The AJ-Conjecture and cabled knots over torus knots

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ABSTRACT

We show that most cabled knots over torus knots in $S^3$ satisfy the AJ-conjecture, namely each $(r,s)$-cabled knot over each $(p,q)$-torus knot satisfies the AJ-conjecture if $r$ is not a number between 0 and $pqs$.

Keywords: Colored Jones polynomials; A-polynomials; the AJ-conjecture; cabled knots.

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1. Introduction

For a knot $K$ in $S^3$, let $J_{K,n}(t)$ denote the $n$-colored Jones polynomial of $K$ with the zero framing, normalized so that for the unknot $U$,

$$J_{U,n}(t) = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

A remarkable result, proved in [4], asserts that for every knot $K$, $J_{K,n}(t)$ satisfies a nontrivial linear recurrence relation. By defining $J_{K,-n}(t) := -J_{K,n}(t)$ and $J_{K,0}(t) = 0$, one may treat $J_{K,n}(t)$ as a discrete function

$$J_{K,-n} : \mathbb{Z} \to \mathbb{Z}[t^{\pm1}],$$

The quantum torus

$$T = \mathbb{C}[t^{\pm1}](L^{\pm1}, M^{\pm1})/(LM - t^2ML)$$

acts on the set of discrete functions $f : \mathbb{Z} \to \mathbb{C}[t^{\pm1}]$ by

$$(Mf)(n) := t^{2n}f(n), \quad (Lf)(n) := f(n + 1).$$
Then linear recurrence relations of $J_{K,n}(t)$ correspond naturally to annihilators of $J_{K,n}(t)$ in $T$. The latter set, which we denote by

$$\mathcal{A}_K := \{ P \in T \mid PJ_{K,n}(t) = 0 \},$$

is obviously a left ideal of $T$, called the recurrence ideal of $K$. The result of [4] cited above states that $\mathcal{A}_K$ is not the zero ideal for every knot $K$.

The ring $T$ can be extended to a principal left ideal domain $\tilde{T}$ by adding inverses of polynomials in $t$ and $M$; that is, $\tilde{T}$ is the set of Laurent polynomials in $L$ with coefficients rational functions of $t$ and $M$ with a product defined by

$$f(t, M)L^a \cdot g(t, M)L^b = f(t, M)g(t, t^{2a}M)L^{a+b}.$$  

The left ideal $\tilde{\mathcal{A}}_K = \tilde{T}\mathcal{A}_K$ is then generated by some nonzero polynomial in $\tilde{T}$, and in particular, this generator can be chosen to be in $\mathcal{A}_K$ and be of the form

$$\alpha_K(t, M, L) = \sum_{i=0}^{d} P_iL^i,$$

with $d$ minimal and with $P_0, \ldots, P_d \in \mathbb{Z}[t, M]$ being coprime in $\mathbb{Z}[t, M]$. This polynomial $\alpha_K$ is uniquely determined up to a sign and is called the (normalized) recurrence polynomial of $K$.

The A-polynomial was introduced in [1]. For a knot $K$ in $S^3$, its A-polynomial $A_K(M, L) \in \mathbb{Z}[M, L]$ is a two-variable polynomial with no repeated factors and with relatively prime integer coefficients, which is uniquely associated to $K$ up to a sign. Note that $A_K(M, L)$ always contains the factor $L - 1$.

The AJ-conjecture was raised in [2] which states that for every knot $K$, its recurrence polynomial $\alpha_K(t, M, L)$ evaluated at $t = -1$ is equal to the A-polynomial of $K$, up to a factor of a polynomial in $M$. The conjecture is obviously of fundamental importance as it predicts a strong connection between two important knot invariants derived from very different backgrounds. This is also a very difficult conjecture; so far only torus knots, some classes of 2-bridge knots and pretzel knots are known to satisfy the conjecture [2, 3, 5–7, 11, 12].

In this paper, we consider the AJ-conjecture for cabled knots over torus knots. Recall that the set of nontrivial torus knots $T(p, q)$ in $S^3$ can be indexed, in a standard way, by pairs of relatively prime integers $(p, q)$ satisfying $|p| > q \geq 2$. Also recall that an $(r,s)$-cabled knot on a knot $K$ in $S^3$ is the knot which can be embedded in the boundary torus of a regular neighborhood of $K$ in $S^3$ as a curve of slope $r/s$ with respect to the meridian/longitude coordinates of $K$ satisfying $(r, s) = 1, s \geq 2$. Note that $r$ can be any integer relatively prime to $s$. We have the following.

**Theorem 1.1.** The AJ-conjecture holds for each $(r, s)$-cabled knot $C$ over each $(p, q)$-torus knot $T$ if $r$ is not an integer between 0 and $pq$s.

A cabling formula for A-polynomials of cabled knots in $S^3$ is given in [9]. In particular when $C$ is the $(r, s)$-cabled knot over the torus knot $T(p, q)$ in $S^3$, its
A-polynomial $A_C(M, L)$ is given explicitly as in (1.1). For a pair of relatively prime integers $(p, q)$ with $q \geq 2$, define $F_{(p,q)}(M, L), G_{(p,q)}(M, L) \in \mathbb{Z}[M, L]$ to be the associated polynomials in variables $M$ and $L$ by:

$$F_{(p,q)}(M, L) := \begin{cases} M^{2p}L + 1 & \text{if } q = 2, p > 0, \\ L + M^{-2p} & \text{if } q = 2, p < 0, \\ M^{2pq}L^2 - 1 & \text{if } q > 2, p > 0, \\ L^2 - M^{-2pq} & \text{if } q > 2, p < 0 \end{cases}$$

and

$$G_{(p,q)}(M, L) := \begin{cases} M^{pq}L - 1 & \text{if } p > 0, \\ L - M^{-pq} & \text{if } p < 0. \end{cases}$$

Then

$$A_C(M, L) = \begin{cases} (L - 1)F_{(r,s)}(M, L)F_{(p,q)}(M^{s^2}, L) & \text{if } s \text{ is odd;} \\ (L - 1)F_{(r,s)}(M, L)G_{(p,q)}(M^{s^2}, L) & \text{if } s \text{ is even.} \end{cases} \quad (1.1)$$

A cabling formula for the $n$-colored Jones polynomial of the $(r, s)$-cabled knot $C$ over a knot $K$ is given in [8] (see also [13]) which in our normalized form is:

$$J_{C,n}(t) = t^{-rs(n^2 - 1)} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4rk(k+1)}J_{K,2k+1}(t). \quad (1.2)$$

In particular the $n$-colored Jones polynomial of the $(p, q)$-torus knot $T$ (which is the $(p, q)$-cabled knot over the unknot $U$) is:

$$J_{T,n}(t) = t^{-pq(n^2 - 1)} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4pk(k+1)}J_{U,2k+1}(t) = t^{-pq(n^2 - 1)} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4pk(k+1)} \frac{t^{4kq+2} - t^{-4k^2-2}}{t^2 - t^{-2}}. \quad (1.3)$$

We divide the proof of Theorem 1.1 into the following cases:

1. $s$ is odd and $q > 2$;
2. $s$ is odd and $q = 2$;
3. $s > 2$ is even;
4. $s = 2$.

In each case, we will find an annihilator of $J_{C,n}(t)$ by applying the formulas (1.3) and (1.2) (where taking the general knot $K$ to be the $(p, q)$-torus knot $T$), and then proceed to prove that it is the recurrence polynomial $\alpha_C(t, M, L)$ of $C$ when $r$ is not an integer between 0 and $pq$, making use of the degree formulas given in Sec. 2. Of course we will also compare $\alpha_C(1, M, L)$ with $A_C(M, L)$ given in (1.1).
to complete the verification of the AJ-conjecture for $C$. For convenience, we often get $\alpha_C(t, M, L)$ in the form $P = \sum_{i=0}^d P_i^1 \in \tilde{A}_C$, with $d$ minimal and with $P_i \in \mathbb{Q}(t, M)$ and with $P(-1, M, L) \neq 0$. Such $P$ only differs from $\alpha_C$ by a factor of a rational function $f(t, M) \in \mathbb{Q}(t, M)$ with $f(-1, M) \neq 0$ and thus is clearly as good as the normalized recurrence polynomial in verification for the AJ-conjecture. We often simply call such $P$ the recurrence polynomial of $C$. Also notice from the formula (1.1) that changing the sign of $r$ or $p$ only changes the $A$-polynomial of $C$ up to a power of $M$, so in checking that $P(-1, M, L) = A_C(M, L)$ up to a factor of a rational function in $M$ we do not need to worry about the sign of $r$ or $p$.

Further investigation of the AJ-conjecture for more general cabled knots, such as iterated torus knots and cabled knots over some hyperbolic knots, is being continued in [10]. In particular for some cabled knots over the figure 8 knot the AJ-conjecture has been verified to be true.

2. Degrees of $J_{T,n}(t)$ and $J_{C,n}(t)$

From now on in this paper, $T$ denotes the $(p, q)$-torus knot and $C$ the $(r, s)$-cabled knot over $T$, with the index convention given in Sec. 1.

For a polynomial $f(t) \in \mathbb{Z}[t^{\pm 1}]$, let $\ell[f]$ and $h[f]$ denote the lowest degree and the highest degree of $f$ in $t$ respectively. Obviously for $f(t), g(t) \in \mathbb{Z}[t^{\pm 1}], \ell[fg] = \ell[f] + \ell[g]$ and $h[fg] = h[f] + h[g]$.

**Lemma 2.1.** (1) When $p > q$,

$$
\ell[J_{T,n}(t)] = -pqn^2 + pq + \frac{1}{2}(1 - (-1)^{n-1})(p - 2)(q - 2),
$$

$$
h[J_{T,n}(t)] = 2(p + q - pq)|n| + 2(pq - p - q).
$$

(2) When $p < -q$,

$$
\ell[J_{T,n}(t)] = 2(p - q - pq)|n| + 2(pq - p + q),
$$

$$
h[J_{T,n}(t)] = -pqn^2 + pq + \frac{1}{2}(1 - (-1)^{n-1})(p + 2)(q - 2).
$$

**Proof.** The formula for $\ell[J_{T,n}(t)]$ in part (1) is proved in [12, Lemma 1.4]. The rest of the lemma can be proved similarly. \hfill \square

Note that $r \neq pqs$ since $r$ is relatively prime to $s$.

**Lemma 2.2.** (1) When $p > q$,

$$
\ell[J_{C,n}(t)] = -pq^2n^2 + (2p^2 - 2pq + 2r - 2rs)n + 2rs - 2r + 2pq - pqs^2
$$

$$
+ \frac{1}{2}(1 - (-1)^{(n-1)})(p - 2)(q - 2), \quad \text{if } r < pqs,
$$
From the formula (1.2) for $J_{C,n}(t)$, we have

$$\ell[J_{C,n}(t)] = -r s n^2 + rs + \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(r - p q s)$$

$$+ \frac{1}{2}[1 - (-1)^{n-1}s](p - 2)(q - 2), \quad \text{if } r > p q s,$$

$$h[J_{C,n}(t)] = -r s n^2 + rs + \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(r - 2 p q + 2 p + 2 q), \quad \text{if } r < 0.$$

(2) When $p < -q$,

$$h[J_{C,n}(t)] = -p q s^2 n^2 + (2 p q s^2 - 2 p q s + 2 r - 2 r s)n + 2 r s - 2 r + 2 p q s - p q s^2$$

$$+ \frac{1}{2}(1 - (-1)^{n-1}s)(p + 2)(q - 2), \quad \text{if } r > p q s,$$

$$h[J_{C,n}(t)] = -r s n^2 + rs + \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(r - p q s)$$

$$+ \frac{1}{2}(1 - (-1)^{n-1}s)(p + 2)(q - 2), \quad \text{if } r < p q s,$$

$$\ell[J_{C,n}(t)] = -r s n^2 + rs + \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(r - 2 p q + 2 p - 2 q), \quad \text{if } r > 0.$$

**Proof.** (1) From the formula (1.2) for $J_{C,n}(t)$ (replacing $K$ there by $T$), we can see that

$$\ell[J_{C,n}(t)] = -r s(n^2 - 1)$$

$$+ \min \left\{ \ell[J_{T,2sk+1}(t)] + 4 r k(k s + 1) \middle| \frac{n - 1}{2} \leq k \leq \frac{n - 1}{2} \right\}.$$ 

By Lemma 2.1(1), we have

$$\ell[J_{T,2sk+1}(t)] + 4 r k(k s + 1)$$

$$= -p q(2 k s + 1)^2 + p q + \frac{1}{2}(1 - (-1)^{2 k s})(p - 2)(q - 2) + 4 r k(k s + 1)$$

$$= (4 r s - 4 p q s^2)k^2 + (4 r - 4 p q s)k + \frac{1}{2}(1 - (-1)^{2 k s})(p - 2)(q - 2).$$

When $n$ is odd, $k$ is integer valued and thus the alternating term vanishes, so the above expression is quadratic in $k$. When $n$ is even, $k$ is half-integer valued and the alternating term is either always equal to zero (when $s$ is even) or is always equal to $(p - 2)(q - 2)$ (when $s$ is odd), and thus the above expression is again quadratic in $k$. So if $r < p q s$, it is minimized at $k = \frac{n - 1}{2}$, which yields the first formula in part (1), and if $r > p q s$, it is minimized at $k = 0$ when $n$ is odd and at $k = -1/2$ when $n$ is even, which yields the second formula in part (1).

Similarly to get the third formula in (1), we look at

$$h[J_{C,n}(t)] = -r s(n^2 - 1)$$

$$+ \max \left\{ h[J_{T,2sk+1}(t)] + 4 r k(k s + 1) \middle| \frac{n - 1}{2} \leq k \leq \frac{n - 1}{2} \right\}.$$
3. Case $s > 2$ is Odd and $q > 2$

3.1. An annihilator $P$ of $J_{C,n}(t)$

By [12, Lemma 1.1], we have

$$J_{C,n+2}(t) = t^{-4rs(n^2+2n+3)} J_{C,n}(t) + (t^{2(r-rs)n-2rs+2r-4pq}n(1)$$

$$- t^{2(-r-rs)n-2rs-2r}) J_{T,s(n+1)-1}(t) + t^{2(r-rs)n-2rs+2r-2pq}(n+1) \delta_{s(n+1)-1}.$$

**Proof.** We know by the cabling formula (1.2)

$$J_{C,n+2}(t) = t^{-rs((n+2)^2-1)} \sum_{k=-\frac{n+1}{2}}^{\frac{n+1}{2}} t^{4rk(k+1)} J_{T,2ks+2}(t)$$

$$= t^{-rs(n^2+2n+3)} \left( \sum_{k=-\frac{n+1}{2}}^{\frac{n+1}{2}} t^{4rk(k+1)} J_{T,2ks+1}(t) + t^{4r(-\frac{n+1}{2})((-\frac{n+1}{2})s+1)} J_{T,-s(n+1)+1}(t) \right).$$
Noting that $J_{T,s(n+1)+1}(t) = -J_{T,s(n+1)-1}(t)$, we have

$$J_{C,n+2}(t) = t^{-rs(n^2+4n+3)} \left( t^{rs(n^2-1)} t^{-rs(n^2-1)} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4rk(ks+1)} J_T,2ks+1(t) \right)$$

$$+ t^{(n+1)^2rs+2r(n+1)} J_{T,s(n+1)+1}(t) - t^{(n+1)^2rs-2r(n+1)} J_{T,s(n+1)-1}(t) \right)$$

$$= t^{-4rsn-4rs} J_{C,n}(t) + t^{2(r-rs)n-2rs+2r} J_{T,s(n+1)+1}(t)$$

$$- t^{2(r-rs)n-2rs-2r} J_{T,s(n+1)-1}(t).$$

Since $J_{T,s(n+1)+1}(t)$ and $J_{T,s(n+1)-1}(t)$ are related by Eq. (3.1) as

$$J_{T,s(n+1)+1} = t^{-4pq(s+1)} J_{T,s(n+1)-1} + t^{-2pq(s+1)} \delta_{s(n+1)-1},$$

we have

$$J_{C,n+2}(t) = t^{-4rsn-4rs} J_{C,n}(t) + t^{2(r-rs)n-2rs+2r-4pq(s+1)}$$

$$- t^{2(r-rs)n-2rs-2r} J_{T,s(n+1)-1}(t) + t^{2(r-rs)n-2rs+2r-2pq(s+1)} \delta_{s(n+1)-1}. \quad \Box$$

**Lemma 3.2.** For all positive integers $m$, we have

$$J_{T,n}(t) = t^{-4pqm(n+1)+4pqm(m+1)} J_{T,n-2m}(t)$$

$$+ \sum_{k=1}^{m} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq} \delta_{n-2k}$$

and in particular, with $s$ any positive integer,

$$J_{T,s(n+3)-1}(t) = t^{-4pqxs^2-8pqxs^2-4pqxs} J_{T,s(n+1)-1}(t) + S_n,$$

**Proof.** We induct on $m$. The base case $m = 1$ follows directly from Eq. (3.1). Then assume the formula holds for some positive integer $m$. Applying equation (3.1) again yields

$$J_{T,n}(t) = t^{-4pqm(n+1)+4pqm(m+1)} J_{T,n-2m}(t)$$

$$+ \sum_{k=1}^{m} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq} \delta_{n-2k}$$

$$= t^{-4pqm(n+1)+4pqm(m+1)} \left( t^{-4pq(n-2m-1)} J_{T,n-2m-2}(t)$$

$$+ t^{-2pq(n-2m-1)} \delta_{n-2m-2} + \sum_{k=1}^{m} t^{(-4pqk+2pq)n+4pqk^2-4pqk+2pq} \delta_{n-2k}$$

$$= t^{-4pqm(n+1)+4pqm(m+1)-4pq(n-2m-1)} J_{T,n-2m-2}(t)$$

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\[ + t^{-4pqmn(n+1)+4pq(m+1)-2pq(n-2m-1)} \delta_{n-2m-2} \]

\[ + \sum_{k=1}^{m} t^{-4pqk+2pq)n+4pqk^2-4pqk+2pq} \delta_{n-2k}. \]

If we compare the terms in the summation to the \( \delta_{n-2m-2} \) term outside, we can easily see that this is precisely the term where \( k = m + 1 \). So moving it inside, we have

\[
J_{T,n}(t) = t^{-4pq(m+1)(n+1)+4pq(m^2+m)+4pq(2m+2)} J_{T,n-2(m+1)}(t)
\]

\[ + \sum_{k=1}^{m+1} t^{-4pqk+2pq)n+4pqk^2-4pqk+2pq} \delta_{n-2k} \]

\[ = t^{-4pq(m+1)(n+1)+4pq(m+1)(m+2)} J_{T,n-2(m+1)}(t)
\]

\[ + \sum_{k=1}^{m+1} t^{-4pqk+2pq)n+4pqk^2-4pqk+2pq} \delta_{n-2k} \]

as needed. Applying the formula at \( s(n+3) - 1 \) gives the particular equation.

We shall now find an annihilator for \( J_{C,n}(t) \). By Lemma 3.1, replacing \( t^{2n} \) with \( M \) gives us

\[
J_{C,n+2}(t) = M^{-2rs} t^{-4rs} J_{C,n}(t) + (M^{r-rs-2pq s} t^{-2rs+2r-4pq s}
\]

\[ - M^{-r-rs} t^{-2rs-2r}) J_{T,n(s(n+1)-1)}(t) + M^{r-rs-2pq s} t^{-2rs+2r-2pq s} \delta_{s(n+1)-1}, \]

and since \( J_{C,n+2}(t) = L^2 J_{C,n}(t) \), we find

\[
(L^2 - M^{-2rs} t^{-4rs}) J_{C,n}(t)
\]

\[ = (M^{r-rs-2pq s} t^{-2rs+2r-4pq s}
\]

\[ - M^{-r-rs} t^{-2rs-2r}) J_{T,n(s(n+1)-1)}(t) + M^{r-rs-2pq s} t^{-2rs+2r-2pq s} \delta_{s(n+1)-1}. \]

In this equation, let

\[
a(t, M) = M^{-r-rs-2pq s} t^{-2rs+2r-4pq s} - M^{-r-rs} t^{-2rs-2r},\]

which is the coefficient of \( J_{T,n(s(n+1)-1)}(t) \), then obviously \( a(t, M) \neq 0 \), and we have

\[
a^{-1}(t, M)(L^2 - M^{-2rs} t^{-4rs}) J_{C,n}(t)
\]

\[ = J_{T,n(s(n+1)-1)}(t) + a^{-1}(t, M) M^{r-rs-2pq s} t^{-2rs+2r-2pq s} \delta_{s(n+1)-1}. \] (3.2)

From Lemma 3.2, we have

\[
(L^2 - t^{-8pq s^2-4pq s} M^{-2pq s^2}) J_{T,n(s(n+1)-1)}(t) = S_n.
\]
So multiplying (3.2) from the left by \((L^2 - t^{-8pqs^2-4pqs}M^{-2pqs^2})\) gives
\[
(L^2 - t^{-8pqs^2-4pqs}M^{-2pqs^2})a^{-1}(t, M)(L^2 - M^{-2rs}t^{-4rs})JC_n(t)
\]
\[
= S_n + (L^2 - t^{-8pqs^2-4pqs}M^{-2pqs^2})a^{-1}(t, M)M^{r-2rs-pqs}t^{-2rs+2r-2pqs}δ_{s(n+1)-1}
\]
\[
= S_n + a^{-1}(t, t^4M)M^{r-2rs-pqs}t^{-6rs+6r-6pqs}δ_{s(n+3)-1}
\]
\[
- a^{-1}(t, M)M^{-2rs-pqs-2pqs^2}t^{-2rs+2r-8pqs^2-6pqs}δ_{s(n+1)-1}. \quad (3.3)
\]
Let \(b(t, M)/(t^2 - t^{-2})\) denote the right-hand side of (3.3). Then \(b(t, M)\) is a rational function in \(t\) and \(M\). We claim that \(b \neq 0\), which we show by checking \(b(-1, M) := \lim_{t \to -1} b(t, M) \neq 0\). Recall
\[
S_n = \sum_{k=1}^{s} M^{-2pqs^2+k}t^{4pqs^2-12pqs^2+6pqs}δ_{s(n+3)-1-2k}.
\]
So we have
\[
\lim_{t \to -1} (t^2 - t^{-2})S_n
\]
\[
= \sum_{k=1}^{s} M^{-2pqs^2+k+pqst^4}M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)}
\]
\[
= (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})\frac{M^{pqst}(1 - M^{-2pqs^2})}{M^{2pqs^2} - 1}. \quad (3.4)
\]
Also
\[
\lim_{t \to -1} (t^2 - t^{-2})(a^{-1}(t, t^4M)M^{r-2rs-pqs}t^{6r-6rs-6pqs}δ_{s(n+3)-1}
\]
\[
- a^{-1}(t, M)M^{r-2rs-pqs-2pqs^2}t^{-6rs+6r-6pqs^2}δ_{s(n+1)-1}
\]
\[
= a^{-1}(-1, M)(M^{r-2rs-pqs} - M^{-r-2rs-pqs-2pqs^2})
\]
\[
\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})
\]
\[
= \frac{1}{M^{r-2rs-2pqs} - M^{-r-2rs}}(M^{r-2rs-pqs} - M^{-r-2rs-pqs-2pqs^2})
\]
\[
\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)}).
\]
Summing up the two limits above, we get
\[
b(-1, M)
\]
\[
:= \lim_{t \to -1} b(t, M)
\]
\[
= \left(\frac{M^{pqst}(1 - M^{-2pqs^2})}{M^{2pqs^2} - 1} + \frac{M^{r-2rs-pqs} - M^{-r-2rs-pqs-2pqs^2}}{M^{r-2rs-2pqs} - M^{-r-2rs}}\right)
\]
\[\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})\]

\[= \frac{(-M^{pq} - M^{-r} + M^{r} - 2pq^2)(M^{ps} - M^{-ps})(M^{qs} - M^{-qs})}{(M^{2pq} - 1)(M^{r} - 2pq^2 - M^{-r})},\]

which is not zero. So \(b \neq 0\) and we conclude that our recurrence (3.3) is inhomogeneous. Therefore,

\[P(t, M, L) = (L - 1)b^{-1}(t, M)(L^2 - t^{-8pq^2 - 4pq^2}M^{-2pq^2})\]

\[\times a^{-1}(t, M)(L^2 - M^{-2rs}t^{-4rs})\]

is an annihilator of \(J_{C,n}(t)\) in \(\tilde{A}_C\).

Up to this point all the results above in this section are valid for general \(C\) over \(T\). From now on in this section, we put in the restriction that \(s\) is odd and \(q > 2\). Once we prove that \(P\) is of minimal degree in \(L\), it will follow that \(P\) is the recurrence polynomial of \(J_{C,n}(t)\) up to normalization. We can check the AJ-conjecture by evaluating \(P\) at \(t = -1\).

\[P(-1, M, L) = b^{-1}(-1, M)a^{-1}(-1, M)(L - 1)(L^2 - M^{-2pq^2})(L^2 - M^{-2rs}),\]

which, up to a nonzero rational function in \(M\), is equal to the A-polynomial of \(C\).

### 3.2. \(P\) is the recurrence polynomial of \(C\)

We now wish to show that the operator \(P\) is the recurrence polynomial of \(C\), up to normalization. It is enough to show that if an operator \(Q = D_4L^4 + D_3L^3 + D_2L^2 + D_1L + D_0\) is an annihilator of \(J_{C,n}(t)\) with \(D_0, \ldots, D_4 \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]\), then \(Q = 0\). Suppose \(QJ_{C,n}(t) = 0\), that is,

\[D_4J_{C,n+4}(t) + D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t) = 0.\]

We wish to show that \(D_i = 0\) for \(i = 0, 1, 2, 3, 4\). Applying our Lemma 3.1, we have

\[0 = D_4J_{C,n+4}(t) + D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)\]

\[= D_4(M^{-2rs}t^{-12rs}J_{C,n+2}(t) + (M^{r} - 2pq^2t^{-6rs} + 6r^2 - 4pq^2)\]

\[\times - (M^{-r} - 6rs - 6r^2)J_{T, s(n+3)}(t) + M^{r} - 2pq^2t^{-6rs} + 4pq^2\delta_{s(n+3)}(t)\]

\[+ D_3(M^{-2rs}t^{-8rs}J_{C,n+1}(t) + (M^{r} - 2pq^2t^{-4rs} - 4rs)\]

\[\times - (M^{-r} - 4rs - 4r^2)J_{T, s(n+2)}(t) + M^{r} - 2pq^2t^{-4rs} + 4pq^2\delta_{s(n+2)}(t)\]

\[+ D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)\]

\[= (D_4M^{-2rs}t^{-12rs} + D_2)(M^{-2rs}t^{-4rs}J_{C,n}(t) + (M^{r} - 2pq^2t^{-2rs} + 2r^2 - 4pq^2)\]

\[\times - (M^{-r} - 2rs - 2r^2)J_{T, s(n+1)}(t) + M^{r} - 2pq^2t^{-2rs} - 2pq^2\delta_{s(n+1)}(t)\]

\[+ D_3(M^{-2rs}t^{-6rs} + 6r^2 - 12pq^2 - (M^{-r} - 6rs - 6r^2)J_{T, s(n+3)}(t)\]

\[\times - 6rs - 6r^2 - 6pq^2\delta_{s(n+3)}(t) + D_4M^{-2rs}t^{-8rs}J_{C,n+1}(t)\]

\[+ M^{r} - 2pq^2t^{-6rs} + 6r^2 - 6pq^2\delta_{s(n+3)}(t) + D_2J_{C,n+1}(t)\]

\[= 0.\]
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\[
+ (M^{r-r_s} - 2pqst^{-4r_s + 4r - 8pqs} - M^{r-r_s} t^{-4r_s - 4r})JT_s(t^{n+2})\frac{1}{t}(t)
\]

\[
+ M^{r-r_s} - pqst^{-4r_s - 4pqs} s(t^{n+2}) - D_1 JT_s(t^{n+1}) + D_0 JT_s(t)
\]

\[
= (D_0 + D_2 M^{2r_s - 4r_s + 4r} + D_4 M^{-4r_s t^{-16r_s}})JT_s(t)
\]

\[
+ (D_1 + D_3 M^{2r_s t^{-8r_s}})JT_s(t^{n+1})
\]

\[
+ D_4 (M^{r-r_s} - 2pqst^{-6r_s + 6r - 12pqs} - M^{r-r_s} t^{-6r_s + 6r})JT_s(t^{n+3})\frac{1}{t}(t)
\]

\[
+ D_3 (M^{r-r_s} - 2pqst^{-4r_s + 4r - 8pqs} - M^{r-r_s} t^{-4r_s + 4r})JT_s(t^{n+2})\frac{1}{t}(t)
\]

\[
+ (D_4 M^{r-r_s} - 2pqst^{-2r_s + 2r - 4pqs} - M^{r-r_s} t^{-2r_s - 2r})JT_s(t^{n+1})\frac{1}{t}(t)
\]

\[
+ (D_4 M^{r-r_s} - 2pqst^{-12r_s} + D_2) (M^{r-r_s} - 2pqst^{-2r_s + 2r - 4pqs} - M^{r-r_s} t^{-2r_s - 2r})JT_s(t^{n+1})\frac{1}{t}(t)
\]

\[
+ (D_4 M^{r-r_s} - 2pqst^{-12r_s} + D_2) M^{r-r_s} -pqst^{-2r_s - 2pqs} - M^{r-r_s} t^{-2r_s - 2r - 4pqs} s(t^{n+1}) - 1
\]

\[
+ D_3 M^{r-r_s} - pqst^{-4r_s - 4pqs} s(t^{n+2}) - 1 + D_4 M^{r-r_s} - pqst^{-6r_s + 6r - 6pqs} s(t^{n+3}) - 1,
\]

and applying Lemma 3.2,

\[
= (D_0 + D_2 M^{2r_s t^{-4r_s} + 4r} + D_4 M^{-4r_s t^{-16r_s}})JT_s(t)
\]

\[
+ (D_1 + D_3 M^{2r_s t^{-8r_s}})JT_s(t^{n+1})
\]

\[
+ (D_4 M^{r-r_s} - 2pqst^{-6r_s + 6r - 12pqs} - M^{r-r_s} t^{-6r_s + 6r})M^{2pqst^{-8pqs} - 4pqs}
\]

\[
+ (D_4 M^{r-r_s} - 2pqst^{-12r_s} + D_2) (M^{r-r_s} - 2pqst^{-2r_s + 2r - 4pqs} - M^{r-r_s} t^{-2r_s - 2r})M^{2pqst^{-8pqs} - 4pqs}
\]

\[
+ M^{r-r_s} t^{-12r_s} JT_s(t^{n+1})\frac{1}{t}(t) + D_1 (M^{r-r_s} - 2pqst^{-4r_s + 4r - 8pqs} - M^{r-r_s} t^{-4r_s + 4r})JT_s(t^{n+2})\frac{1}{t}(t)
\]

\[
+ M^{r-r_s} t^{-4r_s + 4r} JT_s(t^{n+2})\frac{1}{t}(t) + (D_4 M^{r-r_s} - 2pqst^{-12r_s} + D_2) M^{r-r_s} -pqst^{-2r_s - 2pqs} - M^{r-r_s} t^{-2r_s - 2r - 4pqs} s(t^{n+1}) - 1
\]

\[
+ D_3 M^{r-r_s} - pqst^{-4r_s - 4pqs} s(t^{n+2}) - 1 + D_4 M^{r-r_s} - pqst^{-6r_s + 6r - 6pqs} s(t^{n+3}) - 1,
\]

We claim that each $D_i' = 0$, and it then follows that each $D_i = 0$. Indeed, it follows easily from $D_4' = D_4 = 0$ that $D_3 = D_3 = 0$. For the rest, it is enough to show that the two linear equations defined by $D_0' = 0$ and $D_2' = 0$ are linearly independent (with $D_2$ and $D_4$ as variables). We can check that the determinant of the linear system is nonzero, and in particular, we multiply by $(t^2 - t^{-2})$ and then evaluate at $t = -1$ in order to use Eq. (3.4):

\[
(M^{r-r_s} - 2pqst - M^{r-r_s}) \left( M^{r-3r_s - pqst} + M^{r-r_s - pqst} + (M^{r-r_s} - 2pqst - M^{r-r_s}) \right)
\]

\[
\times M^{pqst (1 - M^{-2pqs})} \left( M^{s(p+q)} + M^{s(p-q)} - M^{s(q-p)} - M^{s(q-p)} \right)
\]

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\[-(M^{r-rs-2pq} - 2pq^2 - M^{r-rs-2pq}s^2 + M^{r-3rs-2pq}s - M^{r-3rs})
\times M^{r-2pq} \left( M^{s(p+q)} + M^{s(q-p)} - M^{s(q-p)} \right) \]

\[= \frac{1}{M^{2pq}-1} \left( M^{s(p+q)} + M^{s(p+q)} - M^{s(q-p)} - M^{s(q-p)} \right) \]

\[\times \left( M^{r-2pq} - M^{r-2pq}s + 1 \right) \times \left( M^{2pq} - 1 \right) \]

\[+ \left( M^{r-2pq} - M^{r-2pq}s + \cdots \right) \]

\[= \frac{1}{M^{2pq}-1} \left( M^{s(p+q)} + M^{s(p+q)} - M^{s(q-p)} - M^{s(q-p)} \right) \]

\[\times \left( M^{r-2pq} - M^{r-2pq}s + 1 \right) \times \left( M^{2pq} - 1 \right) \]

\[\quad \times \left( M^{r-2pq} - M^{r-2pq}s + \cdots \right) \]

\[\quad \times \left( M^{2pq} - 1 \right) \]

which is indeed nonzero.

We now prove that if \( r \) is not an integer between 0 and \( pq \), we have \( D'_i = 0 \) for each \( i = 0, 1, 2, 3, 4 \) and thus our annihilator \( P \) is of minimal \( L \)-degree.

We say that a function \( f : \mathbb{Z} \to \mathbb{Z} \) is a quasi-polynomial if there exist periodic functions \( a_0, \ldots, a_d \) each with integral period such that

\[ f(n) = \sum_{i=0}^{d} a_i(n)n^i, \]

and \( f \) is of degree \( d \) if \( a_d \neq 0 \). In particular, we say \( f \) is quasi-quadratic if \( f \) is a quasi-polynomial of degree 2.

**Lemma 3.3.** (1) When \( p > q \) and either \( r < 0 \) or \( r > pq \), we have \( D'_i = 0 \) for \( i = 0, 1, 2, 3, 4 \).

(2) When \( p < -q \) and either \( r > 0 \) or \( r < pq \), we have \( D'_i = 0 \) for \( i = 0, 1, 2, 3, 4 \).

**Proof.** (1) Suppose \( p > q \), \( r > pq \), and some \( D'_i \neq 0 \). Then there must be another nonzero \( D'_j \) such that one of the following equalities hold:

\[ \ell(D'_i J_{C,n}(t)) = \ell(D'_j J_{C,n+1}(t)), \]

\[ \ell(D'_i J_{C,n}(t)) = \ell(D'_j J_{T,s(n+1)}(t)), \]

\[ \ell(D'_i J_{C,n}(t)) = \ell(D'_j J_{T,s(n+2)}(t)), \]

\[ \ell(D'_i J_{C,n+1}(t)) = \ell(D'_j J_{T,s(n+1)}(t)), \]

\[ \ell(D'_i J_{C,n+1}(t)) = \ell(D'_j J_{T,s(n+2)}(t)), \]
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\[ \ell[D_0'J_{C,n+1}(t)] = \ell[D_1'J_{T,s(n+2)}(t)], \]
\[ \ell[D_2'J_{T,s(n+1)-1}(t)] = \ell[D_1'J_{T,s(n+2)}(t)]. \]

That is, two of the summands must share a lowest degree, and since \( \ell[D_0'] \) is only linear in \( n \) for large enough \( n \) while \( \ell[J_{C,n}(t)] \) and \( \ell[J_{T,n}(t)] \) are quasi-quadratic by Lemmas 2.2 and 2.1, we can immediately dispose of the cases involving \( D_0' \).

Subcase 3.1. \( \ell[D_4'J_{C,n}(t)] = \ell[D_3'J_{C,n+1}(t)] \):

From the second formula of Lemma 2.2(1), we have

\[ \ell[D_4'] - \ell[D_3'] = \ell[J_{C,n+1}(t)] - \ell[J_{C,n}(t)] \]
\[ = -2rsn - rs - (-1)^n(s - 2)(r - pqs) + (p - 2)(q - 2), \]

but for sufficiently large \( n \), \( \ell[D_4'] - \ell[D_3'] \) is a linear function in \( n \), while the right-hand side is not a polynomial, so we have a contradiction.

Subcase 3.2. \( \ell[D_4'J_{C,n}(t)] = \ell[D_2'J_{T,s(n+1)-1}(t)] \):

From Lemmas 2.2(1) and 2.1(1), we have

\[ \ell[D_4'] - \ell[D_2'] = \ell[J_{T,s(n+1)-1}(t)] - \ell[J_{C,n}(t)] \]
\[ = s(r - pqs)n^2 + (2pqs - 2pq^2)n - pq(4s^2 - 4s) - rs \]
\[ - \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(r - pqs), \]

which is quasi-quadratic, while the left-hand side is at most linear, giving us another contradiction.

Subcase 3.3. \( \ell[D_4'J_{C,n}(t)] = \ell[D_1'J_{T,s(n+2)-1}(t)] \):

Here we have

\[ \ell[D_4'] - \ell[D_1'] = \ell[J_{T,s(n+2)-1}(t)] - \ell[J_{C,n}(t)] \]
\[ = s(r - pqs)n^2 + (2pqs - 4pq^2)n - pq(4s^2 - 4s) - rs \]
\[ - \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(r - pqs) + (-1)^n(p - 2)(q - 2) \]

again giving us a quasi-quadratic function on the right and a linear function on the left, which is a contradiction.

Subcase 3.4. \( \ell[D_4'J_{C,n+1}(t)] = \ell[D_2'J_{T,s(n+1)-1}(t)] \):

We have

\[ \ell[D_4'] - \ell[D_2'] = \ell[J_{T,s(n+1)-1}(t)] - \ell[J_{C,n+1}(t)] \]
\[ = s(r - pqs)n^2 + (2pqs + 2rs - 2pq^2)n - pq(s^2 - 2s) \]
\[ - \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(r - pqs) + (-1)^n(p - 2)(q - 2) \]
Subcase 3.5. \( \ell[D_3 J_{C, n+1}(t)] = \ell[D_1 J_{T, s(n+2)}(t)] \):

We have

\[
\ell[D_3'] - \ell[D_1'] = \ell[J_{T, s(n+2)}(t)] - \ell[J_{C, n+1}(t)]
= (rs - pqs^2)n^2 + 2pq - 4pq - 4s) \cdot n - pq(4s^2 - 4s)
- \frac{1}{2}(1 - (-1)^n)(s - 2)(r - pqs)
\]

giving us another contradiction.

Subcase 3.6. \( \ell[D_3 J_{T, s(n+1)}(t)] = \ell[D_1 J_{T, s(n+2)}(t)] \):

This time we have

\[
\ell[D_3'] - \ell[D_1'] = \ell[J_{T, s(n+2)}(t)] - \ell[J_{T, s(n+1)}(t)]
= -2pq^2n^2 s + 2pq - 3pqs + \frac{1}{2}(-1)^{(n+1)}(1 - (-1)^n)(p - 2)(q - 2)
= -2pq^2n^2 s + 2pq - 3pqs - (-1)^n(p - 2)(q - 2)
\]

which is alternating on the right and eventually linear on the left, which is a contradiction. This exhausts the possibilities of the case \( r > p q \).

Now assume \( p > q \) and \( r < 0 \). We first consider the highest degrees of the summands in the equation

\[
0 = D_3' J_{C, n}(t) + D_3' J_{C, n+1}(t) + D_3' J_{T, s(n+1)}(t) + D_1' J_{T, s(n+2)}(t) + D_0'
\]

for large positive \( n \)'s. If \( D_3' \neq 0 \) or \( D_1' \neq 0 \), then both of them cannot be zero since by Lemma 2.2(1) \( h[J_{C, n}] \) and \( h[J_{C, n+1}] \) are each quasi-quadratic while by Lemma 2.1(1) \( h[J_{T, s(n+1)}(t)] \) and \( h[J_{T, s(n+2)}(t)] \) are each linear in \( n \) (for positive \( n \)'s), and we must have the following

Subcase 3.7. \( h[D_3' J_{C, n}(t)] = h[D_3' J_{C, n+1}(t)] \):

Then we have, by Lemma 2.2(1),

\[
h[D_1'] - h[D_3'] = h[J_{C, n+1}(t)] - h[J_{C, n}(t)]
= -2rs n - rs - (-1)^n(s - 2)(r - 2pq + 2p + 2q)
\]

which is a linear polynomial for large \( n \) on the left but is not a linear polynomial on the right, giving a contradiction.

So both \( D_3' \) and \( D_1' \) are zero. So we have \( 0 = D_3' J_{T, s(n+1)}(t) + D_1' J_{T, s(n+2)}(t) + D_0' \). We can then analyze the lowest degrees in a similar fashion as above; if one of the \( D_i' \) is not zero, we must have

Subcase 3.8. \( \ell[D_3' J_{T, s(n+1)}(t)] = \ell[D_1' J_{T, s(n+2)}(t)] \):

which is again quasi-quadratic on the right and linear on the left, again a contradiction.
This can be treated similarly as Subcase 3.6. We conclude that each $D'_i = 0$.
This completes the proof of part (1).

Part (2) of the lemma can be proved similarly with the use of Lemmas 2.1(2) and 2.2(2).

**Remark 3.1.** In the proof of Lemma 3.3 we used the condition that $s > 2$ odd and $q > 2$ in several subcases. Some of these subcases will disappear accordingly in later sections when we impose the condition $s$ odd and $q = 2$ or $s > 2$ even or $s = 2$.

4. Case $s > 2$ is Odd and $q = 2$

4.1. An annihilator $P$ of $J_{C,n}(t)$

Define:

$$U_n = \sum_{k=1}^{s}(-1)^{k-1}t^{4n+8s-2-4k} - t^{-4n-8s+2+4k}.$$  

When $q = 2$, we have by [12, Lemma 1.5] the identity

$$J_{T,n+1}(t) = -t^{-(4n+2)p}J_{T,n}(t) + t^{-2p}t^{4n+2} - t^{-4n-2}.  \tag{4.1}$$

Note again that (4.1) is valid for negative $p$ as well.

**Lemma 4.1.** When $q = 2$, for all positive integers $m$, we have

$$J_{T,n}(t) = (-1)^m t^{-4mn+2m} p J_{T,n-m}(t) + \sum_{k=1}^{m} (-1)^{k-1} t^{-(4k-2)p} + (2k^2-2k+2) p t^{-4n-2+4k}.$$  

and in particular, when $s$ is odd,

$$J_{T,s(n+2)-1}(t) = -t^{-4ps^2+4ps-6ps^2} J_{T,s(n+1)-1}(t) + U_n.$$  

**Proof.** Apply the relation (4.1) $m$ times.

Note that the relation (3.2) is valid for general $C$ over $T$. Specializing it at $q = 2$ and $s$ odd, we have

$$a^{-1}(t, M)(L^2 - M^{-2r}t^{-4r}) J_{C,n}(t)$$

$$= J_{T,s(n+1)-1}(t) + a^{-1}(t, M) M^{-r+s-2ps} t^{-2r-2s-4ps} \delta_{s(n+1)-1}.  \tag{4.2}$$

From Lemma 4.1, we get

$$(L + M^{-2ps^2} t^{4ps-6ps^2}) J_{T,s(n+1)-1}(t) = U_n.$$  

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Multiplying (4.2) from the left by \((L + M^{-2ps^2}t^{4ps-6ps^2})\) yields
\[
(L + M^{-2ps^2}t^{4ps-6ps^2})a^{-1}(t, M)(L^2 - M^{-2rs}t^{-4rs})J_{C,n}(t)
= U_n + (L + M^{-2ps^2}t^{4ps-6ps^2})a^{-1}(t, M)M^{r-rs-2ps}t^{-2rs-4ps}δ_{s(n+1)-1}
= U_n + a^{-1}(t, t^2 M)M^{r-rs-2ps}t^{4r-4rs-8ps}δ_{s(n+2)}
+ a^{-1}(t, M)M^{r-rs-2ps}t^{2rs-6ps}δ_{s(n+1)-1}.
\]
(4.3)

As in Sec. 3, let \(b(t, M)/(t^2 - t^{-2})\) denote the right-hand side of (4.3). Then \(b(t, M)\) is a rational function in \(t\) and \(M\). We now show that \(b(-1, M) := \lim_{t \to -1} b(t, M) \neq 0\). Rewrite \(U_n\) as a function of \(t\) and \(M\) by changing \(t^{2n}\) to \(M\):
\[
U_n = \sum_{k=1}^{s} (-1)^{k-1}M^{ps-2psk}t^{2pk^2-8psk+2pk+4ps} \frac{M^{2st_{8s-2-4k}}-M^{2st_{8s+2+4k}}}{t^2 - t^{-2}}.
\]

So we have
\[
\lim_{t \to -1} (t^2 - t^{-2})U_n = \sum_{k=1}^{s} (-1)^{k-1}M^{ps-2psk}(M^{2s} - M^{-2s})
= \frac{(M^{2s} - M^{-2s})(1 + M^{-2ps})}{1 + M^{-2ps}}.
\]
(4.4)

Also
\[
\lim_{t \to -1} (t^2 - t^{-2})(a^{-1}(t, t^2 M)M^{r-rs-2ps}t^{4r-4rs-8ps}δ_{s(n+3)-1}
+ a^{-1}(t, M)M^{r-rs-2ps}t^{2rs-6ps}δ_{s(n+1)-1})
= \frac{M^{r-rs-2ps} + M^{r-rs-2ps-2ps^2}}{M^{r-rs-4ps} - M^{r-rs}}
\times (M^{s(p+2)} + M^{-s(p+2)} - M^{s(2-p)} - M^{-s(2-p)}),
\]

Summing up the two limits above, we get
\[
b(-1, M) := \lim_{t \to -1} b(t, M)
= \frac{(M^{2s} - M^{-2s})(1 + M^{-2ps})}{1 + M^{-2ps}} + \frac{M^{r-rs-2ps} + M^{r-rs-2ps-2ps^2}}{M^{r-rs-4ps} - M^{r-rs}}
\times (M^{s(p+2)} + M^{-s(p+2)} - M^{s(2-p)} - M^{-s(2-p)})
= \frac{(M^{2s} - M^{-2s})(1 + M^{-2ps})M^{rs-ps}(M^{r} - M^{-r})}{(1 + M^{-2ps})(M^{r-rs-4ps} - M^{r-rs})},
\]

which is not zero. So \(b \neq 0\) and we conclude that our recurrence (4.3) is inhomogeneous. Therefore,
\[
P(t, M, L) = (L - 1)b^{-1}(t, M)(L + M^{-2ps^2}t^{4ps-6ps^2})a^{-1}(t, M)(L^2 - M^{-2rs}t^{-4rs})
\]
is an annihilator of \(J_{C,n}(t)\) in \(\tilde{A}_C\).
We now wish to show that the operator $P$ is of minimal degree in $L$, it will follow that $P$ is the recurrence polynomial of $J_{C,n}(t)$ up to normalization. We can check the AJ-conjecture by evaluating $P$ at $t = -1$.

$$P(-1, M, L) = b^{-1}(-1, M)a^{-1}(-1, M)(L - 1)(L + M^{-2ps^2})(L^2 - M^{-2rs}),$$

which, up to a nonzero factor in $Q(M)$, is equal to the A-polynomial of $C$.

4.2. $P$ is the recurrence polynomial of $C$

We now wish to show that the operator $P$ is the recurrence polynomial of $C$. It is enough to show that if an operator $Q = D_3L^3 + D_2L^2 + D_1L + D_0$ with $D_0, \ldots, D_3 \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$ is an annihilator of $J_{C,n}(t)$, then $Q = 0$.

Suppose $QJ_{C,n}(t) = 0$, that is, $D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t) = 0$. We wish to show that $D_i = 0$ for $i = 0, 1, 2, 3$. We have by Lemma 3.1 (specialized at $q = 2$)

$$0 = D_1J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)$$

$$= D_3(M^{-2rs}t^{-4rs}J_{C,n+1}(t)) + (M^{-r-rs-4ps}t^{-4rs+4r-16ps}$$

$$- M^{-r-rs}t^{-4rs-4r}J_{T,s(n+2)-1}(t) + M^{-r-rs-4ps}t^{-4rs-8ps}1\delta_{s(n+2)}(t)$$

$$+ D_2(M^{-2rs}t^{-4rs}J_{C,n}(t)) + (M^{-r-rs-4ps}t^{-2rs+2r-8ps}$$

$$- M^{-r-rs}t^{-2rs-2r}J_{T,s(n+1)-1}(t) + M^{-r-rs-4ps}t^{-2rs-4ps}1\delta_{s(n+1)}(t)$$

$$+ D_1J_{C,n+1}(t) + D_0J_{C,n}(t)$$

$$= (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t) + (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t)$$

and applying Lemma 4.1,

$$= (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t) + (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t)$$

$$+ D_3(M^{-r-rs-4ps}t^{-4rs+4r-16ps} - M^{-r-rs}t^{-4rs-4r})$$

$$\times (-M^{-2ps^2}t^{-4ps-6ps}J_{T,s(n+1)-1}(t) + U_n)$$

$$+ D_2(M^{-r-rs-4ps}t^{-2rs+2r-8ps} - M^{-r-rs}t^{-2rs-2r})J_{T,s(n+1)-1}(t)$$

$$+ D_1J_{C,n+1}(t) + D_0J_{C,n}(t)$$

and applying Lemma 4.1,
When Lemma 4.2.

(1) and \( D = t \) the determinant of the linear system, multiplied by \((-\delta_{(n+1)-1})\) which is indeed nonzero.

Proof. We again wish to show that the two linear equations defined by \( i \) and \( D = t \) are linearly independent (with \( D_2 \) and \( D_3 \) as variables). So let us check the determinant of the linear system, multiplied by \((t^2 - t^{-2})\) and then valued at \( t = -1 \), is nonzero:

\[
(M_{r-r-s-4p} - M_{r-r-r})(-M_{-2p^2})M_{r-r-s-4p}M_{r-r-r}M_{s(p+2)} + M_{-s(p+2)}
- M_{s(2-p)} - M_{-s(2-p)} - (M_{r-r-r-4p} - M_{r-r-r})
\times \left( (M_{r-r-s-4p} - M_{r-r-s})(M_{2s} - M_{-2s})M_{-s(1+2p^2)} \right)
+ M_{r-r-s-4p}M_{s(p+2)}M_{-s(p+2)} - M_{s(2-p)} - M_{-s(2-p)})
\]

\[
= -(M_{r-r-s-4p} - M_{r-r-r})(M_{-2p^2} + 1) \left( M_{r-r-s-2p}M_{s(p+2)} + M_{-s(p+2)}
- M_{s(2-p)} - M_{-s(2-p)} - (M_{r-r-r-4p} - M_{r-r-r}) \frac{M_{2s} - M_{-2s}}{1 + M_{-2p^2}} \right)
\]

\[
= -(M_{r-r-s-4p} - M_{r-r-r})(M_{-2p^2} + 1)(M_{2s} - M_{-2s})
\times (M_{r-r-s-2p}(M_{ps} - M_{-ps}) + (M_{r-r-s-4p} - M_{r-r-r}) \frac{1}{M_{ps} + M_{-ps}})
\]

\[
= \frac{1}{M_{ps} + M_{-ps}}(M_{r-r-s-4p} - M_{r-r-r})(M_{-2p^2} + 1)(M_{2s} - M_{-2s})
\times (M_{r-r-s-2p}M_{ps} - M_{-ps}) + (M_{r-r-s-4p} - M_{r-r-r})
\]

which is indeed nonzero.

The following lemma shows that each \( D_i' = 0 \) if \( r \) is not a number between 0 and \( pqs \).

**Lemma 4.2.** (1) When \( p > q = 2 \) and either \( r < 0 \) or \( r > pqs \), we have \( D_i' = 0 \) for \( i = 0, 1, 2, 3 \).

(2) When \( p < -q \) and either \( r > 0 \) or \( r < pqs \), we have \( D_i' = 0 \) for \( i = 0, 1, 2, 3 \).

**Proof.** The proof is entirely similar to that of Lemma 3.3 (also cf. Remark 3.1).
5. Case $s > 2$ is Even

5.1. An annihilator $P$ of $J_{C,n}(t)$

Recall our definition

$$\delta_j = \frac{t^{2(p+q)(j+1)+2} + t^{-2(p+q)(j+1)+2} - t^2(q-p)(j+1)-2}{t^2 - t^{-2}}$$

and further define:

$$V_n = \sum_{k=1}^{s/2} t^{-4pqsk+2pqsn+4pqk^2-8pqsk+4pq} \delta_{s(n+2)-1-2k}.$$

**Lemma 5.1.** If $s$ is even, then

$$J_{T,s(n+2)-1}(t) = t^{-2pq^2n-3pq^2+2pq} J_{T,s(n+1)-1}(t) + V_n.$$  

**Proof.** Apply Lemma 3.2, setting $m = s/2$. 

Lemma 5.1 yields the relation

$$(L - M^{-pq^2}t^{-3pq^2+2pq}s) J_{T,s(n+1)-1}(t) = V_n.$$  

So applying the operator $(L - M^{-pq^2}t^{-3pq^2+2pq}s)$ to both sides of (3.2) gives:

$$\begin{align*}
(L - M^{-pq^2}t^{-3pq^2+2pq}s) a^{-1}(t, M)(L^2 - M^{-2r^s}t^{-4rs}) J_{C,n}(t) &= V_n + (L - M^{-pq^2}t^{-3pq^2+2pq}s) a^{-1}(t, M) M^{r-rs-pqs} t^{2r-2rs-2pq} \delta_{s(n+1)-1} \\
&= V_n + a^{-1}(t, t^2 M) M^{r-rs-pqs} t^{4r-4rs+4pq} \delta_{s(n+2)-1} \\
&\quad - a^{-1}(t, M) M^{r-rs-pqs} t^{2r-2rs-3pq^2-s(n+1)-1}.
\end{align*}$$

To see this is an inhomogeneous recursion for $J_{C,n}(t)$, let $b(t, M)$ be the right-hand side of this equation and check it is nonzero. As before it suffices to check that $\lim_{t \to 1} b(t, M) \neq 0$, considering $V_n$ and $\delta_{s(n+k)-j}$'s as functions of $t$ and $M$ (changing $t^{2n}$ to $M$). We have

$$b(-1, M) = \lim_{t \to 1} b(t, M) = \lim_{t \to 1} (t^2 - t^{-2})(V_n + a^{-1}(t, t^2 M) M^{r-rs-pqs} t^{4r-4rs+4pq} \delta_{s(n+2)-1}) - a^{-1}(t, M) M^{r-rs-pqs} t^{2r-2rs-3pq^2} \delta_{s(n+1)-1} = \sum_{i=1}^{s/2} M^{-2pq^2(s+1)} + a^{-1}(-1, M) (M^{r-rs-pqs} - M^{r-rs-pqs^2-pqs})$$

$$\times (M^{s(p+q)} + M^{-s(p+q)} - M^{s(q-p)} - M^{-s(q-p)})$$

$$= \left(\frac{M^{-pq^2} - M^{-pq^2-pqs}}{1 - M^{-2pq}} + \frac{M^{-r-rs-pqs} - M^{-r-rs-pqs^2-pqs}}{M^{r-rs-2pq} - M^{-r-rs}}\right)$$

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We now want to show that the operator 

\[ P(t, M, L) = (L-1)b^{-1}(t, M)(L - M^{-pq^2}t^{-3pq^2 + 2pq}) \times a^{-1}(t, M)\left(\frac{L^2 - M^{-2rs}t^{-4rs}}{L - M^{-pq^2}}\right) \]

is an annihilator of \( J_{C,n}(t) \). Assuming \( P \) is of minimal degree in \( L \), we can now check the AJ-conjecture by evaluating \( P \) at \( t = -1 \). We have

\[ P(-1, M, L) = b^{-1}(-1, M)a^{-1}(-1, M)(L - 1)(L - M^{-pq^2})(L^2 - M^{-2rs}), \]

which agrees with the A-polynomial of \( C \), up to a nonzero factor of a rational function in \( \mathbb{Q}(M) \).

### 5.2. \( P \) is the recurrence polynomial of \( C \)

We now want to show that the operator \( P \) is the recurrence polynomial of \( C \). It is enough to prove that if \( Q = D_3L^3 + D_2L^2 + D_1L + D_0 \) is an element in \( \mathcal{A}_C \), then \( Q = 0 \). As in Sec. 4.2 we have

\[
0 = D_3J_{C,n+3}(t) + D_2J_{C,n+2}(t) + D_1J_{C,n+1}(t) + D_0J_{C,n}(t)
\]

\[
= (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t) + (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t)
+ D_3(M^{-rs}pq^2t^{-4rs + 4r - 8pq} - M^{-rs}t^{-4rs - 4r})J_{T,s(n+2)-1}(t)
\]

\[
+ D_2(M^{-rs}pq^2t^{-2rs + 2r - 4pq} - M^{-rs}t^{-2rs - 2r})J_{T,s(n+1)-1}(t)
\]

\[
+ D_2M^{-rs}pq^2t^{-2rs + 2pq}\delta_{s(n+1)-1} + D_3M^{-rs}pq^2t^{-4r + 4s - 4pq}\delta_{s(n+2)-1},
\]

and applying Lemma 5.1,

\[
= (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t) + (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t)
\]

\[
+ D_3(M^{-rs}pq^2t^{-4rs + 4r - 8pq} - M^{-rs}t^{-4rs - 4r})
\]

\[
\times (M^{-pq^2}t^{-3pq^2 + 2pq + 4r}J_{T,s(n+1)-1} + V_n)
\]

\[
+ D_2(M^{-rs}pq^2t^{-2rs + 2r - 4pq} - M^{-rs}t^{-2rs - 2r})J_{T,s(n+1)-1}(t)
\]

\[
+ D_2M^{-rs}pq^2t^{-2rs - 2pq}\delta_{s(n+1)-1} + D_3M^{-rs}pq^2t^{-4r - 4s - 4pq}\delta_{s(n+2)-1}
\]

\[
= (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t) + (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t)
\]

\[
+ (D_3(M^{-rs}pq^2t^{-4rs + 4r - 8pq} - M^{-rs}t^{-4rs - 4r})M^{-pq^2}t^{-3pq^2 + 2pq}
\]

\[
\times (M^{-pq^2}t^{-3pq^2 + 2pq}J_{T,s(n+1)-1} + V_n)
\]

\[
+ (D_2(M^{-rs}pq^2t^{-2rs + 2r - 4pq} - M^{-rs}t^{-2rs - 2r})J_{T,s(n+1)-1}(t)
\]

\[
+ D_2M^{-rs}pq^2t^{-2rs - 2pq}\delta_{s(n+1)-1} + D_3M^{-rs}pq^2t^{-4r - 4s - 4pq}\delta_{s(n+2)-1},
\]

\[= (D_1 + D_3M^{-2rs}t^{-8rs})J_{C,n+1}(t) + (D_0 + D_2M^{-2rs}t^{-4rs})J_{C,n}(t)\]

\[+ (D_3(M^{-rs}pq^2t^{-4rs + 4r - 8pq} - M^{-rs}t^{-4rs - 4r})M^{-pq^2}t^{-3pq^2 + 2pq}
\]

\[\times (M^{-pq^2}t^{-3pq^2 + 2pq}J_{T,s(n+1)-1} + V_n)
\]

\[+ (D_2(M^{-rs}pq^2t^{-2rs + 2r - 4pq} - M^{-rs}t^{-2rs - 2r})J_{T,s(n+1)-1}(t)
\]

\[+ D_2M^{-rs}pq^2t^{-2rs - 2pq}\delta_{s(n+1)-1} + D_3M^{-rs}pq^2t^{-4r - 4s - 4pq}\delta_{s(n+2)-1})
\]
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\[ + D_2 (M^{r-rs-2pqsl-2rs+2r-4pqsl} - M^{-r-rs+t-2rs-2r}) J_{T,s(n+1)-1}(t) \]
\[ + D_2 M^{r-rs-pqsl-2rs-2pqsl} \delta_{s(n+1)-1} + D_3 (M^{r-rs-pqsl-t-4rs+4pqsl} \delta_{s(n+2)-1} \]
\[ + (M^{r-rs-2pqsl-t-4rs+4pqsl} - M^{-r-rs+t-4rs}) V_n) \]
\[ = D'_2 J_{C,n}(t) + D'_2 J_{C,n+1}(t) + D'_3 J_{T,s(n+1)-1}(t) + D'_0. \]

As in the previous section, we show that \( D'_i = 0, i = 1, \ldots, 3 \), implies \( D_i = 0, i = 0, \ldots, 3 \). We just need to show that the two linear equations defined by \( D'_0 = 0 \) and \( D'_1 = 0 \) are linearly independent. Again we just need to check the determinant of the linear system, multiplied by \((t^2 - t^{-2})\) and then valued at \( t = -1 \), is nonzero:

\[ \left( (M^{r-rs-2pqsl} - M^{-r-rs}) M^{-pqsl} M^{r-rs-pqsl} - (M^{r-rs-2pqsl} - M^{-r-rs}) \right) \]
\[ \times \left( M^{r-rs-pqsl} + (M^{r-rs-2pqsl} - M^{-r-rs}) \frac{M^{-pqsl} - M^{-pqsl^2-pqsl}}{1 - M^{-2pqsl}} \right) \]
\[ \times (M^{s(p+q)} + M^{-s(q-p)} - M^{-s(q-p)}) \]
\[ = \left( M^{r-rs-2pqsl} - M^{-r-rs} \right) (M^{r-rs-pqsl^2-pqsl} (1 - M^{-2pqsl}) \]
\[ - M^{r-rs-pqsl} (1 - M^{-2pqsl}) - (M^{r-rs-2pqsl} - M^{-r-rs}) \]
\[ \times (M^{pqsl} - M^{-pqsl^2-pqsl}) (M^{s(p+q)} + M^{-s(q-p)} - M^{-s(q-p)}) \]
\[ = \left( M^{r-rs-2pqsl} - M^{-r-rs} \right) (M^{-pqsl^2} - 1)(M^{r-rs-pqsl} - M^{-r-rs-pqsl}) \]
\[ \times (M^{pqsl} - M^{-pqsl})(M^{qs} - M^{-qs}) \]

which is indeed nonzero.

The following lemma shows that each \( D'_i = 0 \) if \( r \) is not a number between 0 and \( pqsl \).

Lemma 5.2. (1) When \( p > q \) and either \( r < 0 \) or \( r > pqsl \), we have \( D'_i = 0 \) for \( i = 0, 1, 2, 3 \).

(2) When \( p < -q \) and either \( r > 0 \) or \( r < pqsl \), we have \( D'_i = 0 \) for \( i = 0, 1, 2, 3 \).

The proof is similar to that of Lemma 3.3.

6. Case \( s = 2 \)

6.1. An annihilator \( P \) of \( J_{C,n}(t) \)

In this section, we assume that \( C \) is a \((r, 2)\)-cabled knot over a torus knot \( T = T(p, q) \). In

\[ J_{C,n+1}(t) = t^{-2r((n+1)^2-1)} \sum_{k=0}^{\frac{n}{2}} t^{4rk(2k+1)} J_{T,4k+1}(t), \]

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let \( k = -(j + \frac{1}{2}) \), then

\[
J_{C,n+1}(t) = t^{-2r((n+1)^2-1)} \sum_{j=\frac{3}{2}}^{n+1} t^{4r(2j+1)} J_{T,-4j-1}(t)
\]

\[
= -t^{-2r((n+1)^2-1)} \sum_{j=\frac{3}{2}}^{n+1} t^{4r(2j+1)} J_{T,Aj+1}(t)
\]

\[
= -t^{-2r((n+1)^2-1)} \left[ t^{2r(n+1)} J_{T,2n+1} + \sum_{j=\frac{3}{2}}^{n+1} t^{4r(2j+1)} J_{T,Aj+1}(t) \right]
\]

\[
= -t^{-2r((n+1)^2-1)} \left[ t^{2r(n+1)} J_{T,-2n-1} + t^{2r(n^2-1)} J_{C,n}(t) \right]
\]

\[
= t^{-2r n} J_{T,2n+1} - t^{-4r n - 2r} J_{C,n}(t). \quad (6.1)
\]

Turning \( t^{2n} \) into \( M \) and \( J_{C,n+1}(t) \) into \( LJ_{C,n}(t) \), we see that

\[
(L + M^{-2r} t^{-2r}) J_{C,n}(t) = M^{-r} J_{T,2n+1}(t),
\]

or

\[
M^r (L + M^{-2r} t^{-2r}) J_{C,n}(t) = J_{T,2n+1}(t).
\]

We now wish to find an inhomogeneous recurrence for \( J_{T,2n+1}(t) \). Recall equation (3.1):

\[
J_{T,n+2}(t) = t^{-4pq(n+1)} J_{T,n}(t) + t^{-2pq(n+1)} \delta_n,
\]

which implies that

\[
J_{T,2n+3}(t) = t^{-4pq(2n+2)} J_{T,2n+1}(t) + t^{-2pq(2n+2)} \delta_{2n+1}
\]

\[
= M^{-4pq} t^{-8pq} J_{T,2n+1}(t) + M^{-2pq} t^{-4pq} \delta_{2n+1}, \quad (6.2)
\]

and so

\[
(L - M^{-4pq} t^{-8pq}) J_{T,2n+1}(t) = M^{-2pq} t^{-4pq} \delta_{2n+1}.
\]

Letting \( b(t,M)/(t^2 - t^{-2}) = M^{-2pq} t^{-4pq} \delta_{2n+1} \). Then \( b(t,M) \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}] \), which is obviously nonzero, and we obtain an operator \( P(t,M) \) which annihilates \( J_{C,n}(t) \) given by

\[
P(t,M) = (L - 1)b^{-1}(t,M)(L - M^{-4pq} t^{-8pq}) M^r (L + M^{-2r} t^{-2r}).
\]

Assuming \( P \) has the minimal \( L \) degree, we can check the AJ-conjecture. Evaluating \( P(-1,M) \) gives

\[
P(-1,M) = b^{-1}(-1,M)(L - 1)(L - M^{-4pq}) M^r (L + M^{-2r}),
\]

which is equal to the A-polynomial of \( C \) up to a nonzero factor in \( \mathbb{Q}(M) \).
6.2. $P$ is the recurrence polynomial of $C$

Next we show that the operator $P$ given above is a generator of the ideal $\tilde{A}_C$. It amounts to show that if an operator $Q = D_2L^2 + D_1L + D_0$, where each $D_j \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$, is an annihilator of $J_{C,n}(t)$, then $Q = 0$.

So suppose that $QJ_{C,n}(t) = 0$, i.e.

$$D_2J_{C,n+2}(t) + D_1J_{C,n+1} + D_0J_{C,n}(t) = 0. \quad (6.3)$$

Our goal is to show that $D_i = 0$, $i = 0, 1, 2$.

Using (6.1) and (6.2) we can transform (6.3) into

$$0 = D_2(t^{-4rn-2r}J_{T,2n+3}(t) - t^{-4rn-6r}(M^{-r}J_{T,2n+1}(t) - t^{-4rn-2r}J_{C,n}(t)))$$
$$+ D_1(t^{-rn}J_{T,2n+1}(t) - t^{-4rn-2r}J_{C,n}(t)) + D_0J_{C,n}(t)$$
$$= (D_2t^{-8rn-8r} - D_1t^{-4rn-2r} + D_0)J_{C,n}(t)$$
$$+ D_2L^{-2r}t^{-8rn-8r}J_{T,2n+1}(t)$$
$$+ (-D_2t^{-6rn-6r} + D_1t^{-2rn})J_{C,n}(t)$$
$$= (D_2t^{-8r(n+1)} - D_1t^{-4rn-2r} + D_0)J_{C,n}(t)$$
$$+ (D_2(t^{-2r(n+1)}-8pq(n+1)} - t^{-6r(n+1)}) + D_1(t^{-2rn})J_{T,2n+1}(t)$$
$$+ D_2t^{-2r(n+1)}-4pq(n+1)\delta_{2n+1}$$
$$= D_2J_{C,n}(t) + D_1J_{T,2n+1}(t) + D_0.$$

If we can show that $D_i' = 0$, $i = 0, 1, 2$, then it will follow right away that $D_i = 0$, $i = 0, 1, 2$. As in Lemma 3.3, we can show that $D_i' = 0$, $i = 0, 1, 2$, if $r$ is not an integer between 0 and 2pq.

References

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