps^2})}{1 + M^{-2ps}}. \end{aligned} \tag{4.4}$$

Also

$$\begin{aligned}\lim_{t\to-1}(t^2 - t^{-2})(&a^{-1}(t,t^2M)M^{r-rs-2ps}t^{4r-4rs-8ps}\delta_{s(n+3)-1} \\ &+ a^{-1}(t,M)M^{r-rs-2ps-2ps^2}t^{2r-2rs-6ps^2}\delta_{s(n+1)-1}) \\ &= \frac{M^{r-rs-2ps} + M^{r-rs-2ps-2ps^2}}{M^{r-rs-4ps} - M^{-r-rs}} \\ &\quad \times (M^{s(p+2)} + M^{-s(p+2)} - M^{s(2-p)} - M^{-s(2-p)}). \end{aligned}$$

Summing up the two limits above, we get

$$\begin{aligned} b(-1,M) &:= \lim_{t\to-1} b(t,M) \\ &= \frac{(M^{2s} - M^{-2s})M^{-ps}(1 + M^{-2ps^2})}{1 + M^{-2ps}} + \frac{M^{r-rs-2ps} + M^{r-rs-2ps-2ps^2}}{M^{r-rs-4ps} - M^{-r-rs}} \\ &\quad \times (M^{s(p+2)} + M^{-s(p+2)} - M^{s(2-p)} - M^{-s(2-p)}) \\ &= \frac{(M^{2s} - M^{-2s})(1 + M^{-2ps^2})M^{-rs-ps}(M^r - M^{-r})}{(1 + M^{-2ps})(M^{r-rs-4ps} - M^{-r-rs})}, \end{aligned}$$

which is not zero. So $b \neq 0$ and we conclude that our recurrence (4.3) is inhomogeneous. Therefore,

$$P(t,M,L) = (L-1)b^{-1}(t,M)(L + M^{-2ps^2}t^{4ps-6ps^2})a^{-1}(t,M)(L^2 - M^{-2rs}t^{-4rs})$$

is an annihilator of $J_{C,n}(t)$ in $\widetilde{\mathcal{A}}_C$.

Once we prove that P is of minimal degree in L, it will follow that P is the recurrence polynomial of $J_{C,n}(t)$ up to normalization. We can check the AJ-conjecture by evaluating P at $t=-1$.

$$P(-1,M,L)=b^{-1}(-1,M)a^{-1}(-1,M)(L-1)(L+M^{-2ps^2})(L^2-M^{-2rs}),$$

which, up to a nonzero factor in $\mathbb{Q}(M)$, is equal to the A-polynomial of C.

4.2. *P is the recurrence polynomial of C*

We now wish to show that the operator P is the recurrence polynomial of C. It is enough to show that if an operator $Q = D_3L^3 + D_2L^2 + D_1L + D_0$ with $D_0,\ldots,D_3 \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$ is an annihilator of $J_{C,n}(t)$, then $Q=0$.

Suppose $QJ_{C,n}(t) = 0$, that is, $D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t) = 0$. We wish to show that $D_i = 0$ for $i = 0,1,2,3$. We have by Lemma 3.1 (specialized at $q=2$)

$$\begin{aligned}
0 &= D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)\\
&= D_3(M^{-2rs}t^{-8rs}J_{C,n+1}(t) + (M^{r-rs-4ps}t^{-4rs+4r-16ps}\\
&\quad - M^{-r-rs}t^{-4rs-4r})J_{T,s(n+2)-1}(t) + M^{r-rs-2ps}t^{4r-4rs-8ps}\delta_{s(n+2)-1})\\
&\quad + D_2(M^{-2rs}t^{-4rs}J_{C,n}(t) + (M^{r-rs-4ps}t^{-2rs+2r-8ps}\\
&\quad - M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t) + M^{r-rs-2ps}t^{2r-2rs-4ps}\delta_{s(n+1)-1})\\
&\quad + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)\\
&= (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t) + (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t)\\
&\quad + D_3(M^{r-rs-4ps}t^{-4rs+4r-16ps} - M^{-r-rs}t^{-4rs-4r})J_{T,s(n+2)-1}(t)\\
&\quad + D_2(M^{r-rs-4ps}t^{-2rs+2r-8ps} - M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t)\\
&\quad + D_2M^{r-rs-2ps}t^{2r-2rs-4ps}\delta_{s(n+1)-1} + D_3M^{r-rs-2ps}t^{4r-4rs-8ps}\delta_{s(n+2)-1},
\end{aligned}$$

and applying Lemma 4.1,

$$\begin{aligned}
&= (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t) + (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t)\\
&\quad + D_3(M^{r-rs-4ps}t^{-4rs+4r-16ps} - M^{-r-rs}t^{-4rs-4r})\\
&\quad \times (-M^{-2ps^2}t^{4ps-6ps^2}J_{T,s(n+1)-1}(t) + U_n)\\
&\quad + D_2(M^{r-rs-4ps}t^{-2rs+2r-8ps} - M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t)\\
&\quad + D_2M^{r-rs-2ps}t^{2r-2rs-4ps}\delta_{s(n+1)-1} + D_3M^{r-rs-2ps}t^{4r-4rs-8ps}\delta_{s(n+2)-1}\\
&= (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t) + (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t)\\
&\quad + (D_3(M^{r-rs-4ps}t^{-4rs+4r-16ps} - M^{-r-rs}t^{-4rs-4r})(-M^{-2ps^2}t^{4ps-6ps^2})\\
&\quad + D_2(M^{r-rs-4ps}t^{-2rs+2r-8ps} - M^{-r-rs}t^{-2rs-2r}))J_{T,s(n+1)-1}(t)
\end{aligned}$$

$$
\begin{aligned}
&+ D_2 M^{r-rs-2ps} t^{2r-2rs-4ps} \delta_{s(n+1)-1} + D_3((M^{r-rs-4ps} t^{-4rs+4r-16ps} \\
&- M^{-r-rs} t^{-4rs-4r}) U_n + M^{r-rs-2ps} t^{4r-4rs-8ps} \delta_{s(n+2)-1}) \\
= {}& D'_3 J_{C,n}(t) + D'_2 J_{C,n+1}(t) + D'_1 J_{T,s(n+1)-1}(t) + D'_0.
\end{aligned}
$$

We claim that each $D'_i = 0$, and it then follows as in the previous section that each $D_i = 0$. We again wish to show that the two linear equations defined by $D'_0 = 0$ and $D'_1 = 0$ are linearly independent (with D_2 and D_3 as variables). So let us check the determinant of the linear system, multiplied by $(t^2 - t^{-2})$ and then valued at $t = -1$, is nonzero:

$$
\begin{aligned}
&(M^{r-rs-4ps} - M^{-r-rs})(-M^{-2ps^2}) M^{r-rs-2ps} (M^{s(p+2)} + M^{-s(p+2)} \\
&\quad - M^{s(2-p)} - M^{-s(2-p)}) - (M^{r-rs-4ps} - M^{-r-rs}) \\
&\quad \times \left((M^{r-rs-4ps} - M^{-r-rs}) \frac{(M^{2s} - M^{-2s}) M^{-ps} (1 + M^{-2ps^2})}{1 + M^{-2ps}} \right. \\
&\quad \left. + M^{r-rs-2ps} (M^{s(p+2)} + M^{-s(p+2)} - M^{s(2-p)} - M^{-s(2-p)}) \right) \\
&\quad = -(M^{r-rs-4ps} - M^{-r-rs})(M^{-2ps^2} + 1) \left(M^{r-rs-2ps} (M^{s(p+2)} + M^{-s(p+2)} \right. \\
&\quad \left. - M^{s(2-p)} - M^{-s(2-p)}) + (M^{r-rs-4ps} - M^{-r-rs}) \frac{(M^{2s} - M^{-2s}) M^{-ps}}{1 + M^{-2ps}} \right) \\
&\quad = -(M^{r-rs-4ps} - M^{-r-rs})(M^{-2ps^2} + 1)(M^{2s} - M^{-2s}) \\
&\quad \times \left(M^{r-rs-2ps} (M^{ps} - M^{-ps}) + (M^{r-rs-4ps} - M^{-r-rs}) \frac{1}{M^{ps} + M^{-ps}} \right) \\
&\quad = -\frac{1}{M^{ps} + M^{-ps}} (M^{r-rs-4ps} - M^{-r-rs})(M^{-2ps^2} + 1)(M^{2s} - M^{-2s}) \\
&\quad \times (M^{r-rs-2ps} (M^{2ps} - M^{-2ps}) + (M^{r-rs-4ps} - M^{-r-rs})) \\
&\quad = -\frac{1}{M^{ps} + M^{-ps}} (M^{r-rs-4ps} - M^{-r-rs}) \\
&\quad \times (M^{-2ps^2} + 1)(M^{2s} - M^{-2s})(M^{r-rs} - M^{-r-rs})
\end{aligned}
$$

which is indeed nonzero.

The following lemma shows that each $D'_i = 0$ if r is not a number between 0 and pqs.

Lemma 4.2. (1) *When $p > q = 2$ and either $r < 0$ or $r > pqs$, we have $D'_i = 0$ for $i = 0, 1, 2, 3$.*

(2) *When $p < -q$ and either $r > 0$ or $r < pqs$, we have $D'_i = 0$ for $i = 0, 1, 2, 3$.*

Proof. The proof is entirely similar to that of Lemma 3.3 (also cf. Remark 3.1). □

5. Case $s > 2$ is Even

5.1. *An annihilator P of $J_{C,n}(t)$*

Recall our definition

$$\delta_j = \frac{t^{2(p+q)(j+1)+2} + t^{-2(p+q)(j+1)+2} - t^{2(q-p)(j+1)-2} - t^{-2(q-p)(j+1)-2}}{t^2 - t^{-2}}$$

and further define:

$$V_n = \sum_{k=1}^{s/2} t^{-4pqsnk+2pqsn+4pqk^2-8pqsk+4pqs}\delta_{s(n+2)-1-2k}.$$

Lemma 5.1. *If s is even, then*

$$J_{T,s(n+2)-1}(t) = t^{-2pqs^2n-3pqs^2+2pqs}J_{T,s(n+1)-1}(t) + V_n.$$

Proof. Apply Lemma 3.2, setting $m = s/2$. □

Lemma 5.1 yields the relation

$$(L - M^{-pqs^2}t^{-3pqs^2+2pqs})J_{T,s(n+1)-1}(t) = V_n.$$

So applying the operator $(L - M^{-pqs^2}t^{-3pqs^2+2pqs})$ to both sides of (3.2) gives:

$$\begin{aligned}
&(L - M^{-pqs^2}t^{-3pqs^2+2pqs})a^{-1}(t,M)(L^2 - M^{-2rs}t^{-4rs})J_{C,n}(t)\\
&\quad = V_n + (L - M^{-pqs^2}t^{-3pqs^2+2pqs})a^{-1}(t,M)M^{r-rs-pqs}t^{2r-2rs-2pqs}\delta_{s(n+1)-1}\\
&\quad = V_n + a^{-1}(t,t^2M)M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1}\\
&\qquad - a^{-1}(t,M)M^{r-rs-pqs^2-pqs}t^{2r-2rs-3pqs^2}\delta_{s(n+1)-1}.
\end{aligned}$$

To see this is an inhomogeneous recursion for $J_{C,n}(t)$, let $b(t,M)/(t^2 - t^{-2})$ be the right-hand side of this equation and check it is nonzero. As before it suffices to check that $\lim_{t\to -1} b(t,M) \neq 0$, considering V_n and $\delta_{s(n+k)-j}$'s as functions of t and M (changing t^{2n} to M). We have

$$\begin{aligned}
b(-1,M) &= \lim_{t\to -1} b(t,M)\\
&= \lim_{t\to -1}(t^2 - t^{-2})(V_n + a^{-1}(t,t^2M)M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1}\\
&\quad - a^{-1}(t,M)M^{r-rs-pqs^2-pqs}t^{2r-2rs-3pqs^2}\delta_{s(n+1)-1})\\
&= \left(\sum_{i=1}^{s/2} M^{-2pqsj+pqs} + a^{-1}(-1,M)(M^{r-rs-pqs} - M^{r-rs-pqs^2-pqs})\right)\\
&\quad \times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})\\
&= \left(\frac{M^{-pqs} - M^{-pqs^2-pqs}}{1 - M^{-2pqs}} + \frac{M^{r-rs-pqs} - M^{r-rs-pqs^2-pqs}}{M^{r-rs-2pqs} - M^{-r-rs}}\right)
\end{aligned}$$

$$\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})$$
$$= \frac{(1 - M^{-pqs^2})M^{-rs-pqs}(M^r - M^{-r})(M^{ps} - M^{-ps})(M^{qs} - M^{-qs})}{(1 - M^{-2pqs})(M^{r-rs-2pqs} - M^{-r-rs})},$$

which is indeed nonzero. Hence

$$P(t, M, L) = (L - 1)b^{-1}(t, M)(L - M^{-pqs^2}t^{-3pqs^2+2pqs})$$
$$\times a^{-1}(t, M)(L^2 - M^{-2rs}t^{-4rs})$$

is an annihilator of $J_{C,n}(t)$. Assuming P is of minimal degree in L, we can now check the AJ-conjecture by evaluating P at $t = -1$. We have

$$P(-1, M, L) = b^{-1}(-1, M)a^{-1}(-1, M)(L - 1)(L - M^{-pqs^2})(L^2 - M^{-2rs}),$$

which agrees with the A-polynomial of C, up to a nonzero factor of a rational function in $\mathbb{Q}(M)$.

5.2. *P is the recurrence polynomial of C*

We now want to show that the operator P is the recurrence polynomial of C. It is enough to prove that if $Q = D_3L^3 + D_2L^2 + D_1L + D_0$ is an element in $\mathcal{A}_C$, then $Q = 0$. As in Sec. 4.2 we have

$$0 = D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)$$
$$= (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t) + (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t)$$
$$+ D_3(M^{r-rs-2pqs}t^{-4rs+4r-8pqs} - M^{-r-rs}t^{-4rs-4r})J_{T,s(n+2)-1}(t)$$
$$+ D_2(M^{r-rs-2pqs}t^{-2rs+2r-4pqs} - M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t)$$
$$+ D_2M^{r-rs-pqs}t^{2r-2rs-2pqs}\delta_{s(n+1)-1} + D_3M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1},$$

and applying Lemma 5.1,

$$= (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t) + (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t)$$
$$+ D_3(M^{r-rs-2pqs}t^{-4rs+4r-8pqs} - M^{-r-rs}t^{-4rs-4r})$$
$$\times (M^{-pqs^2}t^{-3pqs^2+2pqs}J_{T,s(n+1)-1}(t) + V_n)$$
$$+ D_2(M^{r-rs-2pqs}t^{-2rs+2r-4pqs} - M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t)$$
$$+ D_2M^{r-rs-pqs}t^{2r-2rs-2pqs}\delta_{s(n+1)-1} + D_3M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1}$$
$$= (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t) + (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t)$$
$$+ (D_3(M^{r-rs-2pqs}t^{-4rs+4r-8pqs} - M^{-r-rs}t^{-4rs-4r})M^{-pqs^2}t^{-3pqs^2+2pqs}$$

$$+ D_2(M^{r-rs-2pqs}t^{-2rs+2r-4pqs} - M^{-r-rs}t^{-2rs-2r}))J_{T,s(n+1)-1}(t)$$
$$+ D_2 M^{r-rs-pqs}t^{2r-2rs-2pqs}\delta_{s(n+1)-1} + D_3(M^{r-rs-pqs}t^{4r-4rs-4pqs}\delta_{s(n+2)-1}$$
$$+ (M^{r-rs-2pqs}t^{-4rs+4r-8pqs} - M^{-r-rs}t^{-4rs-4r})V_n)$$
$$= D'_3 J_{C,n}(t) + D'_2 J_{C,n+1}(t) + D'_1 J_{T,s(n+1)-1}(t) + D'_0.$$

As in the previous section, we show that $D'_i = 0$, $i = 1, \ldots, 3$, implies $D_i = 0$, $i = 0, \ldots, 3$. We just need to show that the two linear equations defined by $D'_0 = 0$ and $D'_1 = 0$ are linearly independent. Again we just need to check the determinant of the linear system, multiplied by $(t^2 - t^{-2})$ and then valued at $t = -1$, is nonzero:

$$\Bigg((M^{r-rs-2pqs} - M^{-r-rs})M^{-pqs^2}M^{r-rs-pqs} - (M^{r-rs-2pqs} - M^{-r-rs})$$
$$\times \left(M^{r-rs-pqs} + (M^{r-rs-2pqs} - M^{-r-rs})\frac{M^{-pqs} - M^{-pqs^2-pqs}}{1 - M^{-2pqs}}\right)\Bigg)$$
$$\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})$$
$$= \frac{(M^{r-rs-2pqs} - M^{-r-rs})}{1 - M^{-2pqs}}(M^{r-rs-pqs^2-pqs}(1 - M^{-2pqs})$$
$$- M^{r-rs-pqs}(1 - M^{-2pqs}) - (M^{r-rs-2pqs} - M^{-r-rs})$$
$$\times (M^{-pqs} - M^{-pqs^2-pqs}))(M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})$$
$$= (M^{r-rs-2pqs} - M^{-r-rs})(M^{-pqs^2} - 1)(M^{r-rs-pqs} - M^{-r-rs-pqs})$$
$$\times (M^{ps} - M^{-ps})(M^{qs} - M^{-qs})$$

which is indeed nonzero.

The following lemma shows that each $D'_i = 0$ if r is not a number between 0 and pqs.

Lemma 5.2. (1) *When $p > q$ and either $r < 0$ or $r > pqs$, we have $D'_i = 0$ for $i = 0, 1, 2, 3$.*
(2) *When $p < -q$ and either $r > 0$ or $r < pqs$, we have $D'_i = 0$ for $i = 0, 1, 2, 3$.*

The proof is similar to that of Lemma 3.3.

6. Case $s = 2$

6.1. *An annihilator P of $J_{C,n}(t)$*

In this section, we assume that C is a $(r, 2)$-cabled knot over a torus knot $T = T(p, q)$. In

$$J_{C,n+1}(t) = t^{-2r((n+1)^2-1)} \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} t^{4rk(2k+1)} J_{T,4k+1}(t),$$

let $k = -(j + \frac{1}{2})$, then

$$\begin{aligned}
J_{C,n+1}(t) &= t^{-2r((n+1)^2-1)} \sum_{j=\frac{n-1}{2}}^{-\frac{n+1}{2}} t^{4r(2j+1)j} J_{T,-4j-1}(t) \\
&= -t^{-2r((n+1)^2-1)} \sum_{j=\frac{n-1}{2}}^{-\frac{n+1}{2}} t^{4r(2j+1)j} J_{T,4j+1}(t) \\
&= -t^{-2r((n+1)^2-1)} \left[t^{2rn(n+1)} J_{T,2n+1} + \sum_{j=\frac{n-1}{2}}^{-\frac{n-1}{2}} t^{4r(2j+1)j} J_{T,4j+1}(t) \right] \\
&= -t^{-2r((n+1)^2-1)} [t^{2rn(n+1)} J_{T,-2n-1} + t^{2r(n^2-1)} J_{C,n}(t)] \\
&= t^{-2rn} J_{T,2n+1} - t^{-4rn-2r} J_{C,n}(t). \qquad (6.1)
\end{aligned}$$

Turning t^{2n} into M and $J_{C,n+1}(t)$ into $LJ_{C,n}(t)$, we see that

$$(L + M^{-2r}t^{-2r})J_{C,n}(t) = M^{-r} J_{T,2n+1}(t),$$

or

$$M^r(L + M^{-2r}t^{-2r})J_{C,n}(t) = J_{T,2n+1}(t).$$

We now wish to find an inhomogeneous recurrence for $J_{T,2n+1}(t)$. Recall equation (3.1):

$$J_{T,n+2}(t) = t^{-4pq(n+1)} J_{T,n}(t) + t^{-2pq(n+1)} \delta_n,$$

which implies that

$$\begin{aligned}
J_{T,2n+3}(t) &= t^{-4pq(2n+2)} J_{T,2n+1}(t) + t^{-2pq(2n+2)} \delta_{2n+1} \\
&= M^{-4pq} t^{-8pq} J_{T,2n+1}(t) + M^{-2pq} t^{-4pq} \delta_{2n+1}, \qquad (6.2)
\end{aligned}$$

and so

$$(L - M^{-4pq}t^{-8pq})J_{T,2n+1}(t) = M^{-2pq}t^{-4pq}\delta_{2n+1}.$$

Letting $b(t, M)/(t^2 - t^{-2}) = M^{-2pq}t^{-4pq}\delta_{2n+1}$. Then $b(t, M) \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$, which is obviously nonzero, and we obtain an operator $P(t, M, L)$ which annihilates $J_{C,n}(t)$ given by

$$P(t, M, L) = (L - 1)b^{-1}(t, M)(L - M^{-4pq}t^{-8pq})M^r(L + M^{-2r}t^{-2r}).$$

Assuming P has the minimal L degree, we can check the AJ-conjecture. Evaluating $P(-1, M, L)$ gives

$$P(-1, M, L) = b^{-1}(-1, M)(L - 1)(L - M^{-4pq})M^r(L + M^{-2r}),$$

which is equal to the A-polynomial of C up to a nonzero factor in $\mathbb{Q}(M)$.

6.2. *P is the recurrence polynomial of C*

Next we show that the operator P given above is a generator of the ideal $\widetilde{\mathcal{A}}_C$. It amounts to show that if an operator $Q = D_2L^2 + D_1L + D_0$, where each $D_j \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$, is an annihilator of $J_{C,n}(t)$, then $Q = 0$.

So suppose that $QJ_{C,n}(t) = 0$, i.e.

$$D_2J_{C,n+2}(t) + D_1J_{C,n+1} + D_0J_{C,n}(t) = 0. \tag{6.3}$$

Our goal is to show that $D_i = 0$, $i = 0, 1, 2$.

Using (6.1) and (6.2) we can transform (6.3) into

$$\begin{aligned}
0 &= D_2(t^{-4rn-2r}J_{T,2n+3}(t) - t^{-4rn-6r}(M^{-r}J_{T,2n+1}(t) - t^{-4rn-2r}J_{C,n}(t))) \\
&\quad + D_1(t^{-rn}J_{T,2n+1}(t) - t^{-4rn-2r}J_{C,n}(t)) + D_0J_{C,n}(t) \\
&= (D_2t^{-8rn-8r} - D_1t^{-4rn-2r} + D_0)J_{C,n}(t) \\
&\quad + D_2t^{-2rn-2r}(t^{-8pqn-8pq}J_{T,2n+1}(t) + t^{-4pqn-4pq}\delta_{2n+1}) \\
&\quad + (-D_2t^{-6rn-6r} + D_1t^{-2rn})J_{T,2n+1}(t) \\
&= (D_2t^{-8r(n+1)} - D_1t^{-4rn-2r} + D_0)J_{C,n}(t) \\
&\quad + (D_2(t^{-2r(n+1)-8pq(n+1)} - t^{-6r(n+1)}) + D_1t^{-2rn})J_{T,2n+1}(t) \\
&\quad + D_2t^{-2r(n+1)-4pq(n+1)}\delta_{2n+1} \\
&= D_2'J_{C,n}(t) + D_1'J_{T,2n+1}(t) + D_0'.
\end{aligned}$$

If we can show that $D_i' = 0$, $i = 0, 1, 2$, then it will follow right away that $D_i = 0$, $i = 0, 1, 2$. As in Lemma 3.3, we can show that $D_i' = 0$, $i = 0, 1, 2$, if r is not an integer between 0 and $2pq$.

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