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DEHN FILLINGS OF KNOT MANIFOLDS CONTAINING ESSENTIAL ONCE-PUNCTURED TORI

STEVEN BOYER, CAMERON McA. GORDON, AND XINGRU ZHANG

Abstract. In this paper we study exceptional Dehn fillings on hyperbolic knot manifolds which contain an essential once-punctured torus. Let M be such a knot manifold and let β be the boundary slope of such an essential once-punctured torus. We prove that if Dehn filling M with slope α produces a Seifert fibred manifold, then $\Delta(\alpha,\beta) \le 5$. Furthermore we classify the triples $(M;\alpha,\beta)$ when $\Delta(\alpha,\beta) \ge 4$. More precisely, when $\Delta(\alpha,\beta) = 5$, then M is the (unique) manifold $Wh(-3/2)$ obtained by Dehn filling one boundary component of the Whitehead link exterior with slope $-3/2$, and (α,β) is the pair of slopes $(-5,0)$. Further, $\Delta(\alpha,\beta) = 4$ if and only if $(M;\alpha,\beta)$ is the triple $(Wh(\frac{-2n\pm 1}{n});-4,0)$ for some integer n with $|n|>1$. Combining this with known results, we classify all hyperbolic knot manifolds M and pairs of slopes (β,γ) on ∂M where β is the boundary slope of an essential once-punctured torus in M and γ is an exceptional filling slope of distance 4 or more from β. Refined results in the special case of hyperbolic genus one knot exteriors in S^3 are also given.

1. Introduction

This is the second of four papers in which we investigate the following conjecture of the second-named author (see [Go2, Conjecture 3.4]). Recall that a *hyperbolic knot manifold* is a compact, connected, orientable 3-manifold with torus boundary whose interior admits a complete, finite volume hyperbolic structure.

Conjecture 1.1 (C. McA. Gordon). *Suppose that M is a hyperbolic knot manifold and α,β are slopes on ∂M such that $M(\alpha)$ is Seifert fibred and $M(\beta)$ is toroidal. If $\Delta(\alpha,\beta) > 5$, then M is the figure eight knot exterior.*

Our first result reduces the verification of the conjecture to the case where the Seifert filling is atoroidal.

Theorem 1.2. *Suppose that M is a hyperbolic knot manifold and α,β are slopes on ∂M such that $M(\alpha)$ is a toroidal Seifert fibred manifold and $M(\beta)$ is toroidal. Then $\Delta(\alpha,\beta) \le 4$. Furthermore, if $\Delta(\alpha,\beta) = 4$, then $(M;\alpha,\beta) \cong (N(-\frac{1}{2},-\frac{1}{2});-4,0)$, where N is the exterior of the* 3*-chain link* [MP].

We have that $N(-\frac{1}{2},-\frac{1}{2},-4)$ is Seifert fibred with base orbifold $P^2(2,3)$ and $N(-\frac{1}{2},-\frac{1}{2},0)$ contains an incompressible torus separating $N(-\frac{1}{2},-\frac{1}{2},0)$ into Seifert fibred manifolds with base orbifolds $D^2(2,2)$ and $D^2(2,3)$. (See [MP, Table 2].)

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A *small Seifert* manifold is a 3-manifold which admits a Seifert structure with base orbifold of the form $S^2(a, b, c)$, where $a, b, c \geq 1$. For instance, a closed, atoroidal Seifert manifold is small Seifert.

A small Seifert manifold is a *prism manifold* if its base orbifold is $S^2(2, 2, n)$ for some $n \geq 2$.

Since the distance between a toroidal filling slope and a reducible filling slope is at most 3 ([Oh], [Wu1]), Theorem 1.2 reduces our analysis of Conjecture 1.1 to understanding the case where the Seifert Dehn filling is irreducible and small Seifert. In an earlier paper [BGZ2] we verified the conjecture in the case where M admits no essential punctured torus of boundary slope β which is a fibre or semi-fibre, or which has fewer than three boundary components; more precisely, we showed that in this case $\Delta(\alpha, \beta) \leq 5$. Here we focus on the case where M admits an essential punctured torus with one boundary component.

Let Wh denote the left-handed Whitehead link exterior (see Figure 33). We parameterise the slopes on a boundary component of Wh using the standard meridian-longitude coordinates.

Theorem 1.3. *Let M be a hyperbolic knot manifold and α a slope on ∂M such that $M(\alpha)$ is small Seifert. If M admits an essential, once-punctured torus F of boundary slope β, then $\Delta(\alpha, \beta) \leq 5$. Further, if $\Delta(\alpha, \beta) > 3$, then F is not a fibre and $\pi_1(M(\alpha))$ is finite. More precisely,*

(1) *if $\Delta(\alpha, \beta) = 4$, then $(M; \alpha, \beta) \cong (Wh(\frac{-2n\pm1}{n}); -4, 0)$ for some integer n with $|n| > 1$ and $M(\alpha)$ has base orbifold $S^2(2, 2, |\mp 2n - 1|)$, so $M(\alpha)$ is a prism manifold;*

(2) *if $\Delta(\alpha, \beta) = 5$, then $(M; \alpha, \beta) \cong (Wh(-3/2); -5, 0)$, and $M(\alpha)$ has base orbifold $S^2(2, 3, 3)$.*

Baker [Ba] has proven Theorem 1.3 in the case where $M(\alpha)$ is a lens space. We provide an alternate proof of his result.

Theorem 1.3 is sharp; see the infinite family of examples in §11 for (1) and [MP, Table A.3] for (2). Another family of examples is provided by hyperbolic twist knots. These are genus one knots in the 3-sphere whose exteriors admit small Seifert filling slopes of distance 1, 2, and 3 from the longitudinal slope. Finally, Baker [Ba, Theorem 1.1(IV)] has constructed an infinite family of non-fibred hyperbolic knot manifolds which admit a once-punctured essential torus whose boundary slope is of distance 3 to a lens space filling slope.

Here is an outline of the proof of Theorem 1.3. We begin by showing that the result holds unless, perhaps, M admits an orientation-preserving involution τ with non-empty branch set L contained in the interior of the quotient M/τ, which is a solid torus. The results of [BGZ2] reduce us to the case that L has a very particular form (see Figure 3). On the other hand, τ extends to an involution τ_α of $M(\alpha)$ with branch set L_α contained in the lens space $M(\alpha)/\tau_\alpha$. The fundamental group of $M(\alpha)/\tau_\alpha$ is non-trivial if the distance between α and β is at least 3. Since the involutions on small Seifert manifolds with such quotients are well understood, we can explicitly describe the branch set L_α of τ_α. Comparing this description with the constraints we have already deduced on L leads to the proof of the theorem.

Recall that an *exceptional filling slope* on the boundary of a hyperbolic 3-manifold is a slope γ such that $M(\gamma)$ is not hyperbolic. Geometrisation of 3-manifolds implies that a slope γ is exceptional if and only if $M(\gamma)$ is either reducible, toroidal, or Seifert fibred. Theorem 1.3 combines with [Oh], [Wu1], [Go1], [GW], and Proposition 3.1 to yield the next result.

Theorem 1.4. *Let M be a hyperbolic knot manifold which admits an essential, once-punctured torus F of boundary slope β and let γ be an exceptional filling slope on ∂M.*

(1) $\Delta(\gamma, \beta) \leq 7$.
(2) *If $\Delta(\gamma, \beta) > 3$, then $M(\gamma)$ is either toroidal or has a finite fundamental group.*
(3) *If $\Delta(\gamma, \beta) > 3$ and $M(\gamma)$ is toroidal, then either*
  (a) $\Delta(\gamma, \beta) = 4$ *and* $(M; \gamma, \beta) \cong (Wh(\delta); -4, 0)$ *for some slope δ, or*
  (b) $\Delta(\gamma, \beta) = 5$ *and* $(M; \gamma, \beta) \cong (Wh(-4/3); -5, 0)$ *or* $(Wh(-7/2); -5/2, 0)$, *or*
  (c) $\Delta(\gamma, \beta) = 7$ *and* $(M; \gamma, \beta) \cong (Wh(-5/2); -7/2, 0)$.
(4) *If $\Delta(\gamma, \beta) > 3$ and $\pi_1(M(\gamma))$ is finite, then either*
  (a) $\Delta(\gamma, \beta) = 4$, $(M; \gamma, \beta) \cong (Wh(\frac{-2n\pm 1}{n}); -4, 0)$ *for some integer n with $|n| > 1$, and $M(\gamma)$ has base orbifold $S^2(2, 2, |\mp 2n - 1|)$, or*
  (b) $\Delta(\gamma, \beta) = 5, (M; \gamma, \beta) \cong (Wh(-3/2); -5, 0)$, *and $M(\gamma)$ has base orbifold $S^2(2, 3, 3)$.*

Next we specialize to the case where M is the exterior of a hyperbolic knot in the 3-sphere.

Theorem 1.5. *Let $K \subset S^3$ be a hyperbolic knot of genus one with exterior M_K and suppose p/q is an exceptional filling slope on ∂M_K.*

(1) *$M_K(0)$ is toroidal but not Seifert.*
(2) *$M_K(p/q)$ is either toroidal or small Seifert with hyperbolic base orbifold.*
(3) *If $M_K(p/q)$ is small Seifert with hyperbolic base orbifold, then $0 < |p| \leq 3$.*
(4) *If $M_K(p/q)$ is toroidal, then $|q| = 1$ and $|p| \leq 4$ with equality implying K is a twist knot.*

Here is how the paper is organised. We prove Theorem 1.2 in §2. In §3 we show that there are strong topological constraints on M which must be satisfied if Theorem 1.3 doesn't hold. These constraints will be applied later in the paper to construct an involution on M. In §4 we describe the branching set of an orientation-preserving involution on a small Seifert manifold with quotient space a lens space with non-trivial fundamental group. Using this, in §5 we reduce the proof of Theorem 1.3 to five problems involving links in lens spaces and a problem in which $\Delta(\alpha, \beta) = 4$ and $M(\alpha)$ is a prism manifold. These problems are resolved in §6, §7, §8, §9, §10 and §12, respectively. The infinite family of examples realising distance 4 in Theorem 1.3 is constructed in §11. Theorems 1.4 and 1.5 are dealt with in §13.

2. The case where $M(\alpha)$ is toroidal

In this section we prove Theorem 1.2. Recall from the introduction that N denotes the exterior of the 3-chain link of [MP]. Note that $N(-\frac{1}{2}, -\frac{1}{2})$ is obtained by Dehn filling on $N(-\frac{1}{2})$, which is the exterior of the rational link associated with the rational number 10/3.

To prove Theorem 1.2 we consider all $(M; \alpha, \beta)$ where M is hyperbolic, $M(\alpha)$ and $M(\beta)$ are toroidal and $\Delta(\alpha, \beta) \geq 4$. For $\Delta(\alpha, \beta) \geq 6$ there are only four such $(M; \alpha, \beta)$ [Go1], and in all four cases neither $M(\alpha)$ nor $M(\beta)$ is Seifert fibred.

For $\Delta(\alpha, \beta) = 4$ or 5, the triples $(M; \alpha, \beta)$ are determined in [GW]: there are 14 hyperbolic manifolds M_i, $1 \leq i \leq 14$, each with a pair of toroidal filling slopes α_i, β_i at distance 4 or 5, where M_1, M_2, M_3 and M_{14} have two (torus) boundary components, and the others, one. It is shown in [GW] that a hyperbolic manifold M has two toroidal filling slopes α and β at distance 4 or 5 if and only if $(M; \alpha, \beta) \cong (M_i; \alpha_i, \beta_i)$ for some $1 \leq i \leq 14$, or $(M; \alpha, \beta) \cong (M_i(\gamma); \alpha_i, \beta_i)$ for $i = 1, 2, 3$ or 14 and some slope γ on the second boundary component of M_i. (We adopt the convention that in the above homeomorphisms either $\alpha \mapsto \alpha_i$, $\beta \mapsto \beta_i$ or $\alpha \mapsto \beta_i$, $\beta \mapsto \alpha_i$.) We prove Theorem 1.2 by showing first, for $i \neq 1, 2, 3$ or 14, neither of the toroidal manifolds $M_i(\alpha_i)$ or $M_i(\beta_i)$ is Seifert fibred, second, for $i = 1, 3$ or 14, there is no hyperbolic manifold of the form $M_i(\gamma)$ with either $M_i(\gamma)(\alpha_i)$ or $M_i(\gamma)(\beta_i)$ toroidal Seifert fibred and third, there is a unique example $(M_2(\gamma); \alpha_2, \beta_2)$ (up to homeomorphism) where $M_2(\gamma)$ is hyperbolic, $M_2(\gamma)(\alpha_2)$ and $M_2(\gamma)(\beta_2)$ are toroidal, and one is Seifert fibred; this is the example described in Theorem 1.2.

We first consider the manifolds M_i, $6 \leq i \leq 13$. The toroidal fillings on M_i, $M_i(0)$ and $M_i(\beta_i)$, are described in Lemma 22.2 of [GW]. We adopt the notation introduced in [GW, page 116].

Lemma 2.1. *For $6 \leq i \leq 13$, $M_i(0)$ is not Seifert fibred.*

Proof. $M_i(0)$ is of the form $X(p_1, q_1; p_2, q_2)$; it is the double branched cover of the tangle $Q_i(0)$, which is of the form $T(p_1, q_1; p_2, q_2)$, the union of two Montesinos tangles. Assume the numbering is chosen so that p_1, q_1 are not both 2 (actually this is only an issue when $i = 8$). Then the Seifert fibre φ_1 of $X(p_1, q_1)$ is unique. Since $X(p_1, q_1)$ and $X(p_2, q_2)$ are not both twisted I-bundles, to show that $M_i(0)$ is not Seifert fibred it suffices to show that, in the gluing of $X(p_1, q_1)$ and $X(p_2, q_2)$, φ_1 is not identified with the Seifert fibre φ_2 of $X(p_2, q_2)$. (When $i = 8$, $p_2 = q_2 = 2$ and there are two possible choices for φ_2.) We do this by identifying the image of φ_1 in the boundary of the tangle $T(p_1, q_1)$ and then capping off the tangle $T(p_2, q_2)$ with the corresponding rational tangle. In the double branched cover this corresponds to doing Dehn filling on $X(p_2, q_2)$ along the slope φ_1. If $M_i(0)$ were Seifert fibred, then this Dehn filling would be reducible, and so the corresponding rational tangle filling on $T(p_2, q_2)$ would give a link that is either composite or split. One checks that this is not the case. □

Lemma 2.2. *For $6 \leq i \leq 13$, $M_i(\beta_i)$ is not Seifert fibred.*

Proof. First note that $M_7(\beta_7)$ is of the form $X(2, 3; 2, 2)$. We check that this is not Seifert fibred in the same way as we did for $M_8(0)$ in Lemma 2.1.

When $i \neq 7$, $M_i(\beta_i)$ is the double branched cover of a 2-component link L_i; see [GW, Lemma 22.2]. More specifically, for $i = 6, 8, 9$ or 12, L_i is a cabled Hopf link $C(p_1, q_1; p_2, q_2)$ with $p_1, p_2 > 1$, for $i = 10$ or 11, L_i is the link $C(C; 2, 1)$ (see [GW, page 116]), and for $i = 13$, L_i is the 2-string cable of the trefoil shown in [GW, Figure 22.13(d)]. In all cases, L_i is *toroidal*, i.e. its exterior contains an essential torus. Moreover, the exterior of L_i is not Seifert fibred. Therefore if $M_i(\beta_i)$ were Seifert fibred, then L_i would be a Montesinos link. But the only toroidal Montesinos links are (see [Oe, Corollary 5]) $K(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $K(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, $K(\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4})$, and $K(\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6})$. One easily checks that no L_i is of this form. □

Lemma 2.3. *$M_4(\alpha_4)$ and $M_4(\beta_4)$ are not Seifert fibred.*

Proof. $M_4(\alpha_4)$ and $M_4(\beta_4)$ contain incompressible tori $\widehat{F}_a$ and $\widehat{F}_b$; the corresponding punctured tori F_a and F_b in M_4 have four and two boundary components, respectively. The intersection of F_a and F_b is described by the intersection graphs $\Gamma_a \subset \widehat{F}_a$ and $\Gamma_b \subset \widehat{F}_b$ depicted in Figures 11.9(a) and (b) of [GW], respectively. Note that $\widehat{F}_a$ separates $M_4(\alpha_4)$, into M_B and M_W, say, while $\widehat{F}_b$ is non-separating in $M_4(\beta_4)$. The faces of the graph Γ_b lie alternately in M_B and M_W; we choose the notation so that all the faces of Γ_b that lie in M_B are bigons.

Let f_1, f_2, f_3, and g_1, g_2, g_3 be the faces of Γ_b with edges $G, H; J, K; A, B$; and $D, E; K, P, R; A, G, L$; respectively. Let h_1, h_2, h_3 be the faces of Γ_a with edges $E, N; H, E$; and B, G, N, R; respectively. (The notation refers to the edges illustrated in Figure 11.9 of [GW].)

For computations in $\pi_1(M_B)$ and $\pi_1(M_W)$ we take as "base-point" the rectangle in $\widehat{F}_a$ shown in Figure 11.9(a) of [GW]. Let s, t be the pair of generators of $\pi_1(\widehat{F}_a)$ determined by the downward vertical and rightward horizontal edges of that rectangle, respectively. Let x_1 and x_3 be the elements of $\pi_1(M_B)$ corresponding to the 1-handles $H_{(12)}$ and $H_{(34)}$ in the usual way. The faces f_1, f_2 and f_3 give the relations in $\pi_1(M_B)$:

$$\begin{aligned} &x_1^2 t = 1, \\ &x_3^2 t^{-1} = 1, \\ &s^{-1} x_3 x_1 = 1. \end{aligned}$$

It follows that M_B is Seifert fibred with base orbifold $D^2(2,2)$ and that the classes in $\pi_1(\widehat{F}_a)$ of the Seifert fibres in the two Seifert fibrings of M_B are t and s.

Let x_2 and x_4 be the elements of $\pi_1(M_W)$ corresponding to $H_{(23)}$ and $H_{(41)}$. Then the faces g_1, g_2 and g_3 give the relations in $\pi_1(M_W)$:

$$\begin{aligned} &t x_4 x_2 = 1, \\ &x_2 x_4 t^{-1} x_2 s t = 1, \\ &x_2 x_4^2 t^{-1} = 1. \end{aligned}$$

These show that M_W is Seifert fibred with base orbifold $D^2(2,3)$, the class of the Seifert fibre in $\pi_1(\widehat{F}_a)$ being st^2. Since this is distinct from either of the Seifert fibres of M_B, $M_4(\alpha_4)$ is not Seifert fibred.

We now consider $M_4(\beta_4)$. Let u, v be the pair of generators for $\pi_1(\widehat{F}_b)$ given by the downward vertical and leftward horizontal edges of the rectangle in Figure 11.9(b) of [GW]. (We take this rectangle as "base-point" for computations in $\pi_1(M_4(\beta_4))$.) Let x, y be the elements of $\pi_1(M_4(\beta_4))$ given by the 1-handles $H_{(12)}$ and $H_{(21)}$. The faces h_1, h_2, h_3 give the relations in $\pi_1(M_4(\beta_4))$:

$$\begin{aligned} &x(uv)y^{-1}v^{-1} = 1, \\ &yvx^{-1} = 1, \\ &x^{-1}u^{-1}xux^{-1}(vu)^{-1}y = 1. \end{aligned}$$

The second relation gives $x = yv$, and the first then gives

$$y^{-1}vy = uv^2.$$

The third relation gives

$$(y^{-1}u^{-1}y)u(y^{-1}u^{-1}y)u^{-1}v^{-3} = 1.$$

Now if $M_4(\beta_4)$ were Seifert fibred, the non-separating torus $\widehat{F}_b$ would be horizontal, and so $M_4(\beta_4)$ would be a torus bundle over the circle with fibre $\widehat{F}_b$. Hence $y^{-1}u^{-1}y$ would belong to $\pi_1(\widehat{F}_b)$. But the last relation above shows that if this is the case, then

$$(y^{-1}u^{-1}y)^2 = v^3.$$

Since v^3 is not a square in $\pi_1(\widehat{F}_b)$, this is a contradiction. □

Lemma 2.4. *$M_5(\alpha_5)$ and $M_5(\beta_5)$ are not Seifert fibred.*

Proof. This can be proved in a similar fashion to Lemma 2.3, using [GW, Figure 11.10]. Another way to establish the result is to note that, according to [L2, §6], $M_5 \cong N(1, -\frac{1}{3})$, the toroidal filling slopes α_5, β_5 being -4 and 1. We see that $N(1, -\frac{1}{3}, -4)$ and $N(1, -\frac{1}{3}, 1)$ are not Seifert fibred from Tables 4 and 3 of [MP], respectively. □

We next consider the manifolds M_1, M_2 and M_3, namely the exteriors of the Whitehead link, the 10/3-rational link, and the Whitehead sister (or $(-2,3,8)$-pretzel) link, respectively. These are all obtained by Dehn filling on the 3-chain link: $M_1 \cong N(1)$, $M_2 \cong N(-\frac{1}{2})$, $M_3 \cong N(-4)$. Furthermore, their exceptional slopes and toroidal slopes are as follows (see [MP, Table A.1]):

	exceptional slopes	toroidal slopes
$N(1)$	$\infty, -3, -2, -1, 0, 1$	$-3, 1$
$N(-\frac{1}{2})$	$\infty, -4, -3, -2, -1, 0$	$-4, 0$
$N(-4)$	$\infty, -3, -2, -1, -\frac{1}{2}, 0$	$-\frac{1}{2}, 0$

Lemma 2.5. *In each of the following cases, the manifold $N(\alpha, \beta, \gamma)$ is a toroidal Seifert fibre space if and only if γ is one of the values listed:*

(a) $N(1, -3, \gamma)$: $\gamma = -3, 1,$
$N(1, 1, \gamma)$: $\gamma = -3, -2, -1, 0.$

(b) $N(-\frac{1}{2}, -4, \gamma)$: $\gamma = -\frac{1}{2},$
$N(-\frac{1}{2}, 0, \gamma)$: $\gamma = -\frac{7}{2}.$

(c) $N(-4, -\frac{1}{2}, \gamma)$: $\gamma = -\frac{1}{2},$
$N(-4, 0, \gamma)$: *no* γ.

Proof. This follows by inspecting Tables 2, 3 and 4 of [MP]. We see from these that the only toroidal Seifert fibre spaces $N(\alpha, \beta, \gamma)$ are

(1) $N(-3, 1, 1)$, $N(-3, -\frac{5}{3}, -\frac{5}{3})$, $N(-3, -3, t/u)$ where $t/u \neq -1, -1 + \frac{1}{m}$ or ∞, and

(2) $N(0, \frac{1}{2} + n, -\frac{9}{2} - n)$, $N(1, 1, n)$ where $|n + 1| \leq 1$, $N(-\frac{3}{2}, -\frac{5}{2}, 0)$, and $N(-4, -\frac{1}{2}, -\frac{1}{2})$. □

Note that the values of γ listed in parts (a) and (c) of Lemma 2.5 all belong to the set of exceptional slopes of $N(1)$ and $N(-4)$, respectively. It follows that for $i = 1$ and 3, there is no γ such that $M_i(\gamma)$ is hyperbolic and one of $M_i(\gamma)(\alpha_i)$, $M_i(\gamma)(\beta_i)$ is toroidal Seifert fibred.

In the case $i = 2$, note that by [MP, Proposition 1.5 part (1.4)], there is an automorphism of $N(-\frac{1}{2})$ inducing homeomorphisms

$$N(-\tfrac{1}{2}, -4, -\tfrac{1}{2}) \cong N(-\tfrac{1}{2}, 0, -\tfrac{7}{2}),$$

$$N(-\tfrac{1}{2}, 0, -\tfrac{1}{2}) \cong N(-\tfrac{1}{2}, -4, -\tfrac{7}{2}).$$

Also, we see from [MP, Table 2] that $N(-\frac{1}{2}, 0, -\frac{1}{2})$ is toroidal. Thus part (b) of Lemma 2.5 gives rise to the single example described in Theorem 1.2.

Finally, we take care of M_{14}:

Lemma 2.6. *For no slope γ on the second boundary component of M_{14} is $M_{14}(\gamma)(\alpha_{14})$ or $M_{14}(\gamma)(\beta_{14})$ toroidal Seifert fibred.*

Proof. In [L1] Lee describes a hyperbolic 3-manifold Y with two torus boundary components having (homeomorphic) Dehn fillings $Y(0)$ and $Y(4)$ that contain Klein bottles. In fact $Y(0) \cong Y(4) \cong Q(2,2) \cup Wh$, where $Q(2,2)$ is the Seifert fibre space with base orbifold $D^2(2,2)$ and Wh is the exterior of the Whitehead link. Hence $Y(0) \cong Y(4)$ is toroidal. It follows from the classification in [GW] of the hyperbolic 3-manifolds with toroidal fillings at distance 4 that $Y \cong M_{14}$. (The only other manifolds with two boundary components having toroidal fillings at distance 4 are M_1 and M_2, and there the toroidal fillings are graph manifolds; see e.g. [MP, Table A.1].) It therefore suffices to show that $M_{14}(\gamma)(\alpha_{14})$ is not toroidal Seifert fibred for any slope γ.

The manifold $M = M_{14}(\alpha_{14}) \cong Q(2,2) \cup Wh$ is the double branched cover of the tangle shown in [GW, Figure 22.14(b)]. Thus $M(\gamma) \cong Q(2,2) \cup Wh(\gamma)$. Hence if $M(\gamma)$ is toroidal Seifert fibred then γ must be an exceptional slope for Wh. These slopes (with respect to the parametrization in [MP, Table A.1]) are $\infty, -3, -2, -1, 0$ and 1. Now $Wh(-3)$ and $Wh(1)$ are toroidal non-Seifert, $Wh(\infty) \cong D^2 \times S^1$, and $Wh(-2)$, $Wh(-1)$ and $Wh(0)$ are Seifert fibred with base orbifold $D^2(3,3)$, $D^2(2,4)$ and $D^2(2,3)$, respectively. So we need only consider $M(\gamma)$ for $\gamma = \infty, -2, -1$ and 0; we do this by examining the corresponding rational tangle filling on the tangle shown in [GW, Figure 22.14(b)]. For $\gamma = \infty$, this yields the pretzel knot $K(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, so $M(\infty)$ is atoroidal. For $\gamma = -2, -1$ and 0 we show that the Seifert fibre of $Wh(\gamma)$ does not match the Seifert fibre in either of the two Seifert fibrings of $Q(2,2)$. This is straightforward to check, for example by using the same approach as in the proof of Lemma 2.1. □

3. Background results for the proof of Theorem 1.3

We collect various results in this section and the next which will be used throughout this paper and its sequel [BGZ3]. In what follows, M will be a hyperbolic knot manifold and $b_1(M)$ will denote its first Betti number. In this section we assume that F is an essential, punctured torus of slope β which is properly embedded in M.

For a closed, essential surface S in M we define $\mathcal{C}(S)$ to be the set of slopes δ on ∂M such that S compresses in $M(\delta)$. A slope η on ∂M is called a *singular slope* for S if $\eta \in \mathcal{C}(S)$ and $\Delta(\delta, \eta) \leq 1$ for each $\delta \in \mathcal{C}(S)$. A result of Wu [Wu2] states that if $\mathcal{C}(S) \neq \emptyset$, then there is at least one singular slope for S.

Proposition 3.1. *Suppose that M admits a non-separating, essential, genus 1 surface of boundary slope β which caps-off to a compressible torus in $M(\beta)$. If γ is* *a slope on ∂M such that $M(\gamma)$ is not hyperbolic, then $\Delta(\gamma,\beta) \leq 3$. If $M(\gamma)$ is an irreducible, atoroidal, small Seifert manifold, then $\Delta(\gamma,\beta) \leq 1$.*

Proof. By hypothesis $M(\beta)$ admits a non-separating 2-sphere and so is reducible with first Betti number at least 1. In the case that $b_1(M) \geq 2$, there is a closed essential surface $S \subset \mathrm{int}(M)$ which is Thurston norm minimizing in $H_2(M)$. By [Ga, Corollary], S is essential and Thurston norm minimizing in $H_2(M(\delta))$ for all slopes $\delta \neq \beta$. By [BGZ1, Proposition 5.1], $\Delta(\gamma,\beta) \leq 1$ for any slope γ such that $M(\gamma)$ is not hyperbolic. Suppose then that $b_1(M) = 1$, and note that by hypothesis β is a strict boundary slope. In this case [BCSZ2, Theorem 3.2] implies that β is a singular slope, and so the conclusions of the lemma follow from [BGZ1, Theorem 1.5]. □

Corollary 3.2. *Theorem* 1.3 *holds if M admits a non-separating, essential, genus* 1 *surface of boundary slope β which caps-off to a compressible torus in $M(\beta)$.* □

The torus in $M(\beta)$ obtained by capping-off F with a meridional disk will be denoted $\widehat{F}$. We use M_F to denote the compact manifold obtained by cutting M open along F and $M(\beta)_{\widehat{F}}$ the manifold obtained by cutting $M(\beta)$ open along $\widehat{F}$.

Proposition 3.3. *Suppose that $M(\alpha)$ is a Seifert fibred manifold and $M(\beta)$ is toroidal. Then $\Delta(\alpha,\beta) \leq 3$ as long as one of the following conditions is satisfied:*
(a) *α or β is a singular slope of a closed essential surface in M.*
(b) *$M(\alpha)$ or $M(\beta)$ is reducible.*
(c) (i) *$|\partial F| = 1$ and M_F is not a genus* 2 *handlebody.*
(ii) *$|\partial F| = 2$ and M_F is neither connected nor a union of two genus* 2 *handlebodies.*

Proof. If α or β is a singular slope of a closed essential surface in M, then [BGZ1, Corollary 1.6] shows that $\Delta(\alpha,\beta) \leq 3$, so we are done in case (a).

Assume next that $M(\gamma)$ is reducible, where γ is one of α or β. If $\gamma = \alpha$, then $\Delta(\alpha,\beta) \leq 3$ by [Oh] and [Wu1]. Assume then that $\gamma = \beta$. If $b_1(M) \geq 2$, then $\Delta(\gamma,\beta) \leq 1$ for any exceptional slope γ as in the proof of Proposition 3.1. Assume then that $b_1(M) = 1$. Since $M(\beta)$ is toroidal, it is neither $S^1 \times S^2$ nor a connected sum of lens spaces. Hence [BGZ1, Proposition 6.2] implies that β is a singular slope of a closed essential surface in M. Thus we are done by part (a).

Finally consider part (c) of the proposition. If $|\partial F| = 1$, any compression of ∂M_F in M_F yields one or two tori, so as M is hyperbolic it is not hard to see that M_F is a handlebody, contrary to hypothesis. Thus ∂M_F is incompressible in M_F, and hence in M. Let $S \subset \mathrm{int}(M)$ be the inner boundary component of a collar of ∂M_F in M_F. Then S is incompressible in M, and by construction there is an annulus A in M with boundary components $\partial_1 A$ and $\partial_2 A$, say, where $A \cap S = \partial_1 A$ and $A \cap \partial M = \partial_2 A$ has slope β on ∂M. It follows from [Sh] that S is incompressible in $M(\gamma)$ whenever $\Delta(\gamma,\beta) > 1$. Thus β is a singular slope for S, and so part (a) of this proposition shows $\Delta(\alpha,\beta) \leq 3$. Thus (i) holds.

If $|\partial F| = 2$ and M_F is not connected, then $M = X_1 \cup_F X_2$ where ∂X_j is a genus 2 surface for $j = 1,2$. If ∂X_j compresses in X_j for both j, then X_1 and X_2 are genus 2 handlebodies as M is hyperbolic. Since this possibility is excluded by our hypotheses, ∂X_j is incompressible in X_j for some j. Then it is essential in M but compresses in $M(\beta)$, so as in the previous paragraph, β is a singular slope for ∂X_j. Thus $\Delta(\alpha,\beta) \leq 3$. This completes the proof. □

Theorem 1.2 and Propositions 3.1 and 3.3 yield the following corollary.

Corollary 3.4. *Conjecture* 1.1 *holds as long as it holds when $M(\alpha)$ is an irreducible, atoroidal, small Seifert manifold.* □

Here is a result from [BGZ2]. Recall from §6 of that paper that t_j^+ is the number of *tight components* of $\breve{\Phi}_j^+$.

A 3-manifold is *very small* if its fundamental group does not contain a non-abelian free group.

Proposition 3.5. *Suppose that F is a once-punctured essential genus* 1 *surface of boundary slope β in a hyperbolic knot manifold M which completes to an essential torus in $M(\beta)$ but is not a fibre in M. If $M(\alpha)$ is a small Seifert manifold, then*

$$\Delta(\alpha,\beta) \leq \begin{cases} 6 & \text{if } M(\alpha) \text{ is very small,} \\ 8 & \text{otherwise.} \end{cases}$$

Moreover if $t_1^+ > 0$, then

$$\Delta(\alpha,\beta) \leq \begin{cases} 3 & \text{if } M(\alpha) \text{ is very small,} \\ 4 & \text{otherwise.} \end{cases}$$

Remark 3.6. When $t_1^+ = 0$, $M(\beta)_{\widehat{F}}$ is Seifert with base orbifold an annulus with one cone point [BGZ2, Lemma 7.9].

Proof of Proposition 3.5. The first inequality is the conclusion of [BGZ2, Proposition 13.2]. To deduce the second we use the notation and results of [BGZ2].

Suppose next that $t_1^+ > 0$. Since t_1^+ is even and the number of boundary components of F is bounded below by $\frac{1}{2}t_1^+$, we have $t_1^+ = 2$. Proposition 13.1 of [BGZ2] then shows that $\Delta(\alpha,\beta) \leq 4$. Suppose that $M(\alpha)$ is very small. The first paragraph of the proof of [BGZ2, Proposition 13.1] shows that $\Delta(\alpha,\beta) \leq 3$ if $\overline{\Gamma}_S$ has a vertex of valency 3 or less, while the second shows that the same inequality holds if it doesn't. This completes the proposition's proof. □

4. Involutions on small Seifert manifolds

We collect several results about involutions on small Seifert manifolds in this section.

Lemma 4.1. *Let W be a small Seifert manifold and τ an orientation-preserving involution on W with non-empty fixed point set. Then there is a τ-invariant Seifert structure on W with base orbifold of the form $S^2(a,b,c)$, where $1 \leq a \leq b \leq c$.*

Proof. If W is a lens space, the result follows from [HR]. Assume then that this isn't the case and fix a Seifert structure on W with base orbifold $S^2(a,b,c)$, where $a \leq b \leq c$. The assumption that $\pi_1(W)$ is not cyclic implies that $a \geq 2$ and a,b,c are determined by W.

Let $L \subset W/\tau$ be the branch set of τ. The orbifold theorem implies that the orbifold W/τ is geometric, and since L is a link, W/τ admits a Seifert structure with a 2-dimensional base orbifold [Du]. Thus W admits a τ-invariant Seifert structure. We claim that we can assume this structure has base orbifold $S^2(a,b,c)$. If $b \neq 2$, all Seifert structures on W have this form, so assume $a = b = 2 \leq c$. If the base orbifold of the τ-invariant structure is not $S^2(a,b,c)$, it must be $P^2(d)$ for some integer $d \geq 1$. When $d > 1$, there is a unique singular fibre ϕ in this structure, and it must be invariant under τ. Then τ leaves the exterior E of this fibre invariant, which is a twisted I-bundle over the Klein bottle. By assumption, τ leaves the Seifert structure on E with base orbifold a Möbius band invariant. There is exactly one other Seifert structure on E, up to isotopy, and its base orbifold is $D^2(2,2)$. Moreover, there is at least one such structure which is $\tau|E$-invariant. This structure can be extended across a fibred neighbourhood of ϕ in a τ-invariant fashion, yielding the desired τ-invariant structure on W.

The argument is similar if $d = 1$, for τ induces an involution of the base orbifold P^2 of W, and since any self-map of P^2 has a fixed point, there is a τ-invariant fibre ϕ in W. Now proceed as in the case $d > 1$. □

For our next three results we let W denote a small Seifert manifold and τ an orientation-preserving involution on W with non-empty fixed point set such that the quotient W/τ is a lens space $L(\bar{p}, \bar{q}) \not\cong S^3$. We use L_τ to denote the branch set of τ in $L(\bar{p}, \bar{q})$.

Fix a τ-invariant Seifert structure on W with base orbifold of the form $S^2(a, b, c)$ where $1 \leq a \leq b \leq c$ (Lemma 4.1) and let $\bar{\tau}$ be the involution of $S^2(a, b, c)$ (possibly the identity) induced by τ.

Since the τ-invariant Seifert structure on W has an orientable base orbifold, its fibres can be coherently oriented.

Hodgson and Rubinstein have classified orientation-preserving involutions on lens spaces with non-empty fixed point sets. In particular, their work yields the following result.

Lemma 4.2 ([HR, §4.7])**.** *Suppose that W is the lens space $L(p, q)$ and $W/\tau = L(\bar{p}, \bar{q}) \not\cong S^3$.*

(1) *If p is odd, then L_τ is connected and is either*
   (a) *the core of a solid torus of a genus one Heegaard splitting of $L(\bar{p}, \bar{q})$, or*
   (b) *the boundary of a Möbius band spine of a Heegaard solid torus of $L(\bar{p}, \bar{q})$.*

(2) *If p is even, then L_τ has two components and is either*
   (a) *the union of the cores of the two solid tori of a genus one Heegaard splitting of $L(\bar{p}, \bar{q})$, or*
   (b) *the boundary of an annular spine of a Heegaard solid torus of $L(\bar{p}, \bar{q})$.*

□

Next we suppose that W is not a lens space. In this case $2 \leq a \leq b \leq c$.

Lemma 4.3. *Suppose that W is not a lens space and that τ preserves the orientations of the Seifert fibres of W. Then there is an induced Seifert structure on W/τ such that L_τ is a union of at most three Seifert fibres where at least one of the fibres is regular. Further, $\bar{\tau}$ is either the identity or has two fixed points and*

(1) *if $\bar{\tau}$ is the identity then $a = 2$, $|L_\tau|$ is the number of cone points of $S^2(a, b, c)$ of even order, and the components of L_τ which are regular fibres correspond to the cone points of order 2;*

(2) *if $\bar{\tau}$ is not the identity then L_τ has at most two components, and exactly one of its components is a regular fibre.*

Proof. The hypotheses imply that there is an induced Seifert structure on $L(\bar{p}, \bar{q})$ whose fibres are the images of the fibres of W. Since W has three exceptional fibres, $\bar{\tau}$ fixes precisely one or three cone points. In the latter case, $\bar{\tau}$ is the identity.

Suppose first that $\bar{\tau}$ is the identity on $S^2(a, b, c)$. Since τ has 1-dimensional fixed point set, τ rotates the regular fibres of W by π. Its fixed point set is the union of the fibres of even multiplicity, and therefore L_τ is a union of Seifert fibres. The reader will verify that if a fibre of W has multiplicity k, then its image in $L(\bar{p}, \bar{q})$ has multiplicity $\bar{k} = \frac{k}{\gcd(k,2)}$. Hence as $L(\bar{p}, \bar{q})$ has at most two exceptional fibres, $a = 2$.

Suppose next that $\bar{\tau}$ fixes precisely one cone point of $S^2(a, b, c)$. In this case its fixed point set consists of this cone point and a regular point. Thus the fixed point set of τ is contained in a union of two fibres, so L_τ has at most two components. The reader will verify that each exceptional fibre of W is sent to an exceptional fibre of $L(\bar{p}, \bar{q})$, two of them to the same fibre. Thus the τ-invariant regular fibre of W is sent to a regular fibre of $L(\bar{p}, \bar{q})$. It follows that this fibre lies in the fixed point set of τ, and therefore L_τ contains a regular fibre of $L(\bar{p}, \bar{q})$. □

Lemma 4.4. *Suppose that W is not a lens space and that τ reverses the orientations of the Seifert fibres of W. If $W/\tau = L(\bar{p}, \bar{q}) \not\cong S^3$, then*

(1) *W has base orbifold $S^2(\bar{p}, \bar{p}, m)$, where $m \geq 2$ and the Seifert invariants of the exceptional fibres of order $\bar{p}$ are the same. Hence if W is not a prism manifold, $\bar{p} \neq 2$.*

(2) *There is an integer n coprime with m such that L_τ is isotopic to the closure $K(m/n)$ of an m/n rational tangle in a Heegaard solid torus of W/τ as depicted in Figure 1. In particular,*

$$|L_\tau| = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

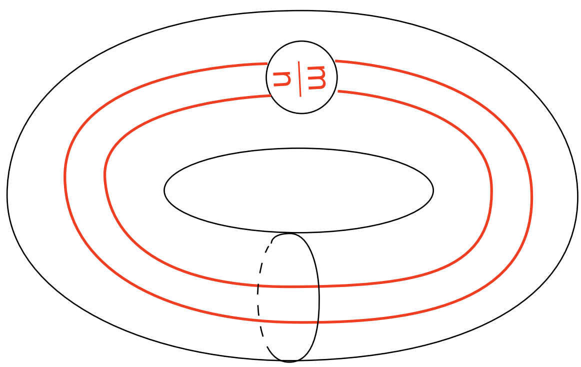

FIGURE 1

Proof. The fixed point set of $\bar{\tau}$ is non-empty, so as it reverses orientation, it is a reflection in an equator of $S^2(a, b, c)$. This equator cannot contain all three cone points, as otherwise τ would be the Montesinos involution on W and therefore $L(\bar{p}, \bar{q})$ would be S^3. Thus it contains exactly one cone point and $\bar{\tau}$ permutes the other two. It follows that up to relabeling, $(a, b, c) = (r, r, m)$ for some integers $r, m \geq 2$. Further, $S^2(r, r, m)/\bar{\tau} = D^2(r; m)$, where $D^2(r; m)$ is the 2-orbifold with underlying space a 2-disk and singular set consisting of a cone point of order r, a corner-reflector point x of order m, and a reflection line $\partial D^2 \setminus \{x\}$. Therefore $L(\bar{p},\bar{q}) = W/\tau \cong L(r,t)$ for some integer t. Thus $r = \bar{p}$, which proves part (1).

A Montesinos-type analysis of the quotient of the τ-invariant solid torus given by the inverse image in W of a small annular neighbourhood of $\mathrm{Fix}(\bar{\tau})$ in $S^2(\bar{p},\bar{p},m)$ shows that the branch set of this quotient is of the form described in part (2). It is well known that this branch set has one component if n is odd and two otherwise, so part (2) holds. □

5. Beginning of the proof of Theorem 1.3

5.1. **Assumptions.** We assume throughout the rest of the paper that M is a hyperbolic knot manifold containing an essential once-punctured torus F of boundary slope β which caps off to an essential torus in $M(\beta)$ (cf. Corollary 3.2) and that $M(\alpha)$ is an atoroidal, irreducible, small Seifert manifold (cf. Corollary 3.4). We assume as well that $\Delta(\alpha,\beta) > 3$, and (therefore) M_F is a genus 2 handlebody by Proposition 3.3.

We will show that under these assumptions, $\Delta(\alpha,\beta) \le 5$, F is not a fibre, $\pi_1(M(\alpha))$ is finite non-cyclic, and

(a) if $\Delta(\alpha,\beta) = 4$, $(M;\alpha,\beta) \cong (Wh(\frac{-2n\pm 1}{n});-4,0)$ for some integer n with $|n| > 1$ and $M(\alpha)$ has base orbifold $S^2(2,2,|\mp 2n-1|)$;

(b) if $\Delta(\alpha,\beta) = 5$, then $(M;\alpha,\beta) \cong (Wh(-3/2);-5,0)$ and $M(\alpha)$ has base orbifold $S^2(2,3,3)$.

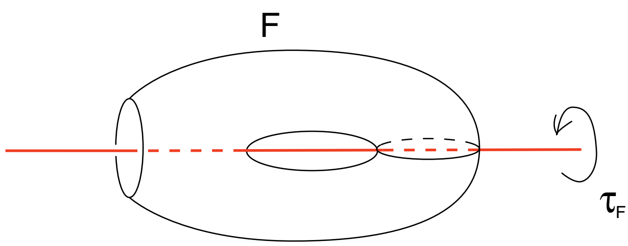

Figure 2

5.2. **An involution on** M**.** There is an involution τ_F on F with exactly three fixed points whose action on ∂F is rotation by π. See Figure 2. Thus F/τ_F is the 2-orbifold $D^2(2,2,2)$. Let $N \cong F \times I$ be a small neighbourhood of F in M and extend τ_F to an involution τ_N in the obvious way. Then $\tau_N|F \times \partial I$ extends to a hyperelliptic involution of ∂M_F. Since M_F is a genus 2 handlebody, the latter extends to an involution τ_{M_F} of M_F. Piecing together τ_N and τ_{M_F} we obtain an orientation-preserving involution $\tau : M \to M$ with non-empty 1-dimensional fixed point set $\widetilde{L} \subset \mathrm{int}(M)$. Further, $V := M/\tau$ is a solid torus containing the branch set L of τ. By construction, this is a hyperbolic link which intersects some meridional disk of V transversely and in three points. When F is a fibre in M, L is braided in V.

Note that L cannot intersect any meridional disk in one point, as M is ∂-irreducible.

The slopes on ∂M can be identified with $\pm$-classes of primitive elements of $H_1(\partial M)$. In particular we assume $\alpha,\beta \in H_1(\partial M)$. Let μ be any dual slope to β.

This means that $1 = \Delta(\mu, \beta) = |\mu \cdot \beta|$. Hence $\{\mu, \beta\}$ form a basis for $H_1(\partial M)$. Write

$$\alpha = p\mu + q\beta, \tag{5.2.1}$$

where p, q are coprime. After possibly changing the signs of μ and β we may assume that

$$p = \Delta(\alpha, \beta). \tag{5.2.2}$$

Without loss of generality we may suppose that $p \geq 1$. The map $M \to V$ is a double cover when restricted to ∂M. It sends β to a slope $\bar{\beta}$, a meridian of V, and sends μ to $\bar{\mu}$, a longitude of V.

For each slope γ on ∂M, τ extends to an involution $\tau_\gamma : M(\gamma) \to M(\gamma)$. Moreover, if $\widetilde{U}_\gamma$ denotes the filling torus in $M(\gamma)$ and $\widetilde{K}_\gamma$ its core, then

$$\mathrm{Fix}(\tau_\gamma) = \begin{cases} \widetilde{L} & \text{if } \Delta(\gamma, \beta) \text{ is odd,} \\ \widetilde{L} \cup \widetilde{K}_\gamma & \text{if } \Delta(\gamma, \beta) \text{ is even.} \end{cases} \tag{5.2.3}$$

It is clear that $\widetilde{U}_\gamma / \tau_\gamma$ is a solid torus U_γ. Denote its core $\widetilde{K}_\gamma / \tau_\gamma$ by K_γ. Thus $M(\gamma)/\tau_\gamma = V \cup_{\bar{\gamma}} U_\gamma$ is a lens space. Indeed, if $\gamma = r\mu + s\beta$, then under the double cover $\partial M \to \partial V$ we have $\gamma \mapsto r\bar{\mu} + 2s\bar{\beta}$. Let $\bar{\gamma} = \frac{1}{\gcd(2,r)}(r\bar{\mu} + 2s\bar{\beta})$ denote the associated slope and L_γ the branch set in $M(\gamma)/\tau_\gamma$. Then

$$(M(\gamma)/\tau_\gamma, L_\gamma) = (V(\bar{\gamma}), L_\gamma) \cong \begin{cases} (L(r, 2s), L) & \text{if } r \text{ is odd,} \\ (L(\frac{r}{2}, s), L \cup K_\gamma) & \text{if } r \text{ is even.} \end{cases}$$

We are interested in the case $\gamma = \alpha$. Set

$$\bar{p} = p/\gcd(p, 2) \quad \text{and} \quad \bar{q} = 2q/\gcd(p, 2) \tag{5.2.4}$$

so that $\bar{\alpha} = \bar{p}\bar{\mu} + \bar{q}\bar{\beta}$ and

$$M(\alpha)/\tau_\alpha \cong L(\bar{p}, \bar{q}).$$

From 5.2.3 we see that

$$|L_\alpha| = \begin{cases} |L| & \text{if } p \text{ is odd,} \\ |L| + 1 & \text{if } p \text{ is even.} \end{cases} \tag{5.2.5}$$

Fix a τ_α-invariant Seifert structure on $M(\alpha)$ with base orbifold $S^2(a, b, c)$ where $1 \leq a \leq b \leq c$ (Lemma 4.1).

Let $\bar{\tau}_\alpha$ be the involution of $S^2(a, b, c)$ (possibly the identity) induced by τ_α.

Lemma 5.1. *Suppose that Assumptions* 5.1 *hold. Suppose as well that $M(\alpha)$ is not a lens space and that τ_α preserves the orientations of the Seifert fibres of $M(\alpha)$. Then there is a Seifert structure on $L(\bar{p}, \bar{q})$ in which L_α is a union of at most three fibres, at least one of which is regular. Further, $L_\alpha = L$ so that $p = \Delta(\alpha, \beta)$ is odd.*

Proof. Lemma 4.3 shows that L is a union of fibres in the induced Seifert structure on $L(\bar{p}, \bar{q})$ and that at least one of these fibres is regular. This implies that $K_\alpha \not\subset L_\alpha$, as otherwise $L = L_\alpha \setminus K_\alpha$ would not be a hyperbolic link in V. Thus $L = L_\alpha$, so p is odd by (5.2.5). ☐

Lemma 5.2. *Suppose that Assumptions* 5.1 *hold. Suppose as well that $M(\alpha)$ is not a lens space and that τ_α reverses the orientations of the Seifert fibres of $M(\alpha)$.*

Then

(1) *$M(\alpha)$ has base orbifold $S^2(\bar{p},\bar{p},m)$, where $m \geq 2$ and the Seifert invariants of the exceptional fibres of order $\bar{p}$ are the same. Hence if $M(\alpha)$ is not a prism manifold, $\Delta(\alpha,\beta) \neq 4$.*

(2) *There is an integer n coprime with m such that L_α is isotopic to the closure $K(m/n)$ of an m/n rational tangle in a Heegaard solid torus of $M(\alpha)/\tau_\alpha$ as depicted in Figure 1. In particular,*

$$|L_\alpha| = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

(3) *$|L| = 1$, m is odd, and $n \equiv p \pmod 2$.*

Proof. Parts (1) and (2) follow from Lemma 4.4.

In order to prove part (3), suppose that $|L| = 2$. Then part (2) shows that $L = L_\alpha$. In particular, p is odd (5.2.5). Consideration of the form of L_α (cf. Figure 1) shows that its two components are isotopic to one another. But since L is transverse to a meridian disk of V and intersects it in three points, the generator γ of $H_1(V(\bar{\alpha})) \cong \mathbb{Z}/\bar{p}$ carried by the core of V satisfies $\gamma = \pm 2\gamma$. Hence $\bar{p} = 3$. But p is odd so $\Delta(\alpha,\beta) = p = \bar{p} = 3$, contrary to our hypotheses. Thus $|L| = 1$.

Next suppose that m is even. Then $L_\alpha = K(m/n)$ is connected, so $L = L_\alpha$ and p is odd, and L is homotopically trivial in $L(\bar{p},\bar{q})$. But L intersects a meridian disk of the Heegaard torus $V \subset L(\bar{p},\bar{q})$ transversely and in three points, so the only way it can be null homotopic is for $3 = \bar{p}$. Since p is odd, $p = 3$, which contradicts our hypotheses. Thus m is odd.

By (2), $|L_\alpha| \equiv n \pmod 2$. Since $|L| = 1$ by (3), Identity (5.2.5) shows that $|L_\alpha| \equiv p \pmod 2$. □

5.3. **Constraints on the branch set** L**.** Here we deduce strong constraints on the form of the branch set L in V.

Lemma 5.3. *Suppose that Assumptions 5.1 hold and that τ_α reverses the orientation of the Seifert fibres of $M(\alpha)$. Let $k \geq 1$ be an integer dividing $\bar{p}$ and consider the k-fold cyclic cover $S^2(\frac{\bar{p}}{k},\frac{\bar{p}}{k},m,m,\ldots,m) \to S^2(\bar{p},\bar{p},m)$ obtained by the k-fold unwrapping of $S^2(\bar{p},\bar{p},m)$ about the two cone points labeled $\bar{p}$. Let $\widetilde{M(\alpha)}_k \to M(\alpha)$ be the associated k-fold cyclic cover where $\widetilde{M(\alpha)}_k$ is Seifert with base orbifold $S^2(\frac{\bar{p}}{k},\frac{\bar{p}}{k},m,m,\ldots,m)$ and the inclusion of a regular fibre of $M(\alpha)$ lifts to $\widetilde{M(\alpha)}_k$. Define $\widetilde{M}_k \to M$ to be the cover obtained by restricting $\widetilde{M(\alpha)}_k \to M(\alpha)$ to M. Then*

(1) *$\partial\widetilde{M}_k$ is connected and F lifts to $\widetilde{M}_k$. In particular, β lifts to a slope $\widetilde{\beta}$ on $\partial\widetilde{M}_k$.*

(2) *α lifts to a slope $\widetilde{\alpha}$ on $\partial\widetilde{M}_k$ such that $\widetilde{M(\alpha)}_k = \widetilde{M}_k(\widetilde{\alpha})$. Further, $\Delta(\widetilde{\alpha},\widetilde{\beta}) = \frac{p}{k}$.*

(3) *$\widetilde{\alpha}$ is the singular slope of a closed essential surface in $\widetilde{M}_k$ if $S^2(\frac{\bar{p}}{k},\frac{\bar{p}}{k},m,m,\ldots,m)$ is hyperbolic with at least four cone points. If this is the case, $p/k \leq 3$.*

Proof. The cover $S^2(\frac{\bar{p}}{k},\frac{\bar{p}}{k},m,m,\ldots,m) \to S^2(\bar{p},\bar{p},m)$ is determined by the homomorphism $\varphi : H_1(S^2(\bar{p},\bar{p},m)) = \langle x,y : \bar{p}x = \bar{p}y = m(x+y) = 0\rangle \to \mathbb{Z}/k$, where $\varphi(x) \equiv -\varphi(y) \equiv 1 \pmod k$.

First note that the homomorphism $H_1(M(\alpha)) \to H_1(V(\bar{\alpha})) \cong \mathbb{Z}/\bar{p}$ kills any class carried by a regular Seifert fibre of $M(\alpha)$ (i.e. there are regular fibres with image an interval). Thus it factors through a homomorphism $\psi : H_1(S^2(\bar{p},\bar{p},m)) \to H_1(V(\bar{\alpha}))$. Since τ_α preserves the fibre of multiplicity m in $M(\alpha)$ but reverses its orientation, $(\bar{\tau}_\alpha)_*(x+y) = -(x+y)$. Thus $2(x+y)$ is sent to zero in $H_1(V(\bar{\alpha}))$, while x is sent to a generator. Since m is odd and $m(x+y) = 0$, $x+y \mapsto 0 \in H_1(V(\bar{\alpha}))$. It follows that φ factors as $H_1(S^2(\bar{p},\bar{p},m)) \xrightarrow{\psi} H_1(V(\bar{\alpha})) \xrightarrow{\cong} \mathbb{Z}/\bar{p} \to \mathbb{Z}/k$. Since $H_1(F)$ lies in the kernel of $H_1(M) \to H_1(V)$ while μ is sent to a generator of $H_1(V)$, we conclude that $\partial\widetilde{M}_k$ is connected and F lifts to $\widetilde{M}_k$. This proves (1).

For (2), note that by construction, there is a basis $\{\widetilde{\mu},\widetilde{\beta}\}$ of $H_1(\partial\widetilde{M}_k)$ where $\widetilde{\mu}$ is sent to $k\mu$ and $\widetilde{\beta}$ is sent to β in $H_1(\partial M)$. Then $\alpha = p\mu + q\beta$ lifts to $(\frac{p}{k})\widetilde{\mu} + q\widetilde{\beta}$. Clearly $\Delta(\widetilde{\alpha},\widetilde{\beta}) = \frac{p}{k}$.

Part (3) is a consequence of [BGZ1, Theorems 1.5 and 1.7]. ☐

Lemma 5.4. *Suppose that Assumptions* 5.1 *hold. Then M is not a once-punctured torus bundle. In particular, Theorem* 1.3 *holds when F is a fibre.*

Proof. We assume that M is a once-punctured torus bundle in order to obtain a contradiction.

There is a 3-braid σ whose closure in V is L. Altering σ by conjugation in $B_3 = \langle \sigma_1,\sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle$ leaves its closure invariant. (Here σ_1,σ_2 are the standard generators of B_3.) There is an isomorphism $B_3 \cong \langle a,b : a^3 = b^2\rangle$ where $a = \sigma_1\sigma_2$ and $b = \sigma_1\sigma_2\sigma_1$. The center of B_3 is generated by a^3 with $B_3/\langle a^3\rangle \cong \mathbb{Z}/2 * \mathbb{Z}/3$. We will use $\bar{\sigma}$ to denote the image of a braid σ in $B_3/\langle a^3\rangle$. Thus $\bar{a}$ has order 3 and $\bar{b}$ has order 2. In particular,

$$\bar{\sigma}_1 = \bar{a}^{-1}\bar{b},$$
$$\bar{\sigma}_2 = \bar{b}\bar{a}^2.$$

The inverse image $\widehat{L}$ of $L \subset V \subset L(\bar{p},\bar{q})$ under the universal cover $S^3 \to L(\bar{p},\bar{q})$ is the closure of the braid $\sigma^{\bar{p}}a^{-3\bar{q}}$. ☐

Claim 5.5. $\widehat{L}$ is not the trivial knot.

Proof of Claim 5.5. If $\widehat{L}$ is trivial then $\sigma^{\bar{p}}a^{-3\bar{q}}$ is conjugate to $\sigma_1\sigma_2, \sigma_1^{-1}\sigma_2^{-1}$, or $\sigma_1\sigma_2^{-1}$ ([BiMe, Classification Theorem, page 27]). The first two cases can be ruled out since they would imply that the exterior of $\widehat{L}$ in the inverse image of V in S^3 is not hyperbolic. On the other hand, in the third case we have $\bar{\sigma}^{\bar{p}} = \bar{\sigma}_1\bar{\sigma}_2^{-1} = \bar{a}^2\bar{b}\bar{a}\bar{b} \in B_3/\langle a^3\rangle \cong \mathbb{Z}/2 * \mathbb{Z}/3$. But this is impossible since $\bar{a}^2\bar{b}\bar{a}\bar{b}$ is not a proper power. ☐

Claim 5.6. τ_α preserves the orientation of the Seifert fibres of $M(\alpha)$. In particular, $\widehat{L}$ is a union of fibres in some Seifert structure on S^3 and p is odd.

Proof of Claim 5.6. Suppose otherwise and consider the $\bar{p}$-fold cyclic cover $\widetilde{M}_{\bar{p}} \to M$ constructed in Lemma 5.3. The base orbifold $S^2(m,m,\dots,m)$ of $\widetilde{M(\alpha)}$ has $\bar{p}$ cone points, each of order $m \geq 3$ by Lemma 5.2(3). If $\bar{p} \geq 4$, Lemma 5.3(3) implies that $\widetilde{M}_{\bar{p}}$ contains a closed essential surface, contrary to [CJR] or [FH]. Hence $\bar{p}$ is 2 or 3, and therefore as $p > 3$, p is 4 or 6. Identity (5.2.5) then combines with parts (2) and (3) of Lemma 5.2 to show that $|L_\alpha| = 2$ and m is odd. It follows that each component of L_α is isotopic to the core of a Heegaard solid torus in $L(\bar{p},\bar{q})$ (cf. Figure 1). In particular this is true of $L = L_\alpha \setminus K_\alpha$. It follows that $\widehat{L}$ is a trivial knot, contrary to the conclusion of Claim 5.5. Thus τ_α preserves the orientation of the Seifert fibres of $M(\alpha)$. The remaining conclusions are a consequence of Lemma 5.1. □

Claim 5.6 implies that $\bar{p} = p$ and $\bar{q} = 2q$.

Since L is a hyperbolic link in V, $\widehat{L}$ is a hyperbolic link in the inverse image of V in S^3. Thus the Schreier normal form for $\sigma^p a^{-6q}$ is generic (cf. [FKP, Theorem 5.2]). On the other hand, by Claim 5.6, $\widehat{L}$ is not a hyperbolic link in S^3, so [FKP, Theorem 5.5] implies that $\sigma^p a^{-6q}$ is conjugate in B_3 to a braid of the form $\sigma_1^c \sigma_2^d$ where $c, d \in \mathbb{Z} \setminus \{0\}$. We must have $\min\{|c|, |d|\} = 1$, as otherwise $\widehat{L}$ would be a connected sum of non-trivial torus links, contrary to the conclusion of Claim 5.6. Thus $\sigma^p a^{-6q}$ is conjugate to $\sigma_1^c \sigma_2^\epsilon$ for some $\epsilon \in \{\pm 1\}$ and non-zero c. The following claim completes the proof of Lemma 5.4.

Claim 5.7. If $p > 3$, $\sigma^p a^{-6q}$ is not conjugate to $\sigma_1^c \sigma_2^\epsilon$ for any $\epsilon \in \{\pm 1\}$.

Proof of Claim 5.7. Suppose that $\sigma^p a^{-6q}$ is conjugate to $\sigma_1^c \sigma_2^\epsilon$ for some $\epsilon \in \{\pm 1\}$. Projecting into $B_3/\langle a^3 \rangle$ shows that $\bar{\sigma}_1^c \bar{\sigma}_2^\epsilon$ is a p^{th}-power in that group. The latter condition is invariant under conjugation and taking inverse, so without loss of generality we can suppose that $\epsilon = 1$. Now

$$\bar{\sigma}_1^c \bar{\sigma}_2 = (\bar{a}^{-1}\bar{b})^c (\bar{b}\bar{a}^{-1}) = \begin{cases} (\bar{b}\bar{a})^{|c|}(\bar{b}\bar{a}^{-1}) & \text{if } c \leq 0, \\ \bar{a} & \text{if } c = 1, \\ \bar{a}^{-1}\bar{b}\bar{a} & \text{if } c = 2, \\ (\bar{a}^{-1}\bar{b})\bar{a}^{-1}(\bar{a}^{-1}\bar{b})^{-1} & \text{if } c = 3, \\ (\bar{a}^{-1}\bar{b}\bar{a})(\bar{a}\bar{b})(\bar{a}^{-1}\bar{b})^{c-4}(\bar{a}^{-1}\bar{b}\bar{a})^{-1} & \text{if } c > 3. \end{cases}$$

Consideration of the normal form for elements of $\mathbb{Z}/2 * \mathbb{Z}/3$ shows that the only values of c which give proper powers in $B_3/\langle a^3 \rangle$ are $c = 1, 2$, or 3.

Say $c = 1$ or 3. Then up to conjugation, $\bar{\sigma}^p = \bar{a}^{\pm 1}$, and therefore $\bar{\sigma} = \bar{a}^{\pm 1}$. Hence $\sigma = a^{3k \pm 1}$ for some integer k. But then it is easy to see that L is boundary-parallel in V, contrary to the fact that $V \setminus L$ is hyperbolic.

Next suppose that $c = 2$. Then $\bar{\sigma}^p = \bar{b}$ up to conjugation, and therefore the same is true of $\bar{\sigma}$. As $a^3 = b^2$, $\sigma = b^{2n+1}$ for some integer n. Then $L \subset \text{int}(V)$ has two components. One is a core curve K_0 of V, while the other is isotopic in $V \setminus K_0$ into ∂V. It follows that there is an essential annulus properly embedded in the exterior of L in $\text{int}(V)$. But this contradicts the fact that L is a hyperbolic link in V.

□ (of Lemma 5.4)

Recall that t_1^+ is the number of tight components of $\breve{\Phi}_1^+$ (cf. [BGZ2, §6]).

Lemma 5.8. *Suppose that Assumptions* 5.1 *hold. Then* $t_1^+ = 0$. *In particular,* $M(\beta)_{\widehat{F}}$ *is Seifert with base orbifold of the form* $A(a)$, *where* A *is an annulus and* $a \geq 2$.

Proof. Lemma 5.4 implies that F is not a fibre, and so Proposition 3.5 and Remark 3.6 show that the lemma holds as long as either $M(\alpha)$ is very small or $\Delta(\alpha, \beta) > 4$. Assume then that $M(\alpha)$ is not very small and that $\Delta(\alpha, \beta) = 4$. The latter equality combines with Lemma 5.1 to show that τ_α reverses the orientations of the fibres of $M(\alpha)$. But then Lemma 5.2(1) implies that $M(\alpha)$ is a prism manifold, contradicting our assumption that $M(\alpha)$ is not very small. Thus the lemma holds. □

Lemma 5.9. *Suppose that Assumptions* 5.1 *hold. Then there are coprime integers* $a \geq 2$ *and* b *as well as a* 3*-braid* σ *such that* L *is isotopic to the link depicted in Figure* 3*.*

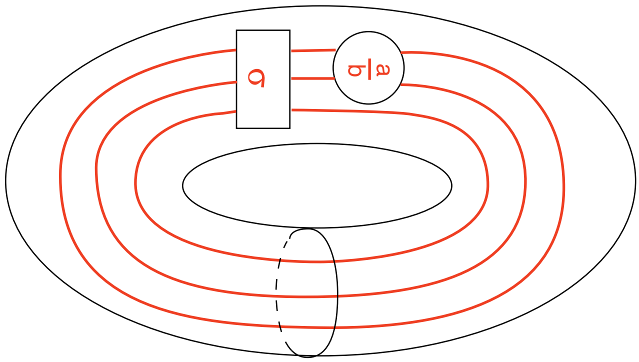

FIGURE 3

Proof. By Lemma 5.8, $M(\beta)_{\widehat{F}}$ is Seifert with base orbifold of the form $A(a)$, where A is an annulus and $a \geq 2$. Consider the involution $\widehat{\tau} : M(\beta)_{\widehat{F}} \to M(\beta)_{\widehat{F}}$ induced by τ_β. Note that $M(\beta)_{\widehat{F}}/\widehat{\tau} = V(\bar{\beta})_{\widehat{F}/\widehat{\tau}} \cong (S^2 \times S^1)_{S^2 \times \{x\}} \cong S^2 \times I$. Now $M(\beta)_{\widehat{F}}$ has a unique Seifert structure which we can suppose is $\widehat{\tau}$-invariant. Let $\overline{\widehat{\tau}}$ be the induced involution on $A(a)$. Note that $\overline{\widehat{\tau}}$ cannot preserve orientation, as otherwise $M(\beta)_{\widehat{F}}/\widehat{\tau} \cong S^2 \times I$ would admit a Seifert structure. Thus it reverses orientation, and since it fixes the cone point and leaves each boundary component invariant, it must be reflection along a pair of disjoint properly embedded arcs, each of which runs from one boundary component to the other. The quotient $A(a)/\overline{\widehat{\tau}}$ is a disk whose boundary contains two disjoint, compact arcs, each a reflector arc, one of which contains the $\mathbb{Z}/a$ cone point. It follows that the branch set in $M(\beta)_{\widehat{F}}/\widehat{\tau} \cong S^2 \times I$ consists of a 2-braid and an $\frac{a}{b}$-rational tangle running from one end to the other which are separated by a properly embedded vertical annulus. See Figure 4.

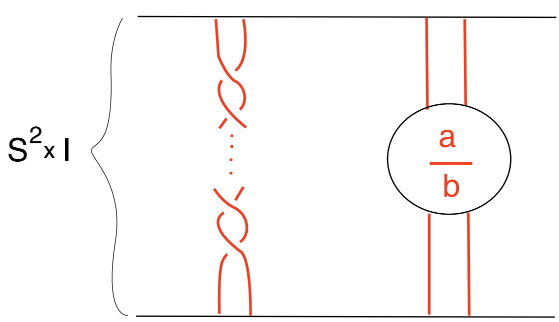

FIGURE 4

We claim that $K_\beta \cap M(\beta)_{\widehat{F}}$ is a component of the 2-braid. To see this, first note that by Lemma 5.8, $\breve{\Phi}_1^+$ has no tight components. Next we refer the reader to the final paragraph of the proof of [BGZ2, Lemma 7.9]. It is shown there that

$M_F = X^+$ is obtained by attaching a solid torus V to the product of an interval I and a once-punctured annulus A_*, where $V \cap (A_* \times I)$ is a pair of annuli which have winding number a in V and components of $\partial A_* \times I$ in $A_* \times I$. This decomposition is invariant under the restriction of $\hat{\tau}$ to M_F, and it is easy to see that the quotient of V contains the $\frac{a}{b}$-rational tangle. Since $(\partial M)_{\partial F} \subset A_* \times I$ is disjoint from V, it follows that $K_\beta \cap M(\beta)_{\widehat{F}}$ is a component of the 2-braid. Thus $L \cap M_F/\tau$ is as depicted in Figure 5, where δ is a 3-braid. It follows that there is a 3-braid σ such that L is as depicted in Figure 3. □

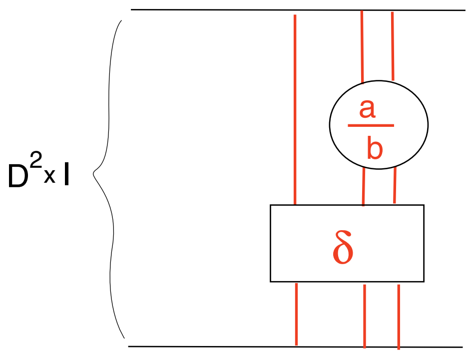

FIGURE 5

5.4. **The lens space case.** The methods of this paper can be used to give a new proof of Ken Baker's theorem: *if M contains a once-punctured essential genus* 1 *surface of boundary slope β and $M(\alpha)$ is a lens space, then* $\Delta(\alpha, \beta) \leq 3$ [Ba]. We begin the proof here and complete it in §8.

Lemma 5.10. *Suppose that Assumptions* 5.1 *hold. If $\pi_1(M(\alpha))$ is cyclic, then $p = 5$, F is not a fibre, and L_α is either the core of a solid torus of a genus one Heegaard splitting of $L(5, 2q)$ or the boundary of a Möbius band spine of a Heegaard solid torus of $L(5, 2q)$.*

Proof. We know that F is not a fibre (Lemma 5.4), so $p = \Delta(\alpha, \beta) \leq 6$ by Proposition 3.5. As $\Delta(\alpha, \beta) = p \geq 4$, $M(\alpha)/\tau_\alpha \cong L(\bar{p}, \bar{q})$ is not S^3. Hence by Lemma 4.2, L_α is a union of Seifert fibres of some Seifert fibring of $L(\bar{p}, \bar{q})$. Since L is hyperbolic in V, K_α cannot be contained in L_α. Thus p is odd by (5.2.3), so $p = \bar{p} = 5$, $\bar{q} = 2q$, and $L = L_\alpha$. Lemma 4.2(1) then shows that L_α is either the core of a solid torus of a genus one Heegaard splitting of $L(5, 2q)$ or the boundary of a Möbius band spine of a Heegaard solid torus of $L(5, 2q)$. □

Remark 5.11. We can complete the proof of Baker's result mentioned above at this point by invoking a theorem of Sangyop Lee [L3] which states that the distance between a toroidal filling slope and a lens space filling slope is at most 4. Nevertheless, we give an independent proof that $\Delta(\alpha, \beta) \neq 5$ (and so $\Delta(\alpha, \beta) \leq 3$) in §8 below.

5.5. **Reduction of the proof of Theorem 1.3.** In this section we reduce the proof of Theorem 1.3 to several problems concerning links. These will be solved in the subsequent sections of the paper. We begin with a slight sharpening of our upper bound for $\Delta(\alpha, \beta)$.

Lemma 5.12. *If Assumptions* 5.1 *hold, then* $\Delta(\alpha, \beta) < 8$*.*

Proof. By Lemma 5.4, F is not a fibre in M. Hence $\Delta(\alpha, \beta) \leq 8$ by Proposition 3.5 (or [LM]). Suppose that $\Delta(\alpha, \beta) = 8$. Then $M(\alpha)$ is not very small by Proposition 3.5. Further, Proposition 3.3 implies that M_F is a genus two handlebody, so we can construct an involution τ as above. Then Lemma 5.1 implies that τ_α reverses the orientations of the Seifert fibres of $M(\alpha)$. Parts (1) and (3) of Lemma 5.2 imply that $M(\alpha)$ has a Seifert structure with base orbifold $S^2(4, 4, m)$, where $m \geq 3$ is odd. Let $\widetilde{M}_2 \to M$ be the 2-fold cover constructed in Lemma 5.3. By part (2) of that lemma, $\widetilde{M}_2(\widetilde{\alpha})$ is Seifert with base orbifold $S^2(4, 4, m, m)$. But then Lemma 5.3(3) implies $4 = \frac{8}{2} \leq 3$, which is false. Thus $\Delta(\alpha, \beta) \neq 8$. □

Lemma 5.13. *Suppose that Assumptions* 5.1 *hold and that* $\Delta(\alpha, \beta) = 4$*. Then* $M(\alpha)$ *is a prism manifold.*

Proof. Since $\Delta(\alpha, \beta)$ is even, $M(\alpha)$ is not a lens space (Lemma 5.10), and so Lemma 5.1 implies that τ_α reverses the orientations of the fibres of $M(\alpha)$. Lemma 5.2(1) now implies that $M(\alpha)$ is a prism manifold. □

Given the last two lemmas, to complete the proof of Theorem 1.3 under Assumptions 5.1, we must consider the possibility that $\Delta(\alpha, \beta) \in \{5, 6, 7\}$ besides the case when $\Delta(\alpha, \beta) = 4$ and $M(\alpha)$ is a prism manifold. We do this by comparing the constraints obtained above on the branch sets L and L_α:

- L lies in V as depicted in Figure 3 (Lemma 5.9);
- when $M(\alpha)$ is not a lens space and τ_α preserves the orientation of the Seifert fibres of $M(\alpha)$, then $\Delta(\alpha, \beta)$ is odd and L_α is the union of at most three fibres of some Seifert structure on $L(\bar{p}, \bar{q})$ (Lemma 5.1);
- when $M(\alpha)$ is not a lens space and τ_α reverses the orientation of the Seifert fibres of $M(\alpha)$, then L_α lies in some Heegaard solid torus of $L(\bar{p}, \bar{q})$ as depicted in Figure 1 (Lemma 4.4);
- when $M(\alpha)$ is a lens space, then $\Delta(\alpha, \beta) = 5$ and L_α is either the core of a Heegaard solid torus of $L(5, 2q)$ or the boundary of a Möbius band spine of a Heegaard solid torus of $L(5, 2q)$ (Lemma 5.10).

The proof of Theorem 1.3 therefore reduces to proving the following claims:

(1) If τ_α preserves the orientation of the Seifert fibres and $M(\alpha)$ is not a lens space, then $\Delta(\alpha, \beta) = 5$ and $(M; \alpha, \beta)$ is homeomorphic to $(Wh(-3/2); -5, 0)$.

(2) The links contained in the universal cover S^3 of $L(7, \bar{q})$ which are depicted in Figure 17 and Figure 18 are not equivalent when $\Delta(\alpha, \beta) = 7$, $|L| = 1$, m is odd, and $n \equiv 1 \pmod 2$.

(3) The link depicted in Figure 3 considered as lying in a Heegaard solid torus in $L(5, 2q)$ is not isotopic to either the core of a Heegaard solid torus or the boundary of a Möbius band spine of a Heegaard solid torus.

(4) The links contained in a Heegaard solid torus in $L(3, \bar{q})$ depicted in Figure 1 and Figure 3 are not equivalent.

(5) The links contained in the universal cover S^3 of $L(5, \bar{q})$ which are depicted in Figure 26 and Figure 27 are not equivalent when $\Delta(\alpha, \beta) = 5$, $|L| = 1$, m is odd, and $n \equiv 1 \pmod 2$.
(6) $\Delta(\alpha, \beta) = 4$ and $M(\alpha)$ is a prism manifold if and only if $(M; \alpha, \beta) \cong (Wh(\frac{-2n\pm1}{n}); -4, 0)$ for some integer n with $|n| > 1$.

These will be proved in §6, §7, §8, §9, §10 and §12, respectively.

6. The case where τ_α preserves the orientation of the Seifert fibres, $M(\alpha)$ is not a lens space, and $\Delta(\alpha, \beta) \in \{5, 7\}$

In this section we suppose that Assumptions 5.1 hold and show that if τ_α preserves the orientation of the Seifert fibres, $M(\alpha)$ is not a lens space, and $\Delta(\alpha, \beta) \in \{5, 7\}$, then $\Delta(\alpha, \beta) = 5$ and $(M; \alpha, \beta)$ is homeomorphic to $(Wh(-3/2); -5, 0)$.

By hypothesis, $M(\alpha)$ is small Seifert with exactly three singular fibres. It is not a prism manifold by [L2] and so has a unique Seifert structure. Recall that $M(\alpha)/\tau_\alpha = V(\bar{\alpha})$ is the lens space $L(\bar{p}, \bar{q}) = L(p, 2q)$ and the branch set of τ_α in $L(p, 2q)$ is a link denoted by L_α. As p is odd, $L_\alpha = L$ (cf. (5.2.5)).

Suppose that L_α is a Seifert link with respect to the induced Seifert fibration on $L(p, 2q) = M(\alpha)/\tau_\alpha$. We need to show that $p = 5$ and $(M; \alpha, \beta)$ is homeomorphic to $(Wh(-3/2); -5, 0)$.

By Lemma 5.1, at least one component of L is a regular fibre of $L(p, 2q)$. Let K be such a component and denote by X the exterior of L in $L(p, 2q)$. Then X has the induced Seifert fibration with $|\partial X| = |L|$ boundary components, each a torus. Let T_K be the component of ∂X corresponding to the knot K.

Lemma 6.1. *There is an essential separating vertical annulus $(A, \partial A) \subset (X, T_K)$ which cuts X into two components X_1 and X_2 such that each X_i is either a torus cross interval or a fibred solid torus whose core is a singular fibre of X of order larger than* 2.

Proof. The lemma follows from Lemma 4.3 and its proof. Let $\bar{\tau}_\alpha$ be the induced map on the orbifold $S^2(a, b, c)$ of $M(\alpha)$ where each of a, b, c is ≥ 2. Then $\bar{\tau}_\alpha$ is either the identity or an involution with two fixed points. Let $\sigma_1, \sigma_2, \sigma_3$ denote the singular fibres of $M(\alpha)$ and let their orders be a, b, c, respectively.

First assume that $\bar{\tau}_\alpha$ is the identity map. Then Lemma 4.3(1) implies that at least one of a, b, c, say a, is 2 and the fixed point set of τ_α in $M(\alpha)$ is the union of those σ_i with even orders. In particular, σ_1 belongs to the fixed point set of τ_α and its image in $L(p, 2q)$ is a regular fibre. Note that if σ_2, respectively σ_3, does not belong to the fixed point set of τ_α, then b, respectively c, is odd and the image of σ_2, respectively σ_3, in $L(p, 2q)$ is a fibre of $L(p, 2q)$ of order b, respectively c. Hence the sum of $|\partial X| = |L|$ and the number of the singular fibres of X equals 3. Since the surface underlying the base orbifold of X is planar, the lemma follows in this case.

Next assume that $\bar{\tau}_\alpha$ is an involution. Then two of the singular fibres of $M(\alpha)$, say σ_1 and σ_2, have the same order $a = b$. Both are mapped to a common singular fibre in $L(p, 2q)$ of order a. Since $M(\alpha)$ is not a prism manifold, $a = b > 2$.

By Lemma 4.3(2), the fixed point set of τ_α in $M(\alpha)$ consists of a regular fibre and possibly the remaining singular fibre σ_3. If σ_3 does not belong to $\mathrm{Fix}(\tau_\alpha)$, then its image in $L(p, 2q)$ is a singular fibre of order $2c \geq 4$ and therefore the sum of

$|\partial X| = |L|$ and the number of the singular fibres of X again equals 3. As in the previous case, the lemma follows from this. $\square$

Recall that K_α is the core circle of the filling solid torus in $V(\bar{\alpha}) = L(p, 2q)$. The exterior Y of K_α in X is also the exterior of L in V and so is hyperbolic. Let $T_V = \partial V \subset \partial Y$.

The solid torus V has a meridian disk D which intersects L in three points such that $P = D \cap Y$ is an essential thrice-punctured disk in Y. Let $d_V = \partial P \cap T_V$ and let c_1, c_2, c_3 be the three components of ∂P contained in $\partial Y \setminus T_V$. Note that d_V has the slope $\bar{\beta}$ in T_V, and each c_i is a meridian curve of some component of L.

Among all annuli satisfying the conditions of Lemma 6.1, we choose one, denoted A, which intersects T_V in the minimal number of components. Since Y is hyperbolic, $A \cap T_V$ is non-empty. The surface $Q = A \cap Y$ is essential in Y. Since A is separating in X, $\partial Q \cap T_V$ consists of an even number, say n, of simple essential loops in T_V of slope $\bar{\alpha}$. Let a_1, a_2 be the two components of ∂Q in T_K, and let $b_1, ..., b_n$ be the components of ∂Q in T_V numbered so that they occur successively around d_V. Each a_i is a Seifert fibre of X, and each b_j has slope $\bar{\alpha}$ on T_V. If c_j is a meridian curve of K, then the distance between c_j and a_i is 1 since K is a regular fibre of $L(p, 2q)$.

Now define the labeled intersection graphs Γ_P and Γ_Q as usual. We may consider d_V, c_1, c_2, c_3, a_1, a_2, $b_1, ..., b_n$ as the boundaries of the fat vertices of these graphs. Each b_i, $i = 1, ..., n$, has valency $p = \Delta(\bar{\alpha}, \bar{\beta}) = \Delta(\alpha, \beta)$, and the valency of d_V is np. Note that the valency of a_1 is equal to the valency of a_2 and is equal to the number of c_i's which are meridians of K. Further, the valency of c_i is either 2 or 0 depending on whether or not c_i is a meridian curve of K.

We call the edges in Γ_Q connecting some b_i to some b_j *B-edges*, and call the edges in Γ_P connecting d_V to itself *D-edges*. Similarly we define A-edges, C-edges, AB-edges, and CD-edges. Note that an arc in $P \cap Q$ is a B-edge in Γ_Q if and only if it is a D-edge in Γ_P, is an A-edge in Γ_Q if and only if it is a C-edge in Γ_P, and is an AB-edge in Γ_Q if and only if it is a CD-edge in Γ_P.

Every D-edge is positive, so by the parity rule, every B-edge is negative. By construction, no D-edge in Γ_P is boundary parallel in P. Thus there are at most three different D-edges in the reduced graph $\overline{\Gamma}_P$ (cf. Figure 6).

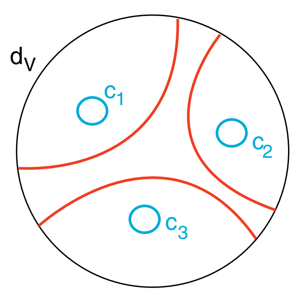

FIGURE 6. The maximal possible D-edges in $\overline{\Gamma}_P$

Lemma 6.2. *There can be no S-cycle in Γ_P consisting of D-edges.*

Proof. Suppose otherwise that $\{e_1, e_2\}$ is an S-cycle in Γ_P consisting of D-edges with label pair $\{j, j+1\}$. We may assume that the bigon face E between e_1 and e_2 lies on the X_1-side of A.

Let H be the portion of the filling solid torus of $L(p, 2q)$ lying in X_1 which contains $\hat{b}_j$ and $\hat{b}_{j+1}$. In Γ_Q, $e_1 \cup b_j \cup e_2 \cup b_{j+1}$ cannot be contained in a disk region D_* of A as otherwise a regular neighbourhood of $D_* \cup E \cup H$ in X_1 would be a punctured projective space. Thus $e_1 \cup b_j \cup e_2 \cup b_{j+1}$ contains a core circle of A (cf. Figure 7).

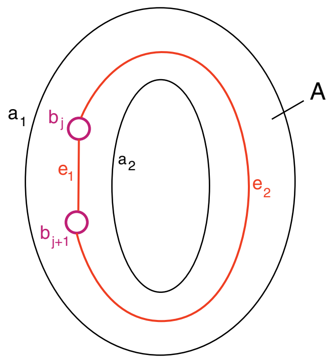

FIGURE 7. The corresponding cycle $\{e_1, e_2\}$ in Γ_Q

Let U be a regular neighbourhood of $E \cup H \cup A$ in X_1. Then U is a solid torus and the frontier of U in X_1 is an annulus $(A', \partial A') \subset (X, T_K)$ for which $\partial A'$ is parallel to ∂A in T_K and which intersects T_V in $n-2$ components. By construction, A' is inessential in X_1 and therefore X_1 cannot be a torus cross interval. It follows that X_1 is a fibred solid torus of X. Since A' has winding number 2 in the solid torus U, the singular fibre of X_1 has order 2, contrary to Lemma 6.1. Thus the lemma holds. □

Note that Γ_P has at most six CD-edges and thus Γ_P has at least $(np-6)/2$ D-edges, so there is a family of at least $(np-6)/6$ mutually parallel D-edges. By Lemma 6.2 we have $(np-6)/6 \le n/2$. Hence $n \le 6/(p-3)$, and therefore $p = 5$ and $n = 2$. If Γ_P has a C-edge, it would have only one family of parallel D-edges, and this family would have at least three edges, contrary to the fact that no two D-edges can be parallel in Γ_P by Lemma 6.2. Also, Γ_P has at least four CD-edges, as otherwise there would be four D-edges, two of which would form an S-cycle. Thus Γ_P has either six or four CD-edges.

We first consider the case when there are exactly four CD-edges. In this case we have three D-edges in Γ_P, no two of which can be parallel. Hence Γ_P may be assumed to be as illustrated in Figure 8, i.e. c_1 and c_2 are contained in T_K and c_3 is contained $\partial X \setminus T_K$. Thus $|L| = |\partial X| = 2$, and we may assume that X_1 is a solid torus and X_2 is a torus cross interval. In particular, c_3 is contained in X_2.

Consider the face f given in Figure 8. From this figure we see that f and c_3 are on the same side of A (since A is separating in X), and thus f is contained in X_2. Let T_* be the component of ∂X_2 containing A, and H that part of the filling solid torus of $L(p, 2q)$ contained in X_2. We use $\partial_0 H$ to denote $\partial H \cap T_V$. It is evident that the boundary ∂f of f is contained in $T_* \cup \partial_0 H$. Also note that $\partial f \cap T_*$

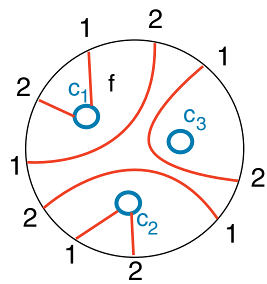

FIGURE 8. Γ_P when $\Delta(\alpha, \beta) = 5$, $n = 2$ and 4 CD-edges

cannot be contained in a disk in T_*, as otherwise X_2 would contain a projective space as a summand. Thus $\partial f \cap T_*$ is contained in an annulus A_* of T_*. A regular neighbourhood W of $H \cup f \cup T_*$ in X_2 is a Seifert fibred space whose base orbifold is an annulus with a cone point of order 2. Since X_2 is a torus cross interval, the frontier of W in X_2 is an incompressible torus in X_2. But this torus cannot be parallel to T_* in X_2, contradicting the fact that X_2 is a torus cross interval. Thus the case when there are exactly four CD-edges does not arise.

We now know that Γ_P must have six CD-edges. Hence there are exactly two D-edges in Γ_P and they are not parallel. It follows that Γ_P is as illustrated in Figure 9 (1) or (2). (Without loss of generality, we may assume that the labels around d_V are as shown in these figures and that the vertices c_1, c_2 and c_3 are numbered as given there.) Therefore $L = K$ and both X_1 and X_2 are solid tori.

We are going to show that part (1) of Figure 9 cannot arise and that in the case of part (2) of Figure 9 the dual graph Γ_Q may be assumed to be as shown in part (6) of Figure 10.

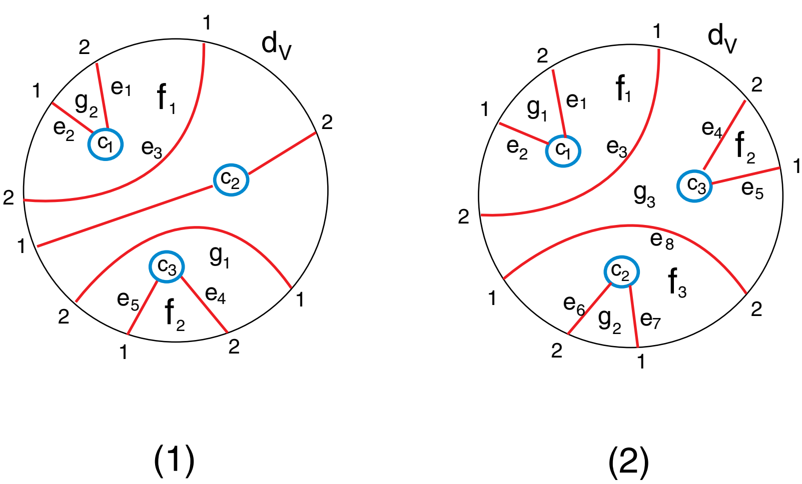

FIGURE 9. Γ_P when $p = 5$, $n = 2$ with 6 CD-edges

Lemma 6.3. *The graph Γ_P cannot be as shown in part (1) of Figure 9.*

Proof. Suppose otherwise that Γ_P is given by part (1) of Figure 9. Since A is a separating annulus, the faces f_1, f_2 of Γ_P lie on the same side of A, say in X_1, and the faces g_1, g_2 lie in X_2.

Let H be the part of the filling solid torus of $L(p, 2q)$ contained in X_1 and set $\partial_0 H = \partial H \cap T_V$. The boundary edges of f_1 consist of two CD-edges e_1, e_2 and one D-edge e_3. Without loss of generality, we may assume that the label of the edge e_1 at the vertex c_1 is 2. In Γ_Q, the boundary edges of f_1 may be assumed to be as illustrated in part (1) of Figure 10. Note that the boundary ∂f_1 of f_1, including the corners, lies in $\partial X_1 \cup \partial_0 H$. Further, $\partial f_1 \cap \partial X_1$ is contained in an annulus A_* of ∂X_1 whose slope has distance 1 from that of ∂A. Note as well that $\partial f_1 \cap (\partial X_1 \setminus A)$ is an essential arc in the annulus $(\partial X_1 \setminus A)$. A regular neighbourhood U of $H \cup f_1 \cup A_*$ in X_1 is a solid torus whose frontier in X_1 is an annulus $A_\#$ of winding number 2 in U. Thus $A_\#$ must be parallel to $\partial X_1 \setminus A_*$ through $X_1 \setminus U$. It follows that the fundamental group of X_1 is carried by U and thus has presentation

$$\langle x, t : x^2 t = 1 \rangle,$$

where we take a fat base point in A containing $b_1 \cup b_2 \cup (\partial f_1 \cap A) \cup (\text{all } AB\text{-edges})$, x is a based loop formed by a cocore arc of $\partial_0 H$, and t is a based loop formed by a cocore arc of $\partial X_1 \setminus A$.

Now consider the face f_2. We claim that the label of the edge e_4 at the vertex c_3 cannot be 2. Otherwise in Γ_Q, the boundary edges of f_2, e_4 and e_5 would be as depicted in part (2) or part (3) of Figure 10. In either case, the face f_2 would add the relation $xts = 1$ to the presentation for $\pi_1(X_1)$ above, where s is the element represented by a core circle of the annulus A. Thus the fundamental group of the solid torus X_1 would be generated by $s = x$. But s can be considered as a regular fibre of X. So the singular fibre of X_1 would have order one, which contradicts Lemma 6.1.

Thus the label of e_4 at c_3 is 1. It follows that in Γ_Q, the edges e_4 and e_5 are as shown in part (4) of Figure 10, and the face f_2 adds the relation $xt^{-1}s = 1$ to the presentation for $\pi_1(X_1)$, where s is the element represented by a core circle of the annulus A. Therefore $s = x^{-3}$. Since s can be considered as a regular fibre of X and x can be considered as a core circle of the solid torus X_1, the singular fibre in X_1 has order 3.

By the same argument, we see that the existence of the faces g_1 and g_2 in part (1) of Figure 9 implies that the singular fibre in X_2 has order 3. Hence the two singular fibres of X both have order 3, which implies that the order of the lens space $L(p, 2q)$ is divisible by 3. But the lens space has order $p = \Delta(\alpha, \beta) = 5$, yielding a contradiction. So part (1) of Figure 9 cannot arise. □

So Γ_P must be as shown in part (2) of Figure 9. Note that the faces f_1, f_2, f_3 lie on the same side of A, say in X_1, and the faces g_1, g_2, g_3 in X_2. Arguing similarly as in the proof of Lemma 6.3, we see that in the dual graph Γ_Q the edges e_1, e_2, e_3, e_4 and e_5 may be assumed to be as shown in part (4) of Figure 10.

We now consider the face g_3. Note that ∂g_3 must be contained in an annulus A' of ∂X_2 whose slope has distance 1 from that of ∂A and that $\partial g_3 \cap (\partial X_2 \setminus A)$ is an essential arc in the annulus $(\partial X_2 \setminus A)$. Thus e_8 is parallel to e_3 in Γ_Q. By combining this with the argument given in Lemma 6.3, we see that the graph Γ_Q must be as depicted in part (5) or part (6) of Figure 10.

Lemma 6.4. *Figure* 10*(5) is impossible.*

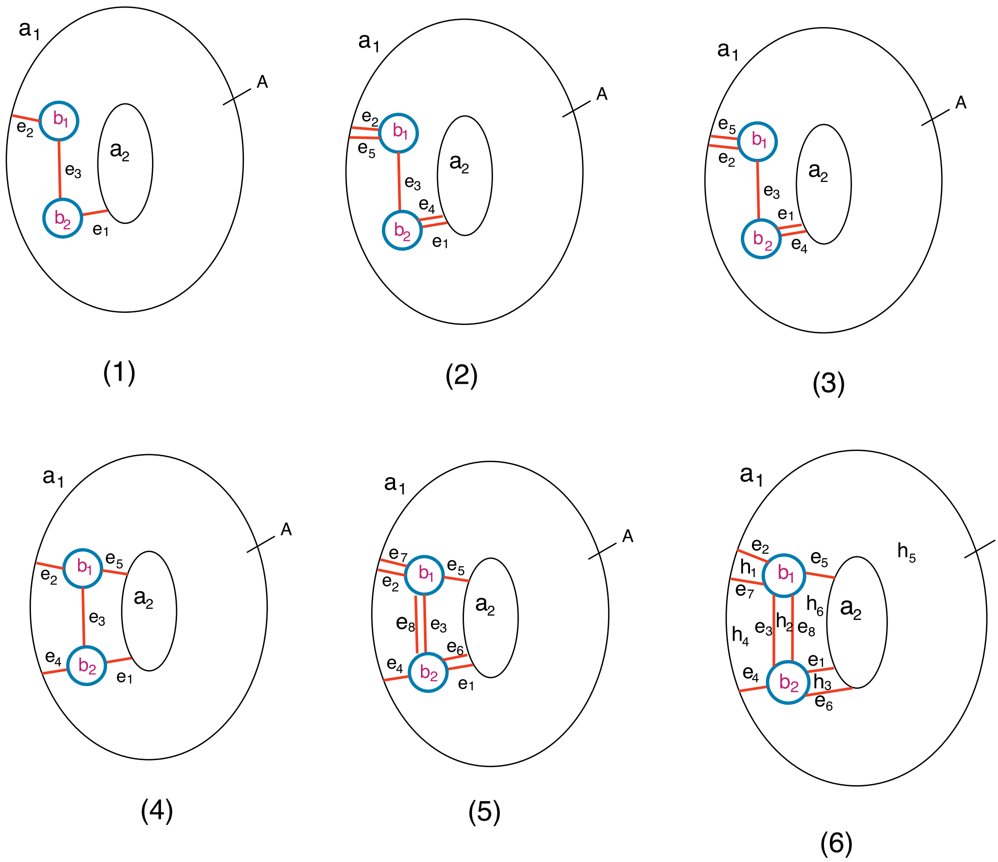

FIGURE 10. About the graph Γ_Q

Proof. In Figure 9(2), let p_0, p_1, p_2, p_3, p_4 be the points labeled 1 on d_V, in cyclic order around d_V. These are points of intersection of b_1 with d_V on the torus T_V. It follows that the corresponding points appear around b_1 in the order $p_0, p_d, p_{2d}, p_{3d}, p_{4d}$, for some d coprime to $\Delta = \Delta(\alpha, \beta) = 5$. The point p_i is the endpoint of an edge $e_{j(i)}$. Then, denoting p_i by the label $j(i)$ of the corresponding edge, the cyclic order of the p_i's around d_V in Figure 9(2) is 28753. In the graph Γ_Q in Figure 10(5), the order of the corresponding points is 82753. Since these cyclic orderings are not related in the manner described above, Γ_Q cannot be as illustrated in Figure 10(5). □

Remark 6.5. In Figure 10(6) the order is 27385, which is of the required form, with $d = 2$.

So far we have shown that $p = \Delta(\alpha, \beta) = 5$ and the graphs Γ_P and Γ_Q must be as shown in part (2) of Figure 9 and part (6) of Figure 10, respectively. In the rest of this section we are going to show that these conditions determine the triple (M, α, β) uniquely up to homomorphism, and thus it must be the triple $(Wh(-3/2); -5, 0)$.

The surface Q separates Y into Y_1 and Y_2, say, where $Y_i \subset X_i$, $i = 1, 2$. Let N be a regular neighbourhood of $T_V \cup T_K \cup P \cup Q$ in Y, and let $\partial_0 N = \partial N \setminus (T_V \cup T_K)$. Then $\partial_0 N = \partial_1 N \cup \partial_2 N_2$, where $\partial_i N \subset Y_i$, $i = 1, 2$.

Lemma 6.6. *For $i = 1$ and 2, $\partial_i N$ has two components, each a 2-sphere.*

Proof. By Remark 6.5, the curves d_V, b_1, b_2 on the torus T_V are as shown in Figure 11. They decompose T_V into rectangles $R_1, \ldots, R_5, S_1, \ldots, S_5$, where the R_i's lie in Y_1 and the S_i's in Y_2. In Figure 11 a point of intersection of $b_1 \cup b_2$ with d_V is labeled with the edge of which it is an endpoint. Similarly, the curves a_1, a_2, c_1, c_2, c_3 decompose the torus T_K into rectangles $T_1, T_2, T_3, U_1, U_2, U_3$, where the T_j's lie in Y_1 and the U_j's in Y_2. See Figure 12.

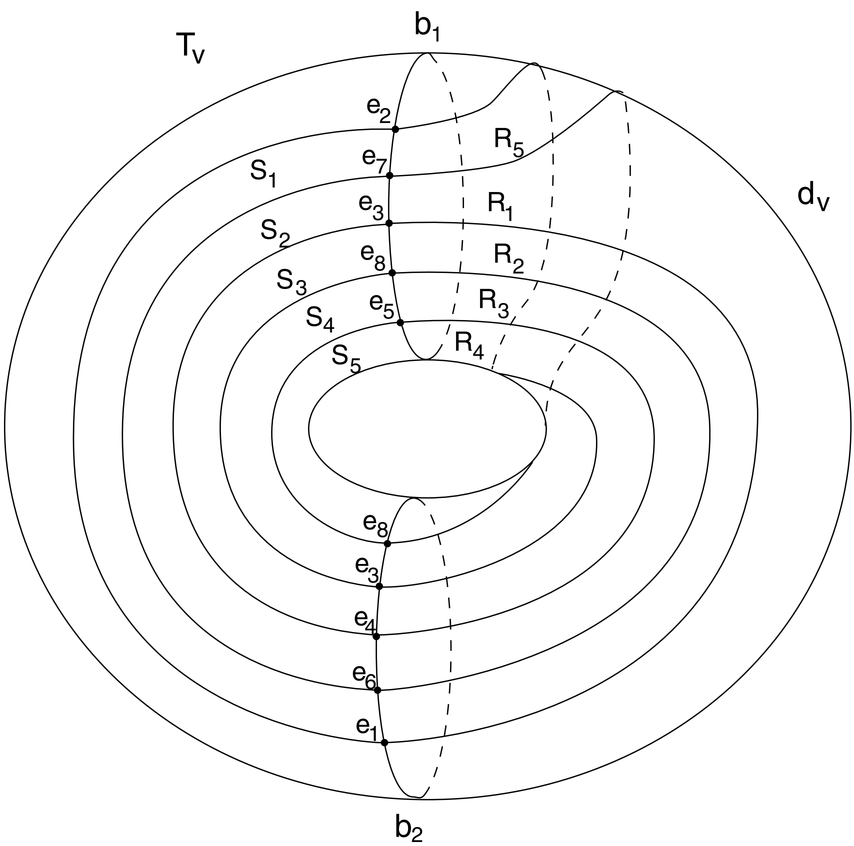

FIGURE 11

The faces of the graph Γ_P are $f_1, f_2, f_3, g_1, g_2, g_3$, where the f_i's lie in Y_1 and the g_i's lie in Y_2; see Figure 9(2). Let the faces of Γ_Q be $h_1, \ldots, h_6$, as shown in Figure 10(6).

The regular neighbourhood N is the union of product neighbourhoods $T_V \times [0,1], T_K \times [0,1], P \times [-1,1]$ and $Q \times [-1,1]$, in the obvious way, where $T_V = T_V \times \{0\}, T_K = T_K \times \{0\}, P = P \times \{0\}$, and $Q = Q \times \{0\}$. Corresponding to R_i is a 2-cell contained in $(T_V \times \{1\}) \cap \partial_0 N$, which we continue to denote by R_i; similarly

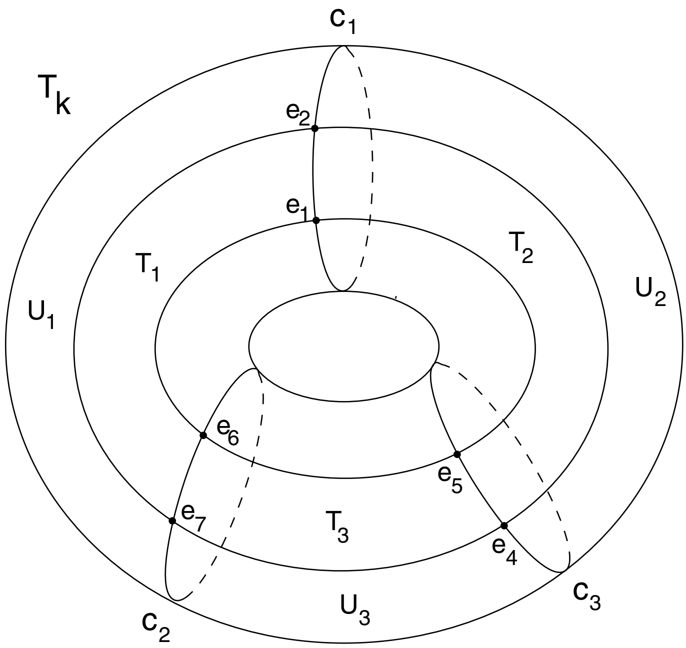

FIGURE 12

for S_i, T_j and U_j. A face f_i of Γ_P gives rise to two 2-cells $f_i^+ \subset (P \times \{1\}) \cap \partial_0 N$ and $f_i^- \subset (P \times \{-1\}) \cap \partial_0 N$, and similarly for the g_i's and the faces h_k of Γ_Q. Since h_k^+ (say) $\subset \partial_1 N$ and $h_k^- \subset \partial_2 N$, there will be no confusion in denoting $h_k^\pm$ by h_k.

By carefully examining the identifications between these various 2-cells one sees that $\partial_1 N$ has two components Σ_1 and Σ_1', and $\partial_2 N$ has two components Σ_2 and Σ_2', composed of the following 2-cells:

Σ_1**:** $f_1^+, f_3^-, h_1, h_2, h_3, R_2, R_5, T_1$,
Σ_1'**:** $f_1^-, f_2^+, f_2^-, f_3^+, h_4, h_5, h_6, R_1, R_3, R_4, T_2, T_3$,
Σ_2**:** $g_1^+, g_2^-, h_1, h_3, S_1, U_1$,
Σ_2'**:** $g_1^-, g_2^+, g_3^+, g_3^-, h_2, h_4, h_5, h_6, S_5, S_2, S_3, S_4, U_2, U_3$.

The precise patterns of identification are shown in Figures 13, 14, 15 and 16, respectively. In particular, $\Sigma_1, \Sigma_1', \Sigma_2, \Sigma_2'$ are 2-spheres. □

Remark 6.7. One can see that $\Sigma_1, \Sigma_1', \Sigma_2, \Sigma_2'$ are 2-spheres without completely determining the identification patterns of their constituent 2-cells, by means of the following Euler characteristic computation.

First note that

$$\chi(P \cup Q) = \chi(P) + \chi(Q) - \chi(P \cap Q) = (-2) + (-2) - 8 = -12.$$

Also, $(P \cup Q) \cap T_V$ consists of three circles, meeting in a total number of 10 points. So $\chi((P \cup Q) \cap T_V) = -10$. Similarly, $\chi((P \cup Q) \cap T_K) = -6$. Therefore

$$\chi(N) = \chi((P \cup Q) \cup (T_V \cup T_K)) = (-12) + 0 - ((-10) + (-6)) = 4.$$

Hence $\chi(\partial N) = 8$.

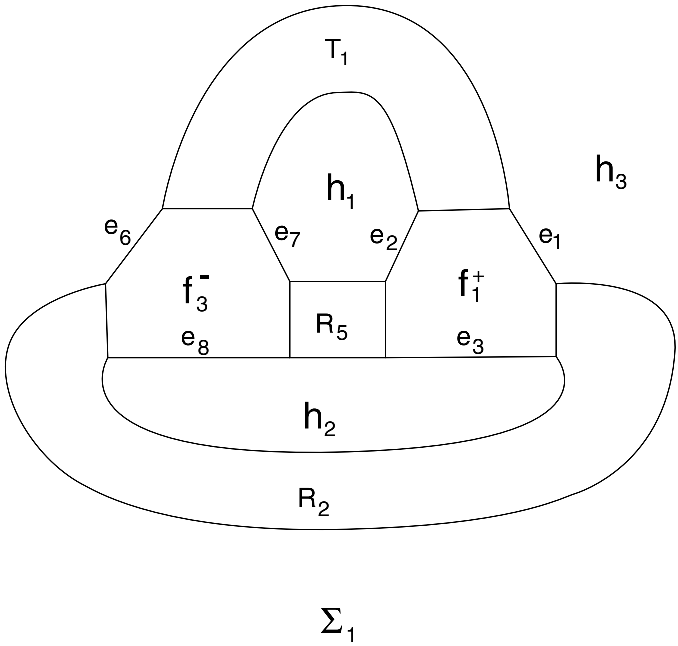

FIGURE 13

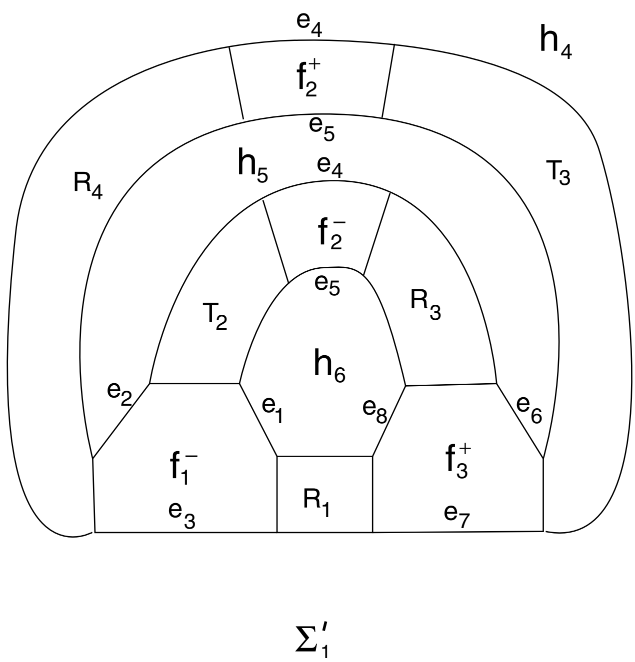

FIGURE 14

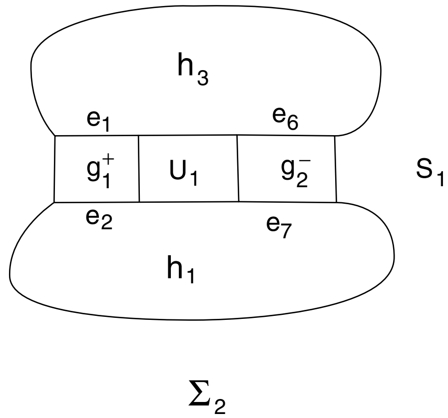

FIGURE 15

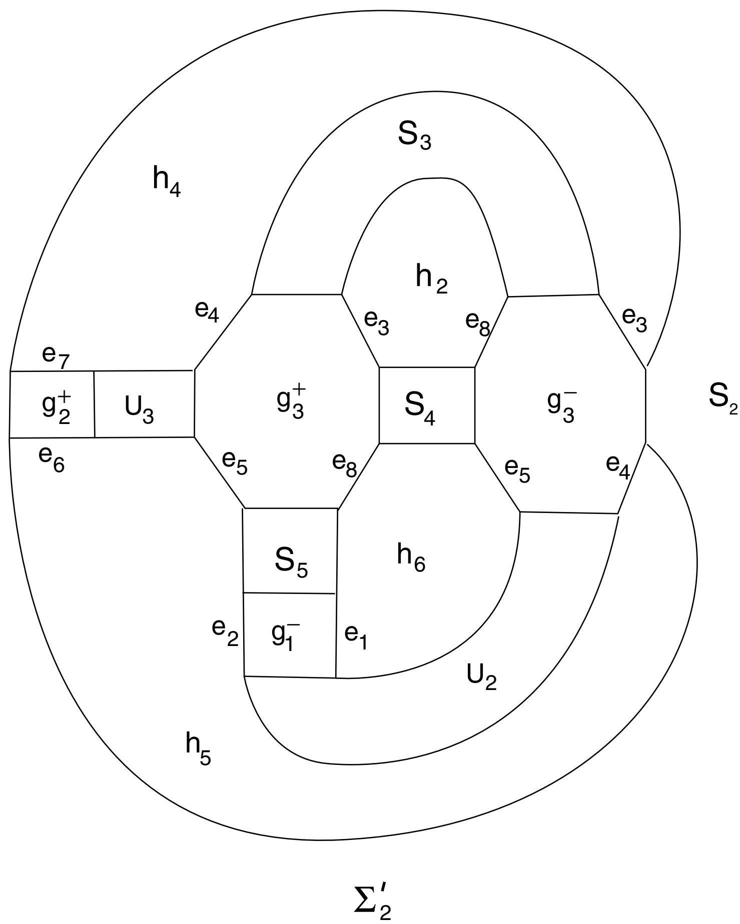

FIGURE 16

Now one can easily check that each of $\partial_1 N$ and $\partial_2 N$ has at most two components. Hence each must have exactly two components, both 2-spheres.

Proof that $(M;\alpha,\beta)$ *is homeomorphic to* $(Wh(-3/2);-5,0)$. Since Y is irreducible the components of $\partial_0 N$ bound 3-balls in Y. Hence the triple $(Y;P,Q)$ is uniquely determined up to homeomorphism, by Figures 9(2) and 10(6). Since the curves c_j are meridians of L, the pair (V,L), together with the slopes $\bar{\alpha},\bar{\beta}$, is uniquely determined. Passing to the double branched cover, we have that $(M;\alpha,\beta)$ is uniquely determined.

In [MP, Table A3] it is shown that -5-filling on the hyperbolic manifold $Wh(-3/2)$ is Seifert fibred with base orbifold $S^2(2,3,3)$, while 0-filling gives a manifold containing a non-separating torus. In fact, it is easy to see that $Wh(-3/2)$ contains an essential once-punctured torus with boundary slope 0. Hence $(M;\alpha,\beta)\cong(Wh(-3/2);-5,0)$. □

7. The case $\Delta(\alpha,\beta)=7$ and the involution τ_α reverses the orientations of the Seifert fibres of $M(\alpha)$

In this section we suppose that Assumptions 5.1 hold and show that it is impossible for $\Delta(\alpha,\beta)$ to be 7 and for τ_α to reverse the orientations of the Seifert fibres of $M(\alpha)$. We assume otherwise in order to obtain a contradiction.

A *tangle* will be a pair $\mathcal{T}=(R,t)$, where R is S^3 minus the interiors of a disjoint union of 3-balls and t is a properly embedded 1-manifold. Let $\widetilde{\mathcal{T}}=(X,\tilde{t})$ be the double branched cover of $\mathcal{T}$. In our examples each boundary component S of R will meet t in either 4 or 6 points, and hence the corresponding boundary component $\widetilde{S}$ of X is either a torus or a surface of genus 2, respectively.

An *essential disk* in $\mathcal{T}$ is a properly embedded disk D in R such that either

(i) $D\cap t=\emptyset$ *and* ∂D *does not bound a disk in* $\partial R\setminus t$, or
(ii) D *meets* t *transversely in a single point and* ∂D *does not bound a disk in* ∂R *containing a single point of* t.

It follows from the $\mathbb{Z}/2$-equivariant Disk Theorem ([GLi], [KT], [YM]) that X contains an essential disk $\widetilde{D}$, i.e., a properly embedded disk such that $\partial\widetilde{D}$ is essential in ∂X, if and only if $\mathcal{T}$ contains an essential disk D.

If S is a boundary component of R such that $|S\cap t|=4$, a *marking* of S is a specific identification of $(S,S\cap t)$ with $(S^2,\{NE,NW,SW,SE\})$. We can then attach a rational tangle $\mathcal{R}(\gamma)$ to $\mathcal{T}$ along S with respect to this marking, where $\gamma\in\mathbb{Q}\cup\{1/0\}$.

By Lemma 4.4(1), $M(\alpha)=M(7/q)$ has base orbifold $S^2(7,7,m)$ for some odd integer $m\geq 3$. As in Lemma 5.3, let $\widetilde{M}_7$ be the 7-fold cyclic cover of M. Then $\partial\widetilde{M}_7$ is a single torus, and both α and β lift to slopes $\tilde{\alpha}$ and $\tilde{\beta}$ in $\partial\widetilde{M}_7$, i.e. $\widetilde{M}_7(\tilde{\alpha})$ is a 7-fold cyclic cover of $M(\alpha)$ and $\widetilde{M}_7(\tilde{\beta})$ is a 7-fold cyclic cover of $M(\beta)$. Furthermore the involution τ on M lifts to an involution $\tilde{\tau}$ on $\widetilde{M}_7$ and $\tilde{V}=\widetilde{M}_7/\tilde{\tau}$ is a 7-fold cyclic cover of $M/\tau=V$. So $\tilde{V}$ is a solid torus. The involution $\tilde{\tau}$ extends to an involution $\tilde{\tau}_{\tilde{\alpha}}$ on $\widetilde{M}_7(\tilde{\alpha})$ such that $\widetilde{M}_7(\tilde{\alpha})/\tilde{\tau}_{\tilde{\alpha}}=S^3$ is the 7-fold cyclic cover of the lens space $M(\alpha)/\tau_\alpha=L(7,2q)$. Let L_7 be the inverse image of L in S^3. Then by Lemma 5.9, L_7 is as shown in Figure 17, where the box with an integer r in it stands for r full horizontal twists, and by Lemma 4.4(2), L_7 is also as shown in Figure 18, where the box with an integer r' in it stands for r' full horizontal twists.

Since $p = 7$, n is odd by Lemma 5.2(3). Hence from Figure 18 we see that L_7 is a single knot. So to get a contradiction, we just need to show that the two knots K and K' shown in Figures 17 and 18, respectively, are inequivalent.

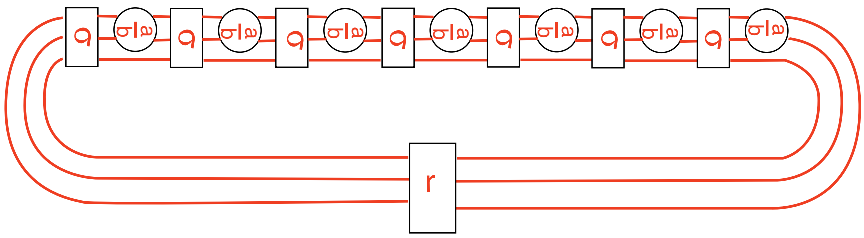

FIGURE 17

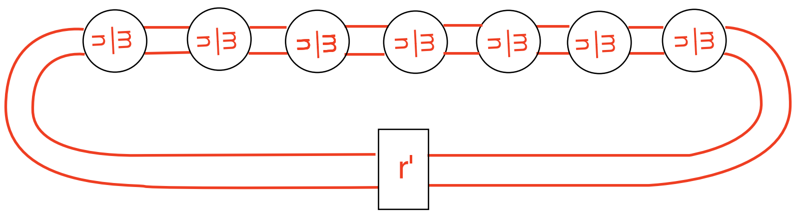

FIGURE 18

Theorem 7.1. *The knots K and K' are inequivalent.*

Let W, W' be the double cover of S^3 branched over K, K', respectively. We shall show that W and W' are not homeomorphic. Note that W' is a Seifert fibred manifold with base orbifold $S^2(m, m, m, m, m, m, m)$. We will examine W and show that it cannot be such a Seifert manifold.

Let $\mathcal{T} = (R, t)$ be the tangle shown in Figure 19. Let the boundary components of R be S, S_1, S_2, S_3 as shown. Note that $|t \cap S| = 6$ and $|t \cap S_i| = 4$, $i = 1, 2, 3$. Let X be the double branched cover of $\mathcal{T}$. Then $\partial X = G \amalg \coprod_{i=1}^{3} T_i$, where G is the double branched cover of $(S, S \cap t)$ and T_i is the double branched cover of $(S_i, S_i \cap t)$, $i = 1, 2, 3$; thus G has genus 2 and the T_i are tori.

Remark 7.2. The permutation induced by σ takes 1 to 2 or 3, since K is connected.

Proposition 7.3. *$X(a/b, a/b, a/b)$ is either*

(1) *boundary-irreducible, or*

(2) *the boundary connected sum of two copies of a Seifert fibred manifold with base orbifold $D^2(a, d)$, $d > 1$, or*

(3) *a handlebody of genus* 2.

We prove Proposition 7.3 by successively filling along T_1, T_3 and T_2.

Lemma 7.4. *G is incompressible in X.*

Proof. Because of Remark 7.2 above, the arrangement of the components of t with respect to the boundary components of R is as illustrated schematically in Figure 20. It follows easily that $\mathcal{T} = (R, t)$ cannot contain any essential disk D with $\partial D \subset S$. □

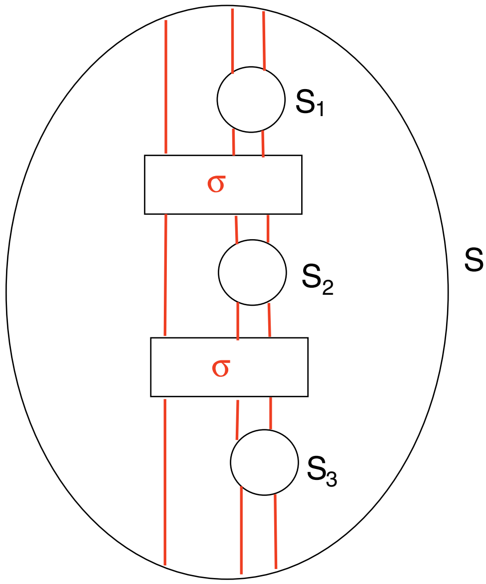

FIGURE 19

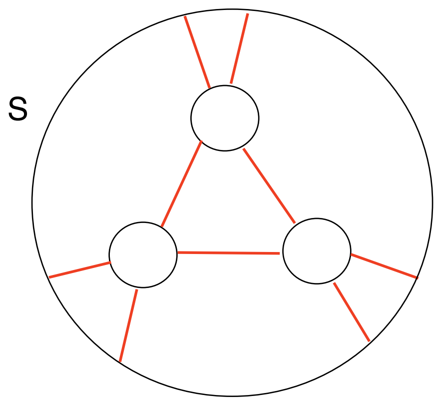

FIGURE 20

In the sequel, a "$*$" will indicate that the corresponding boundary component is left unfilled.

Lemma 7.5. *G is incompressible in $X(a/b, *, *)$.*

Proof. There is an essential annulus $A_1 \subset R$, disjoint from t, with one boundary component in S and the other having slope $0/1$ on S_1; see Figure 21. A component of the inverse image of A_1 in X is an essential annulus with one boundary component on G and the other having slope $0/1$ on T_1. Since $\Delta(a/b, 0/1) = a > 1$, it follows from [Sh] and Lemma 7.4 that G is incompressible in $X(a/b, *, *)$. $\square$

Lemma 7.6. *G is incompressible in $X(a/b, *, a/b)$.*

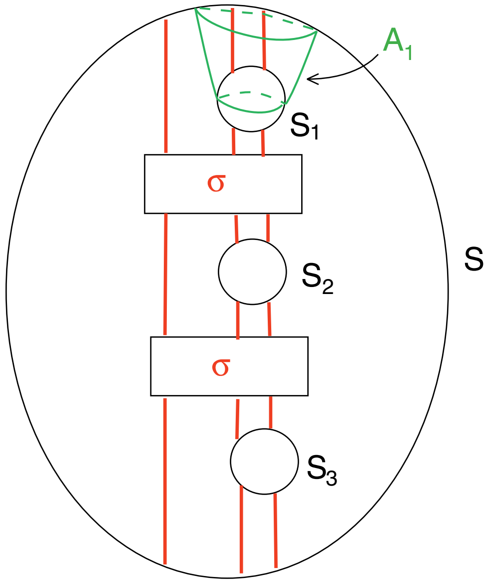

FIGURE 21

Proof. There is an essential annulus $A_3 \subset R(a/b, *, *)$ with one boundary component on S and the other having slope $0/1$ on S_3. The result now follows as in the proof of the previous lemma. □

Proof of Proposition 7.3. There is an essential disk in $\mathcal{T}(a/b, 0/1, a/b)$, meeting $t(a/b, 0/1, a/b)$ in a single point; see Figure 22. Therefore G is compressible in $X(a/b, 0/1, a/b)$. Since $\Delta(a/b, 0/1) = a > 1$, it follows from Lemma 7.6 and [Wu2] that either G is incompressible in $X(a/b, a/b, a/b)$ or there is an essential annulus $A \subset X(a/b, *, a/b)$ with one boundary component on G and the other having slope r/s on T_2, where $\Delta(r/s, 0/1) = \Delta(r/s, a/b) = 1$. We may assume the latter, in which case, by Dehn twisting $X(a/b, *, a/b)$ along A, we have that $X(a/b, a/b, a/b) \cong X(a/b, 0/1, a/b)$. From Figure 22 we see that $X(a/b, 0/1, a/b)$ is the boundary connected sum of two copies of Y, the double branched cover of the tangle shown in Figure 23.

The disk D shown in Figure 23 separates the tangle into two rational tangles $\mathcal{R}, \mathcal{R}'$ and lifts to an annulus $A \subset Y$ which separates Y into two solid tori U and U', the double branched covers of $\mathcal{R}, \mathcal{R}'$, respectively. Note that A has a winding number a in U. Also, it is easy to see (by Remark 7.2) that A is not meridional on U'. Hence Y is either a Seifert fibre space with base orbifold $D^2(a, d)$, for some $d > 1$, or a solid torus, giving conclusions (2) and (3), respectively. □

Let Z be the double branched cover of the tangle (Q, s) shown in Figure 24. Then ∂Z has one torus component and two genus two components.

Lemma 7.7. *$Z(a/b)$ has incompressible boundary.*

Proof. For $i = 0, 1$, there is an annulus $A_i \subset Q$, disjoint from s, with one boundary component on S_i and the other having slope $0/1$ on S, as shown in Figure 24. Since $\Delta(a/b, 0/1) = a > 1$, the result follows as in the proof of Lemma 7.5. □

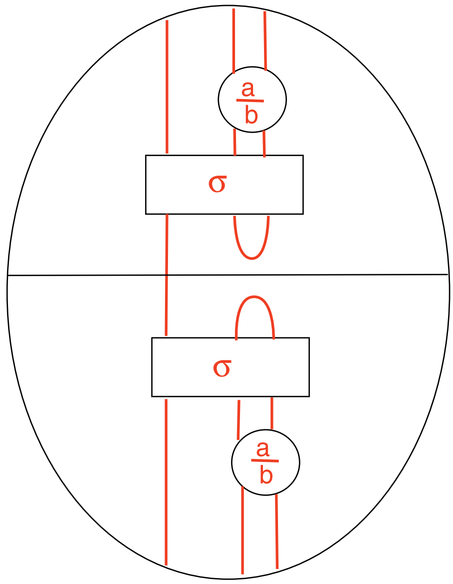

FIGURE 22

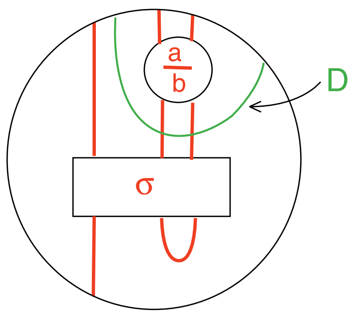

FIGURE 23

Note that filling (Q, s) along S with the rational tangle $\mathcal{R}(1/0)$ gives a product tangle. Hence $Z \cong G \times I - \text{int } N(C)$, where G is a surface of genus two and C is a simple closed curve $\subset G \times \{1/2\}$.

Proposition 7.8. *The double branched cover W of (S^3, K) either*

(1) *contains a separating incompressible surface of genus* 2, *or*

(2) *contains four disjoint tori, each cutting off a manifold which is Seifert fibred over $D^2(a, d)$, $d > 1$, or*

(3) *has Heegaard genus at most* 3.

Proof. From Figure 25 we see that $W \cong P \cup_G Z(a/b) \cup_{G'} P'$, where P and P' are copies of $X(a/b, a/b, a/b)$.

Case (1) of Proposition 7.3, together with Lemma 7.7, gives conclusion (1).

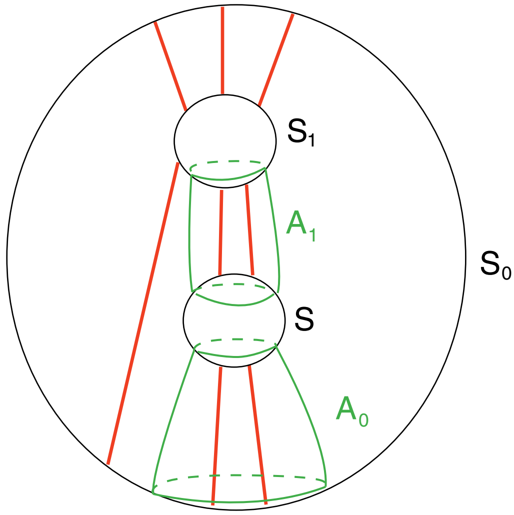

FIGURE 24

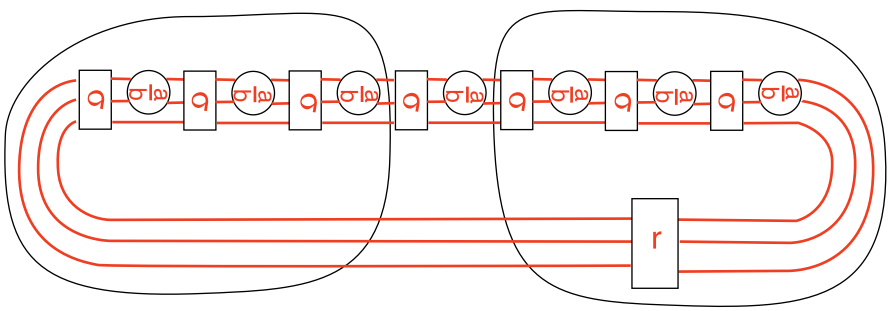

FIGURE 25

In Case (2) of Proposition 7.3, each of P, P' contains two disjoint tori, each cutting off a manifold which is Seifert fibred over $D^2(a, d)$, and we have conclusion (2).

In Case (3) of Proposition 7.3, P and P' are handlebodies of genus 2. Also, by the remark after the proof of Lemma 7.7, $Z(a/b)$ is obtained from $G \times I$ by Dehn surgery on a curve in $G \times \{1/2\}$. Hence W is obtained from a closed manifold with a Heegaard splitting of genus 2 by a Dehn surgery on a curve in the Heegaard surface. Since such a curve has tunnel number at most 2, W has Heegaard genus at most 3. □

Proof of Theorem 7.1. To get a contradiction, suppose $W \cong W'$.

Recall that W' is the double branched cover of (S^3, K') and is a Seifert fibred space with base orbifold $S^2(m, m, m, m, m, m, m)$.

In Case (1) of Proposition 7.8, W' would contain a separating incompressible surface of genus 2. This surface would have to be horizontal, and would then separate W' into two twisted I-bundles. Thus W' would contain a non-orientable surface. But since W' is the double branched cover of a knot in S^3, $H_1(W'; \mathbb{Z}/2) = 0$, a contradiction.

In Case (2) of Proposition 7.8, the tori in question are incompressible (otherwise W' would have base orbifold $S^2(a, d, r)$ for some $r \geq 1$). Hence they are vertical in W'. But since W' has only 7 exceptional fibres, this is clearly impossible.

Finally, since W' has base orbifold $S^2(m, m, m, m, m, m, m)$, every irreducible Heegaard splitting of W' is either horizontal or vertical by [MSch]. It also follows from [MSch] that when W' has an irreducible horizontal Heegaard splitting, its genus is larger than 6, and that any irreducible vertical Heegaard splitting of W' has genus 6. Hence Case (3) of Proposition 7.8 is impossible. □

8. The case $\Delta(\alpha, \beta) = 5$ and $M(\alpha)$ is a lens space

In this section we suppose that Assumptions 5.1 hold and show that $M(\alpha)$ cannot be a lens space, thus completing our proof of Baker's theorem [Ba]. As we noted at the end of §5, it suffices to show that the link depicted in Figure 3, considered as lying in a Heegaard solid torus in $L(5, 2q)$, is not isotopic to either the core of a Heegaard solid torus or the boundary of a Möbius band spine of a Heegaard solid torus.

The proof of the following lemma is straightforward.

Lemma 8.1. *Let V_1 be a Heegaard solid torus in a lens space $L(p, q)$ and let K be either a core of V_1 or a $(2, k)$-cable of a core of V_1. In the first case assume that p is odd. Then the double branched cover of $(L(p, q), K)$ is a lens space.* □

Remark 8.2. The condition that p be odd in the first case is needed to guarantee the existence of a double branched cover. Furthermore, in that case we have $L(p, q) \cong L(p, 2r) \cong L(p, 2r')$, where $4rr' \equiv 1 \pmod{p}$, and then the double branched cover is homeomorphic to either $L(p, r)$ or $L(p, r')$.

Lemma 8.3. *Let Q be a once-punctured torus bundle over S^1, with β the boundary slope of the fibre, and let γ be a slope on ∂Q such that $Q(\gamma)$ is reducible. Then $\Delta(\beta, \gamma) = 1, 2, 3, 4$ or 6.*

Proof. We consider separately three possibilities for Q.

(1) *Q is hyperbolic.* Here $\Delta(\beta, \gamma) = 1$ by [BZ1, Lemma 4.1].

(2) *Q is Seifert fibred.* In this case the monodromy of the bundle has finite order, d, say, where $d = 1, 2, 3, 4$ or 6. If $Q(\gamma)$ is reducible, then γ is the Seifert fibre slope, and hence $\Delta(\beta, \gamma) = d$.

(3) *Q is toroidal and not Seifert fibred.* Let T_0 be the once-punctured torus fibre of Q. Here the monodromy of the bundle is $\pm$ the r^{th} power of a Dehn twist along an essential loop x in T_0, where $r \neq 0$ and $+/-$ denotes composition with the identity and the elliptic involution, respectively. The free group $\pi_1(T_0)$ has basis $\{x, y\}$ with $[\partial T_0] = [x, y] = xyx^{-1}y^{-1}$. Then $\pi_1(Q)$ has presentation

(i) $$\langle x, y, t : t^{-1}xt = x,\ t^{-1}yt = yx^r \rangle$$

or

(ii) $$\langle x, y, t : t^{-1}xt = (xy)x^{-1}(xy)^{-1},\ t^{-1}yt = x(x^{-r}y^{-1})x^{-1} \rangle$$

in the $+/-$ cases mentioned above. In both cases $\pi_1(\partial Q) = \langle t, [x, y] \rangle$.

For the proof in this case we will use the following lemma.

Lemma 8.4. *If $A * B$ is a non-trivial free product quotient of $\pi_1(Q)$, then $4t = 0 \in H_1(A * B)$.*

Proof. Let $A*B$ be a quotient of $\pi_1(Q)$ with $A \neq 1 \neq B$. We adopt the convention that a word in x, y and t denotes the image in $A * B$ of the corresponding element of $\pi_1(Q)$.

Case (i). Here x and t commute. Hence either

(a) x and t are powers of some element z, or
(b) x and t lie in a conjugate of a factor.

In subcase (a) we have $x = z^m$, $t = z^n$, say. The second relation in the presentation (i) gives $z^{-n}yz^n = yz^{rm}$, and therefore $y^{-1}z^n y = z^{n-rm}$. By applying an inner automorphism of $A * B$ we may assume that z is represented by a cyclically reduced word in the factors. It follows that $|n| = |n - rm|$, otherwise we have two cyclically reduced words, z^n and z^{n-rm}, of different lengths in the same conjugacy class. Hence either $m = 0$ or $y^{-1}z^n y = z^{-n}$. If $m = 0$ then $x = 1$, and so $A * B$ is a quotient of $\langle y, t : t^{-1}yt = y\rangle \cong \mathbb{Z} \times \mathbb{Z}$, a contradiction. If $y^{-1}z^n y = z^{-n}$ then $y^{-1}ty = t^{-1}$, and so $2t = 0 \in H_1(A * B)$.

In subcase (b) we may assume, by applying an inner automorphism of $A*B$, that $x, t \in A$. Then $y^{-1}t^{-1}y = x^r t^{-1} \in A$. But $t^{-1} \in A$, and hence $y \in A$. Therefore $B = 1$, a contradiction.

Case (ii). Let $s = txy$. Then $\pi_1(Q)$ has the presentation

$$\langle x, y, s : s^{-1}xs = x^{-1},\ s^{-1}ys = y^{-1}x^{-r}\rangle.$$

Since x and s^2 commute, either

(a) x and s^2 are powers of some element z, or
(b) x and s^2 lie in a conjugate of a factor.

In subcase (a), suppose $x = z^m$, $s^2 = z^n$. The second relation in the presentation of $\pi_1(Q)$ implies $s^{-2}ys^2 = x^r y x^r$, i.e. $z^{-n}yz^n = z^{rm}yz^{rm}$, giving $y^{-1}z^{(n+rm)}y = z^{n-rm}$. As in Case (i) we may assume that z is cyclically reduced, and hence $|n + rm| = |n - rm|$, i.e. either $m = 0$ or $n = 0$. If $m = 0$ then $x = 1$, and so $A * B$ is a quotient of the Klein bottle group $\langle y, s : s^{-1}ys = y^{-1}\rangle$, which is easily seen to imply $A * B \cong \mathbb{Z}_2 * \mathbb{Z}_2$. If $n = 0$ then $s^2 = 1$. Hence $2s = 0 \in H_1(A * B)$. But in $H_1(Q)$, $s = t + x + y$, $2x = 0$, and $4y = 0$. Therefore $4t = 0 \in H_1(A * B)$.

In subcase (b) we may assume that $x, s^2 \in A$. Hence $s \in A$. From the second relation in the above presentation of $\pi_1(Q)$ we get $(ys^{-1})^2 = x^{-r}s^{-2} \in A$. Therefore $ys^{-1} \in A$, and hence $y \in A$. This implies that $B = 1$, a contradiction. □

We now complete the proof of Lemma 8.3.

Let $\Delta = \Delta(\beta, \gamma)$. Then $\pi_1(Q(\gamma))$ is obtained from $\pi_1(Q)$ by adding the relation $t^\Delta[x, y]^q = 1$, for some integer q coprime to Δ. It is easy to see from the presentations (i) and (ii) that $H_1(Q(\gamma)) \ncong \mathbb{Z}$. Therefore $Q(\gamma)$ is a non-trivial connected sum and hence $\pi_1(Q(\gamma))$ is a non-trivial free product. The relation $t^\Delta[x, y]^q = 1$ shows that t has order Δ in $H_1(Q(\gamma))$. Hence by Lemma 8.4, Δ divides 4. □

Now we complete the proof that $M(\alpha)$ cannot be a lens space under the assumption that the conditions 5.1 hold. Suppose otherwise. By Lemma 5.10, $M(\alpha)/\tau_\alpha \cong L(5, 2q)$, L is either the core of a Heegaard solid torus in $L(5, 2q)$ or a $(2, k)$-cable of such a core, and furthermore $L(5, 2q)$ has a genus 1 Heegaard splitting $V \cup V_0$ such that L is isotopic to a curve in V of the form shown in Figure

3, where a and b are coprime integers with $a \geq 2$ and σ is a 3-braid. We will show that these conditions on L lead to a contradiction.

Remove from the solid torus V in Figure 3 the interior of the 3-ball B containing the a/b-rational tangle. We then get a tangle $\mathcal{T}$ in $Y = (V - \text{int } B) \cup V_0 = L(5, 2q) \setminus \text{int } B$. Let X be the double branched cover of $(Y, \mathcal{T})$.

Since $\mathcal{T}(a/b) = L$, by Lemma 8.1 we have

- $X(a/b)$ is a lens space.

Also, clearly $\mathcal{T}(0/1) = (\text{core of } V) \# (\text{knot in } S^3)$, so

- $X(0/1) \cong L(5, r) \# N$ for some closed 3-manifold N.

Lemma 8.5. *$X(1/k)$ is irreducible for all $k \in \mathbb{Z}$.*

Proof. $\mathcal{T}(1/k)$ is (the 3-braid $\sigma_1^k \sigma$ in V) $\cup V_0$. Hence $X(1/k) = Q_k \cup \widetilde{V}_0$, where Q_k is the double branched cover of $(V, \sigma_1^k \sigma)$ and $\widetilde{V}_0$ is a solid torus. Now Q_k is a T_0-bundle over S^1, where T_0 is the double branched cover of (D^2, 3 points), i.e. a once-punctured torus. Let β be the boundary slope of the fibre of Q_k; note that β projects to the meridian μ of V. Let $\mu_0, \tilde{\mu}_0$ be the meridians of $V_0, \widetilde{V}_0$, respectively. Since $\Delta(\mu, \mu_0) = 5$, we have $\Delta(\beta, \tilde{\mu}_0) = 5$. Hence by Lemma 8.3, $X(1/k)$ is irreducible. □

There is a $\mathbb{Z}/2$-action on X with quotient $Y = L(5, 2q) \setminus \text{int } B$. It follows easily that X is not a solid torus. We consider the following three possibilities for X.

(1) *X is reducible.* Here we must have $X \cong X' \# X(a/b)$, where $X'(a/b) \cong S^3$. By Lemma 8.5, $X'(1/k) \cong S^3$ for infinitely many k, and hence X' is a solid torus with meridian $0/1$. Since $\Delta(a/b, 0/1) = a > 1$, this contradicts the fact that $X'(a/b) \cong S^3$.

(2) *X is irreducible and not Seifert fibred.* Since $\Delta(a/b, 0/1) = a > 1$, the forms of $X(a/b)$ and $X(0/1)$ stated above contradict [CGLS] if $N \cong S^3$ and [BZ2, Corollary 1.4] otherwise.

(3) *X is Seifert fibred with incompressible boundary.*

If X is not the twisted I-bundle over the Klein bottle, let φ be the slope on ∂X of the Seifert fibre in the unique Seifert fibring of X. If X is the twisted I-bundle over the Klein bottle, let φ be the slope of the Seifert fibre in the Seifert structure on X with orbifold $D^2(2, 2)$. In both cases, φ is the only slope on ∂X such that $X(\varphi)$ is a non-trivial connected sum. Therefore, if $N \not\cong S^3$, then $\varphi = 0/1$. But $X(a/b)$ is a lens space, and so $\Delta(a/b, 0/1) = 1$, contradicting our assumption that $a > 1$. Hence $N \cong S^3$, and so $\Delta(a/b, \varphi) = \Delta(0/1, \varphi) = 1$. In particular $\varphi = 1/s$ for some integer s. Therefore $X(1/s)$ is reducible. But this contradicts Lemma 8.5. □

9. The case $\Delta(\alpha, \beta) = 6$ and the involution τ_α reverses the orientations of the Seifert fibres of $M(\alpha)$

In this section we suppose that Assumptions 5.1 hold and show that it is impossible for $\Delta(\alpha, \beta)$ to be 6 and for τ_α to reverse the orientations of the Seifert fibres of $M(\alpha)$. We assume otherwise in order to obtain a contradiction. Here $M(\alpha)/\tau_\alpha = L(3, q) \cong L(3, 1)$. By Lemma 5.9, L is as shown in Figure 3. By Lemma 5.2 parts (2) and (3), n is even, m is odd, $|L| = 1$, and $L_\alpha = L \cup K_\alpha$ is as shown in Figure 1. Since L is a component of L_α, we see that L is a core of some

Heegaard solid torus of $L(3,1)$. Hence the double branched cover of $(L(3,1), L)$ is homeomorphic to $L(3,1)$.

Let $Y, \mathcal{T}, X$ be as in the previous section, with $L(5,2q)$ replaced by $L(3,1)$. Again as in that proof, here we have $X(a/b) \cong L(3,1)$ and $X(0/1) \cong L(3,1) \# N$ for some closed 3-manifold N. In the current situation we only have the following weaker version of Lemma 8.5.

Lemma 9.1. *$X(1/k)$ is irreducible for infinitely many $k \in \mathbb{Z}$.*

Proof. As in the proof of Lemma 8.5, $\mathcal{T}(1/k)$ is (the 3-braid $\sigma_1^k\sigma$ in V) $\cup V_0$, and $X(1/k) = Q_k \cup \widetilde{V}_0$, where Q_k is the double branched cover of $(V, \sigma_1^k\sigma)$ and $\widetilde{V}_0$ is a solid torus. Now Q_k is a T_0-bundle over S^1, where T_0 is a once-punctured torus. If $\rho \in B_3$, let $\tilde{\rho}$ denote the corresponding homeomorphism $T_0 \to T_0$. Then $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are Dehn twists about a pair of curves in T_0 with intersection number 1. With respect to this basis, $\tilde{\rho}$ defines an element of $SL_2(\mathbb{Z})$. Note that since L is connected, σ is not a power of σ_1. The elements of $SL_2(\mathbb{Z})$ corresponding to $\tilde{\sigma}_1^k$ and $\tilde{\sigma}$ are therefore $\left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, say, where $c \neq 0$. Then the matrix corresponding to $\tilde{\sigma}_1^k\tilde{\sigma}$ has trace $a + d + kc$, which has absolute value greater than 2 for all but at most five values of k. For such k the manifold Q_k is therefore hyperbolic.

Let β be the boundary slope of the fibre of Q_k; note that β projects to the meridian μ of V. Let $\mu_0, \tilde{\mu}_0$ be the meridians of V_0, $\widetilde{V}_0$, respectively. Since $\Delta(\mu, \mu_0) = 3$, we have $\Delta(\beta, \tilde{\mu}_0) = 3$. If Q_k is hyperbolic, then by [BZ1, Lemma 4.1] $Q_k(\gamma)$ reducible implies $\Delta(\beta, \gamma) = 1$. Therefore $X(1/k) = Q_k(\tilde{\mu}_0)$ is irreducible for infinitely many k. □

As in the previous section, we have possibilities (1), (2) and (3) for X. Cases (1) and (2) are ruled out exactly as before (applying Lemma 9.1 instead of Lemma 8.5). In case (3) we may conclude that both $X(a/b)$ and $X(0/1)$ are $L(3,1)$, $X(1/s)$ is reducible for some integer s and $\Delta(\beta, \tilde{\mu}_0) = 3$. The proof of Lemma 8.3 shows that the monodromy of the once-punctured torus bundle Q_s has order 3. Therefore Q_s has base orbifold $D^2(3,3)$, and so $X(1/s) \cong Q_s(\tilde{\mu}_0) \cong L(3,q_1) \# L(3,q_2)$. This implies that X has base orbifold $D^2(3,3)$. But then no two distinct fillings on X can give the lens space $L(3,1)$, yielding a contradiction.

10. The case $\Delta(\alpha, \beta) = 5$ and the involution τ_α reverses the orientations of the Seifert fibres of $M(\alpha)$

In this section we suppose that Assumptions 5.1 hold and show that it is impossible for $\Delta(\alpha, \beta)$ to be 5 and for τ_α to reverse the orientations of the Seifert fibres of $M(\alpha)$. We assume otherwise in order to obtain a contradiction.

As in §7, we just need to show that the two knots, K, K', shown in Figures 26 and 27, respectively, are inequivalent in S^3.

Theorem 10.1. *The knots K and K' are inequivalent.*

As in §7, we will show that the double branched covers W, W' of $(S^3, K), (S^3, K')$ are not homeomorphic.

Here we consider the tangle $\mathcal{T} = (R, t)$ shown in Figure 28, with double branched cover X. Let the boundary components of R be S, S_1, S_2 (see Figure 28) and the corresponding boundary components of X be G, T_1, T_2, so that T_1 and T_2 are tori and G has genus two.

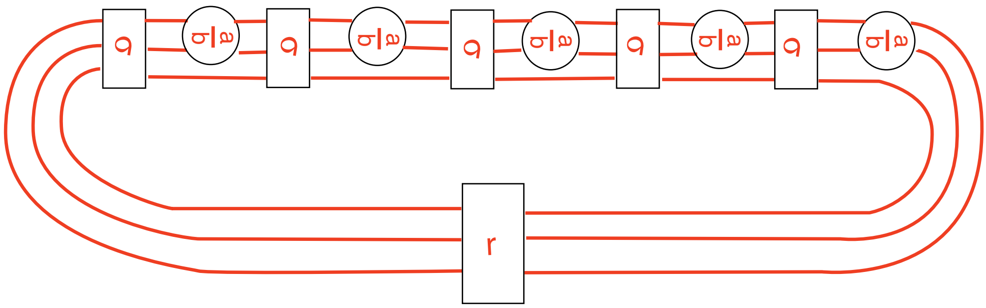

FIGURE 26

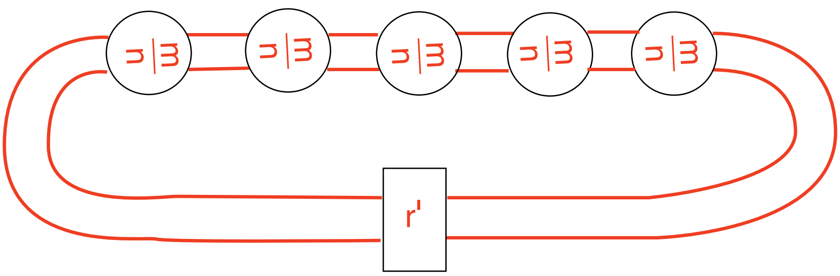

FIGURE 27

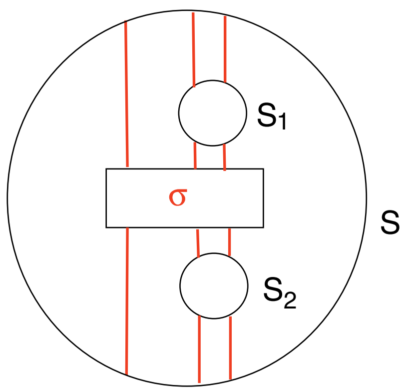

FIGURE 28

Lemma 10.2. *If G is compressible in X then $\mathcal{T}$ is isotopic to the tangle shown in Figure* 29.

Proof. Since $t \cup S_1 \cup S_2$ is connected, any essential disk D in $\mathcal{T}$ with $\partial D \subset S$ must meet t in a single point. Hence D meets the unique strand of t connecting S_1 and S_2, decomposing $\mathcal{T}$ into two tangles $\mathcal{T}_1$ and $\mathcal{T}_2$. We claim that each of $\mathcal{T}_1$ and $\mathcal{T}_2$ is a product tangle. To see this, note that deleting the strand of t that joins S_2 to S and runs through the braid σ gives the tangle shown in Figure 30. It follows that $\mathcal{T}_1$ is as stated. Similarly, $\mathcal{T}_2$ is also a product tangle. $\square$

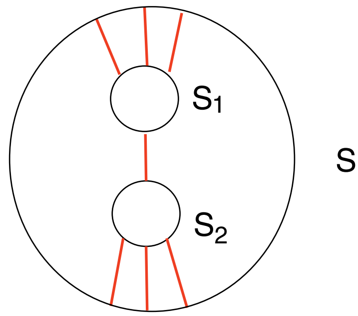

FIGURE 29

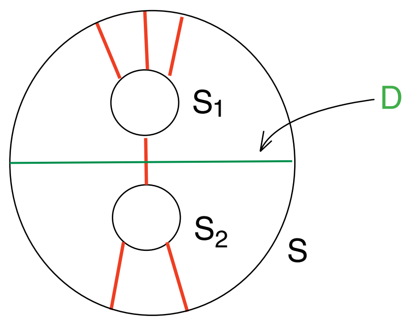

FIGURE 30

Corollary 10.3. *If G is compressible in X then $X({}^a/_b, {}^a/_b)$ is a genus 2 handlebody.*

Lemma 10.4. *If G is incompressible in X then G is incompressible in $X({}^a/_b, {}^a/_b)$.*

Proof. This is exactly like the proof of Lemma 7.6 in §7, using the annuli A_1 and A_2 shown in Figure 31. □

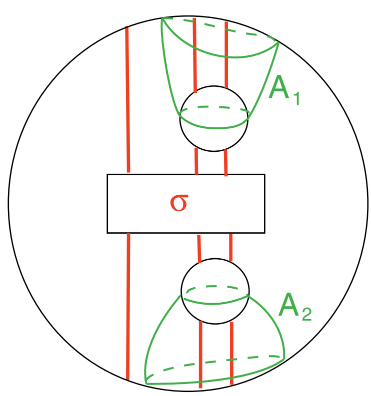

FIGURE 31

Proposition 10.5. *W either*
(1) *contains a separating incompressible surface of genus 2, or*
(2) *has Heegaard genus at most 3.*

Proof. From Figure 32 we see that $W \cong U \cup_G Z(a/b) \cup_{G'} U'$, where U and U' are copies of $X(a/b, a/b)$.

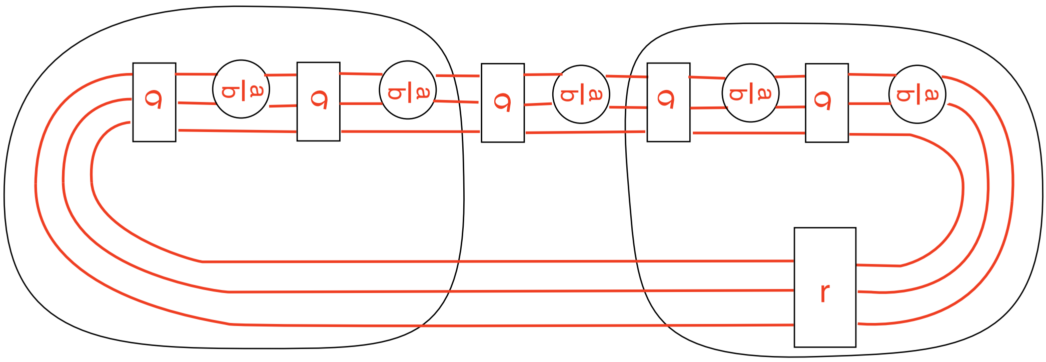

FIGURE 32

If G is incompressible in X, then we get conclusion (1) by Lemmas 10.4 and 7.7.
If G is compressible in X, then we get conclusion (2) by Corollary 10.3 and the proof of part (3) of Proposition 7.8. □

Proof of Theorem 10.1. Assume $W \cong W'$. Since W' is a Seifert fibre space over S^2 with 5 exceptional fibres, we get a contradiction to Proposition 10.5 as in the proof of Theorem 7.1 in cases (1) and (3) of Proposition 7.8. □

11. A FAMILY OF EXAMPLES REALIZING $\Delta(\alpha, \beta) = 4$

In this section we show that distance 4 between a prism manifold filling slope and a once-punctured torus slope can be realized on infinitely many hyperbolic knot manifolds.

Let Wh be the exterior of the Whitehead link with standard meridian-longitude coordinates on ∂Wh. We use $Wh(\gamma)$ to denote the manifold of Dehn filling one boundary component of Wh with slope γ, and $Wh(\gamma, \delta)$ the manifold of Dehn filling one boundary component with slope γ and the other with slope δ.

Theorem 11.1. *For each integer n with $|n| > 1$, $Wh(\frac{-2n\pm 1}{n})$ is a hyperbolic knot manifold whose 0-slope is the boundary slope of an essential once-punctured torus and whose -4-slope yields a prism manifold whose base orbifold is $S^2(2, 2, |\mp 2n - 1|)$.*

Proof. It is well known that $Wh(\gamma)$ is hyperbolic for each $\gamma \notin \{-1, -2, -3, -4, 0, 1/0\}$. That $Wh(\gamma)$, $\gamma \neq 1/0$, contains an essential once-punctured torus with boundary slope 0 is obvious from the Whitehead link diagram.

The Whitehead link admits an involution τ as shown in Figure 33. This involution restricts to an involution, still denoted τ, on Wh and then extends to an involution τ_γ on $Wh(\gamma)$ and to an involution $\tau_{\gamma,\delta}$ on $Wh(\gamma, \delta)$ for all slopes γ and δ. The quotient space under τ is shown in Figure 34. Note that the branch

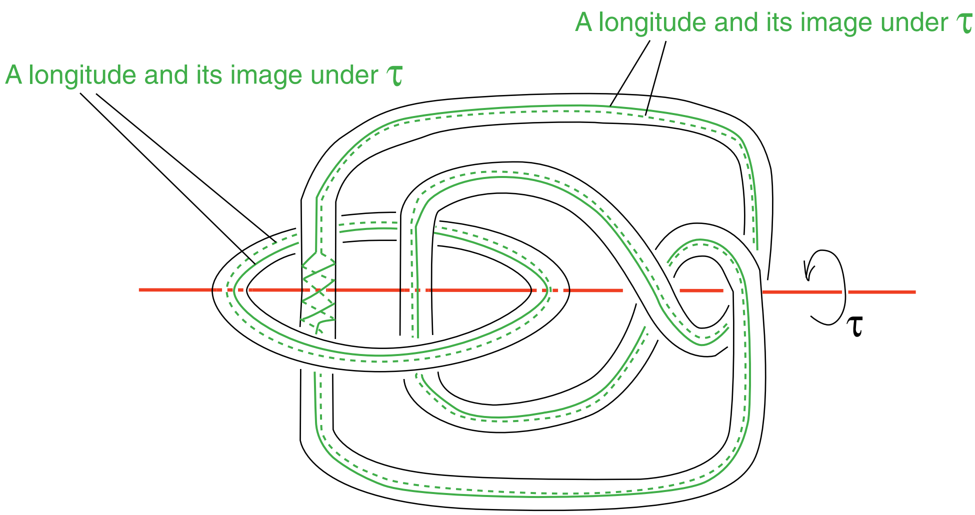

FIGURE 33

set of $Wh(\gamma)/\tau_\gamma$ is obtained by removing the two 1/0-tangles in Figure 34 and then filling one γ-tangle. Figure 35 shows the branch set in $Wh(-4)/\tau_{-4}$ and Figure 36 shows the branch set in $Wh(\frac{-2n\pm1}{n},-4)/\tau_{\frac{-2n\pm1}{n},-4}$. As the branch set in $Wh(\frac{-2n\pm1}{n},-4)/\tau_{\frac{-2n\pm1}{n},-4}=S^3$ is a Montesinos link of type $(2,2,\frac{\mp2n-1}{2})$, the double branched cover $Wh(\frac{-2n\pm1}{n},-4)$ is a prism manifold whose base orbifold is $S^2(2,2,|\mp2n-1|)$. □

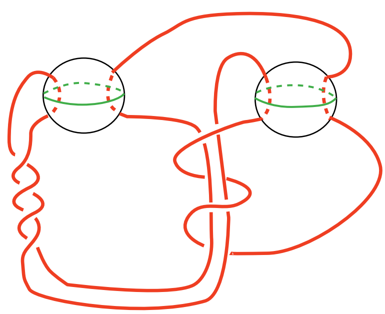

FIGURE 34

12. THE CASE WHEN $\Delta(\alpha,\beta)=4$ AND $M(\alpha)$ IS A PRISM MANIFOLD

In this section we show

Theorem 12.1. *Let M be a hyperbolic knot exterior containing an essential once-punctured torus with slope β. If $M(\alpha)$ is a prism manifold with $\Delta(\alpha,\beta)=4$, then M is one of the examples given in §11, that is,* $(M;\alpha,\beta)\cong(Wh(\frac{-2n\pm1}{n});-4,0)$ *for some integer n with $|n|>1$.*

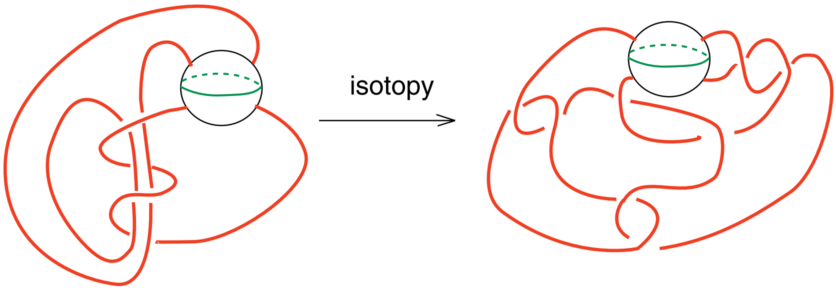

FIGURE 35

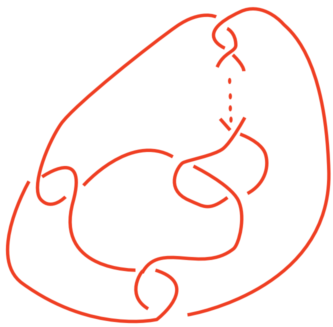

FIGURE 36

Let F be an essential once-punctured torus in M with slope β. Choose a Klein bottle $\hat{P}$ in $M(\alpha)$ which has the minimal number of intersection components with ∂M and let $P = M \cap \hat{P}$. Then $p = |\partial P| > 0$ since M is hyperbolic. The punctured Klein bottle P is essential in M, i.e. it is incompressible and boundary-incompressible in M. The proof of this statement is essentially contained in [Te2, Proofs of Lemmas 2.1 and 2.2], and we only need to add the condition that $M(\alpha)$ is a prism manifold which is thus irreducible and does not contain a projective plane.

As usual, the two surfaces F and P define two labeled intersection graphs which we denote by Γ_F and Γ_P. Then neither Γ_F nor Γ_P contain trivial loops ([Te2, Lemma 3.1] with the same proof). The graph Γ_F has a unique vertex whose valency is $4p$, and the graph Γ_P has p vertices each having valency 4. Note that every edge of Γ_F is positive since F is orientable and has only one boundary component.

Lemma 12.2. (1) *When $p \geq 2$, Γ_F has no S-cycle.*

(2) *When $p \geq 3$, Γ_F has no generalized S-cycle (see* [Te2] *for its definition).*

(3) *Γ_F cannot have more than $\frac{p}{2} + 1$ mutually parallel edges.*

Proof. Part (1) is [Te2, Lemma 3.2] with the same proof, part (2) is [Te2, Lemma 3.3] with a similar argument plus the fact that $M(\alpha)$ does not contain a projective plane, and part (3) is [LT, Lemma 6.2 (4)] with the same proof. □

Lemma 12.3. $p = 1$.

Proof. The lemma was proved in [Te2, Lemma 5.2] when M was a genus one non-cabled knot exterior in S^3, in which case p was an odd integer. In our situation, we need to extend the argument of [Te2, Lemma 5.2] slightly, using Lemma 12.2 (3) instead of [Te2, Lemma 3.4].

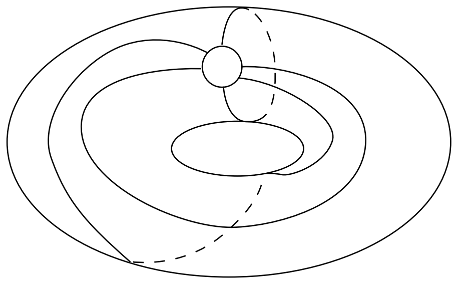

FIGURE 37

Suppose otherwise that $p \geq 2$. The reduced graph $\overline{\Gamma}_F$ is a subgraph of the graph shown in Figure 37 ([Go1, Lemma 5.1]). In particular $\overline{\Gamma}_F$ has at most three edges. Suppose these edges of $\overline{\Gamma}_F$ have weights w_k, $k = 1, 2, 3$, some of which may possibly be zero. Then $2(w_1 + w_2 + w_3) = 4p$. Let $e_1, ..., e_{w_k}$ be a parallel family of consecutive edges in Γ_F. Reading the labels around the vertex of Γ_F, we see that the labels of the edges $e_1, e_2, ..., e_{w_k}$ are as illustrated in Figure 38.

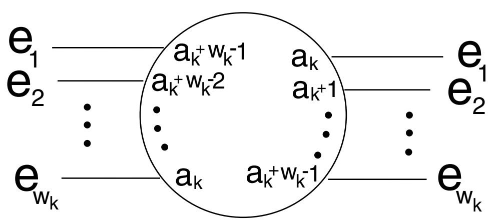

FIGURE 38

By Lemma 12.2, $w_k = 0$ or 1, $1 \leq k \leq 3$. (More precisely, this follows from Lemma 12.2 (1) if w_k is even, Lemma 12.2 (2) if w_k is odd and $p \geq 3$, and Lemma 12.2 (3) if w_k is odd and $p = 2$.) This is a contradiction. □

So Γ_F has exactly two edges and both are level edges (i.e. having the same label at the two endpoints of the edge). Let e_1, e_2 be the two edges of Γ_F and of Γ_P. Note that each e_i is an orientation-reversing loop in P by the parity rule.

Since $\Delta(\alpha, \beta) = 4$, if the endpoints of the two edges around the vertex ∂F are labeled consecutively by $1, 2, 3, 4$, the labels around ∂P are also consecutive. It follows from this fact that if the two edges in Γ_F are not parallel, then the two edges in Γ_P must be parallel. Also, combining this fact with the proof of [Te2, Lemma 4.1], we have that the two edges e_1 and e_2 cannot be parallel in both Γ_P

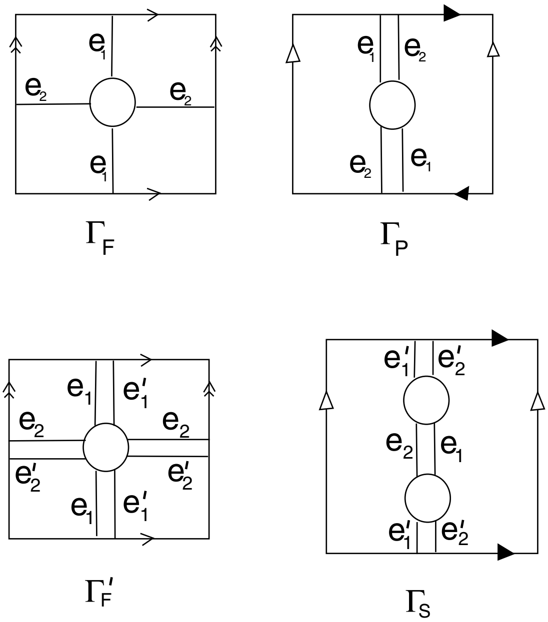

FIGURE 39

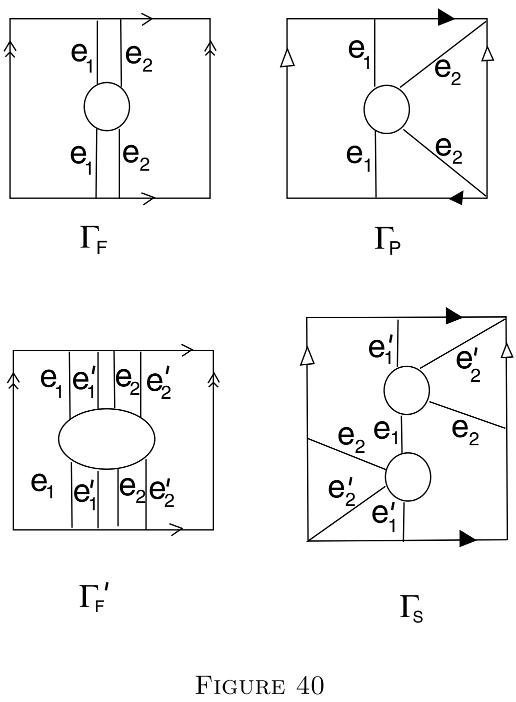

FIGURE 40

and Γ_F. So there are only two possible configurations for the pair of graphs Γ_F and Γ_P, which we illustrate in Figure 39 and Figure 40, respectively.

Let S be the frontier of a thin regular neighbourhood of P in M. Then S is a separating twice-punctured torus in M. The surfaces F and S define two labeled intersection graphs Γ'_F and Γ_S. Note that Γ'_F is obtained by doubling the edges of Γ_F and Γ_S double covers Γ_P. See Figures 39 and 40 for illustrations of the graphs $\Gamma_F, \Gamma_P, \Gamma'_F$ and Γ_S.

The surface S separates M into two components which we denote by X^+ and X^-, where X^- is a twisted I-bundle over P. Note that $\widehat{X}^-$ is a twisted I-bundle over the Klein bottle $\widehat{P}$ and $\widehat{X}^+$ is a solid torus since $M(\alpha) = \widehat{X}^- \cup \widehat{X}^+$ is a prism manifold. Let H^ϵ denote the part of the filling solid torus of $M(\alpha)$ contained in $\widehat{X}^\epsilon$, $\epsilon \in \{\pm\}$, and let $\partial_0 H^\epsilon = \partial H^\epsilon \cap \partial M$.

We first show

Lemma 12.4. *The case given by Figure* 39 *cannot occur.*

Proof. The 4-gon face of Γ'_F, which we denote by f, is contained in X^+ and its boundary edges form a Scharlemann cycle of order 4. From Figure 39 we see that ∂f is a non-separating curve in the genus two surface $S \cup \partial_0 H^+$ and $\partial f \cap S$ is contained in an essential annulus A in $\widehat{S}$. Let U be a regular neighbourhood of $A \cup H^+ \cup f$ in $\widehat{X}^+$. Then U is a compact 3-manifold with ∂U a torus, and the fundamental group of U has the presentation

$$\langle x, t : x^3 t x t = 1 \rangle,$$

where we take a fat base point in $\widehat{S}$ containing $\partial S \cup e_1 \cup e_2$, x is a based loop formed by a cocore arc of $\partial_0 H^+$ and t is represented by a core circle of A. Let $y = xt$; then

$$\pi_1(U) = \langle x, y : x^2 y^2 = 1 \rangle.$$

So U is Seifert fibred with base orbifold $D(2,2)$. Thus U contains a Klein bottle. But U is contained in the solid torus $\widehat{X}^+$. This gives a contradiction. □

So the case of Figure 40 must occur. In this case we are going to show that M is obtained by Dehn filling one boundary component of the Whitehead link exterior.

In this case, the bigon faces of Γ'_F between e_1 and e'_1 and between e_2 and e'_2 lie in X^-, and the bigon face between e'_1 and e_2, which we denote by B, is contained in X^+. Let Q be a regular neighbourhood of $S \cup \partial_0 H^+ \cup B$ in X^+, and $\widehat{Q} = Q \cup H^+$. Then it's easy to see that $\widehat{Q}$ is a Seifert fibred manifold whose base orbifold is an annulus with a single cone point of order 2. The boundary of $\widehat{Q}$ consists of two tori, one of which is the torus $\widehat{S}$. Let T_0 be the other component. Note that T_0 is contained in the interior of X^+. Since $\widehat{X}^+$ is a solid torus, T_0 must bound a solid torus in $\widehat{X}^+ \setminus \widehat{Q}$, which we denote by N.

Lemma 12.5. *The Seifert structure of $\widehat{Q}$ does not match with the Seifert structure of $\widehat{X}^-$ whose base orbifold is $D(2,2)$.*

Proof. The S-cycle $\{e_1, e'_1\}$ in Γ'_F implies that as a cycle in Γ_S, $e_1 \cup e'_1$ is a fibre of the Seifert structure of $\widehat{X}^-$ whose base orbifold is $D(2,2)$. Similarly the S-cycle $\{e'_1, e_2\}$ in Γ'_F implies that as a cycle in Γ_S, $e'_1 \cup e_2$ is a fibre of the Seifert structure of $\widehat{Q}$. Obviously from Figure 40 these two cycles have different slopes in $\widehat{S}$. □

Let $W = X^- \cup_S Q$. Note that $M = W \cup_{T_0} N$. So we just need to show that W is the Whitehead link exterior. We use the notation $W(\partial M, \gamma)$ to denote the Dehn filling of W along a slope γ in $\partial M \subset \partial W$.

Lemma 12.6. (1) *W is irreducible.*

(2) *The twice-punctured torus S is incompressible in W.*

(3) *$F \cap W$ has a component which is an essential once-punctured annulus in W with the puncture lying in ∂M of slope β and with the boundary of the annulus lying in T_0.*

(4) *$W(\partial M, \alpha)$ contains an essential torus which is $\widehat{S}$.*

Proof. By the construction of Q, one can easily see that Q is irreducible and S is incompressible in Q. Obviously X^- is irreducible and S is incompressible in X^-. Thus S is incompressible in $W = X^- \cup_S Q$ and W is irreducible. So we get (1) and (2).

Part (3) follows from the graph Γ'_F shown in Figure 40 and the construction of Q. In fact, the exterior in F of the annulus which is the annulus face of Γ'_F shrunk slightly into the interior of the face is the required punctured annulus. It is incompressible in W because it is an essential subsurface of F. It is boundary incompressible in W because it has only one intersection component with ∂M and M does not contain an essential disk with slope β.

For (4), we just need to note that $W(\partial M, \alpha) = \widehat{X}^- \cup_{\widehat{S}} \widehat{Q}$. □

Lemma 12.7. *W is hyperbolic.*

Proof. We already know that W is irreducible (Lemma 12.6(1)). Obviously W cannot be Seifert fibred since $M = W \cup N$ is hyperbolic. So we just need to show that W is atoroidal. Suppose otherwise that W contains an essential torus T. Note that T is separating since M is hyperbolic.

Note that Q (a compression body) is of the form $T_0 \times [0, 1]$ union a 1-handle attached to $T_0 \times 1$. It is now easy to see that any incompressible torus in Q is isotopic into $T_0 \times [0, 1]$, and therefore boundary parallel. Hence T cannot be contained in Q. Obviously X^- is atoroidal because it is a twisted I-bundle over a punctured Klein bottle. So T cannot be contained in X^- either. Therefore T must intersect S. As S is incompressible in W (Lemma 12.6(2)), we may assume that every component of $S \cap T$ is a circle which is essential in both T and S. As S is separating, $T \cap S$ has an even number of components. We may further assume that each component of $T \setminus (S \cap T)$ is an essential annulus in (X^-, S) or in (Q, S) (using isotopy of T to eliminate inessential ones), and thus can be further assumed to be a vertical annulus in the characteristic I-bundle of (X^-, S) or (Q, S). Note that the characteristic I-bundle for the pair (Q, S) is isotopic to a regular neighbourhood of $B \cup \partial_0 H^+$ in Q such that the horizontal boundary of the I-bundle is a twice-punctured annulus ϕ contained in S such that $\widehat{\phi}$ is an essential annulus in $\widehat{S}$, and the vertical boundary of the I-bundle has two components: one is $\partial_0 H^+$ and the other is the frontier of the I-bundle in Q. So we may assume that $S \cap T$ is contained in ϕ.

Let A be a component of $T \setminus (T \cap S)$. It's easy to see that ∂A is $\widehat{S}$-essential, for otherwise A would be isotopic to $\partial_0 H^\epsilon$ and T would be parallel to ∂M. Now if A is contained in Q, its two boundary components are either isotopic in ϕ to the two inner boundary components of ϕ respectively or bound an annulus in ϕ which separates ϕ into two once-punctured annuli. Moreover, A is a vertical annulus in the Seifert fibred structure of $\widehat{Q}$. If A is contained in X^-, it is a vertical annulus in one of the two Seifert fibred structures of $\widehat{X}^-$. So the Seifert structure of $\widehat{Q}$ matches a Seifert structure of $\widehat{X}^-$. By Lemma 12.5, the Seifert structure of $\widehat{X}^-$ must be the one whose base orbifold is a Möbius band. Thus if a component A of

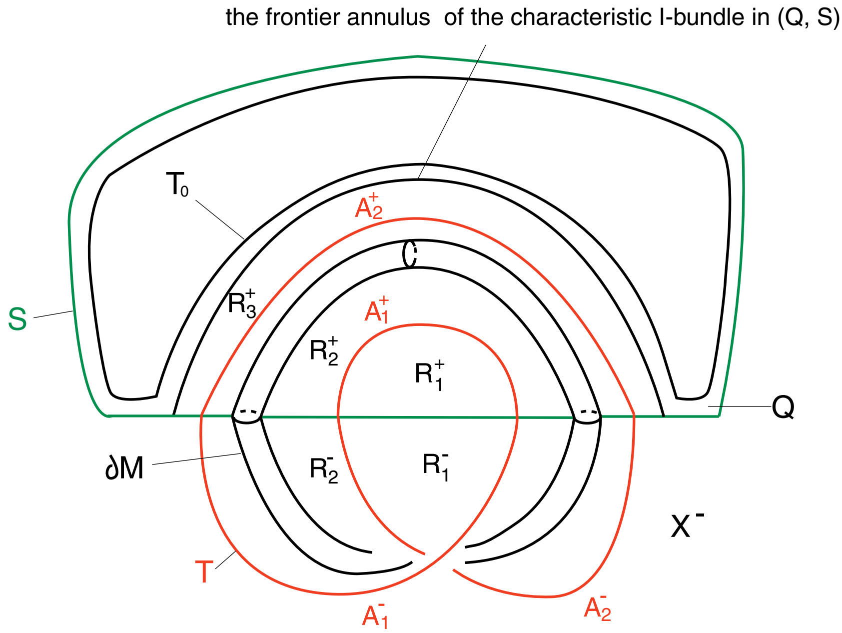

FIGURE 41

$T \setminus (S \cap T)$ is contained in X^-, it is a non-separating annulus in X^-. In particular, if A is contained in X^-, ∂A cannot be parallel in S. For otherwise the union of A with the annulus in S bounded by ∂A would be a Klein bottle in $W \subset M$, giving a contradiction.

With the above information we have obtained on the components of $T \setminus (S \cap T)$ we see that the following case must occur: $T \setminus (S \cap T)$ has exactly four components, two in Q which we denote by A_1^+ and A_2^+, and two in X^- which we denote by A_1^- and A_2^-, and they are connected as shown in Figure 41. More specifically, the annuli A_1^+ and A_2^+ separate Q into three components R_1^+, R_2^+, R_3^+ such that R_1^+ is a solid torus in which A_1^+ has winding number 2, R_2^+ contains $\partial_0 H^+$ and is a product I-bundle over a once-punctured annulus, and R_3 is a regular neighbourhood of T_0. The annuli A_1^- and A_2^- separate X^- into two components R_1^-, R_2^- such that R_1^- contains $\partial_0 H^-$ and is a product I-bundle over a once-punctured annulus, and R_2^- is a solid torus (cf. Figure 41). Moreover, $R_2^+ \cup R_2^-$ is a once-punctured annulus bundle over S^1 with finite order monodromy and thus is Seifert fibred. In fact, one can see that the monodromy has order two. On the other hand, $R_1^+ \cup R_1^- \cup R_3^+$ is Seifert fibred over an annulus with one cone point of order two. Hence W is a graph manifold. But $M = W \cup N$ is hyperbolic. We get a contradiction. □

Lemma 12.8. *$W(\partial M, \beta)$ contains an essential annulus which is the cap-off of the once-punctured annulus given in part* (3) *of Lemma* 12.6*.*

Proof. Note that the punctured annulus given in part (3) of Lemma 12.6 is non-separating in W. So it caps off to a non-separating annulus in $W(\partial M, \beta)$. If this annulus is inessential in $W(\partial M, \beta)$, then it must be compressible, from which we may get a compressing disk for T_0 in $W(\partial M, \beta)$. That is, β becomes a boundary-reducing Dehn filling slope on ∂M for W. On the other hand, α is a toroidal filling slope on ∂M for W by Lemma 12.6(4). Hence by [GL], we have $\Delta(\alpha, \beta) \leq 2$. But this contradicts the assumption that $\Delta(\alpha, \beta) = 4$. Thus the above annulus is essential in $W(\partial M, \beta)$. □

Now we have shown that W is hyperbolic, and for $(W, \partial M)$, α is a toroidal filling slope and β an annular filling slope. Furthermore $W(\partial M, \beta)$ contains an essential annulus whose intersection with ∂M has only one component. Applying [GW2, Theorem 1.1], we see that W is the Whitehead link exterior.

So $W \cong Wh$. By tubing off the once-punctured annulus in W (given by Lemma 12.6(3)) with an annulus in T_0, we get a once-punctured torus in $(W, \partial M)$ with slope β. So β corresponds to the zero slope with respect to the standard coordinates on ∂Wh. Similarly we see that α is the slope -4. As $\widehat{Q}$ is Seifert fibred over an annulus with a single cone point, $\widehat{X}^+ = \widehat{Q} \cup_{T_0} N$ is a solid torus if and only if the filling slope on T_0 is distance one from the Seifert slope of $\widehat{Q}$ on T_0. This Seifert slope is unique. From the examples given in §11, we see that the Seifert slope of $\widehat{Q}$ on T_0 is -2 and those examples are the only examples realizing Theorem 1.3(1). That is, we have $(M; \alpha, \beta) \cong (Wh(\frac{-2n \pm 1}{n}); -4, 0)$ for some integer n with $|n| > 1$.

13. Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Let M be a hyperbolic knot manifold containing an essential once-punctured torus F_β with boundary slope β. Let γ be an exceptional slope on ∂M.

We may suppose that the capped-off torus $\hat{F}_\beta$ is incompressible in $M(\beta)$ by Proposition 3.1. Now $M(\gamma)$ is either reducible, small Seifert, or toroidal. In the first case $\Delta(\beta, \gamma) = 1$ by [BZ1, Lemma 4.1], while in the second case Theorem 1.3 implies that $\Delta(\beta, \gamma) \leq 5$ with equality only if $(M; \gamma, \beta) \cong (Wh(-3/2); -5, 0)$ and $M(\gamma)$ has base orbifold $S^2(2, 3, 3)$, and $\Delta(\beta, \gamma) = 4$ only if $(M; \gamma, \beta) \cong (Wh(\frac{-2n \pm 1}{n}); -4, 0)$ for some integer n with $|n| > 1$ and $M(\gamma)$ has base orbifold $S^2(2, 2, |\mp 2n - 1|)$.

So suppose that $M(\gamma)$ is toroidal. We then have a punctured torus F_γ in M with boundary slope γ such that the capped-off torus $\hat{F}_\gamma$ in $M(\gamma)$ is incompressible. Assume that n_γ, the number of boundary components of F_γ, is minimal over all such punctured tori. Similarly, assuming for the moment only that $M(\beta)$ is toroidal, we have a punctured torus F_β in M with boundary slope β and n_β boundary components. Triples $(M; F_\beta, F_\gamma)$ of this kind with $\Delta(\beta, \gamma) \geq 4$ are classified in [Go1] (in the case $\Delta(\beta, \gamma) \geq 6$) and [GW] (in the case $\Delta(\beta, \gamma) = 4$ or 5). In particular, it is shown in [GW] that if M is a hyperbolic knot manifold with a once-punctured torus slope β and a toroidal slope γ with $\Delta(\beta, \gamma) = 4$, then $(M; \gamma, \beta) \cong (Wh(\delta); -4, 0)$ for some slope δ on the other boundary component of Wh. This proves part (3)(a) of the theorem.

The only examples with $n_\beta = 1$ and $\Delta(\beta, \gamma) \geq 5$ are $M = Wh(-5/2)$, with $\Delta(\beta, \gamma) = 7$ [Go1], and $M = M_5$ or M_{10} in [GW], with $\Delta(\beta, \gamma) = 5$. In fact the only examples with $\Delta(\beta, \gamma) = 5$ where $M(\beta)$ (say) contains a non-separating torus are M_5, M_{10} and M_{11} (see [GW, Lemma 23.1]). Now in [MP] three examples of hyperbolic knot manifolds are given, each with a pair of toroidal fillings at distance 5, one of which contains a non-separating torus: these are $Wh(-7/2), Wh(-4/3)$ and $N(-5, 5)$, described in Tables A.3, A.4 and A.9, respectively. By comparing the description in these tables of the second toroidal filling at distance 5 with that given in [GW, Lemma 22.2], we see that $Wh(-7/2) = M_{10}$, $N(-5,5) = M_{11}$, and (hence) $Wh(-4/3) = M_5$. It is well known that $Wh(\delta)$ contains a once-punctured essential torus of slope 0. The determination of the slopes γ, β as listed in parts (3)(b) and (3)(c) has been done by Martelli and Petronio. See [MP, Tables A.2 and A.3]. □

Proof of Theorem 1.5. Let $K \subset S^3$ be a hyperbolic knot of genus one with exterior M_K and suppose p/q is an exceptional filling slope on ∂M_K where $q \geq 1$.

Hyperbolic genus one knots in the 3-sphere do not admit reducible surgery slopes [BZ1], so an exceptional surgery slope is either toroidal or irreducible, atoroidal, small Seifert. If K is fibred, it is necessarily the figure eight knot, and the theorem holds in this case. Assume that K is not a fibred knot. Then

(a) $M_K(0)$ is not fibred [Ga],
(b) K admits no L-space surgery [Ni],
(c) K is not a Eudave-Muñoz knot [E-M].

A genus one Seifert surface for K completes to an essential torus in $M_K(0)$ [Ga]. Suppose that $M_K(0)$ is Seifert fibred. As its first homology group is infinite cyclic, its base orbifold must have underlying space S^2 and $M_K(0)$ must have a non-zero Euler number. Thus it admits a non-separating, horizontal surface, which implies $M_K(0)$ fibres over the circle, contrary to (a). Thus $M_K(0)$ is not Seifert fibred, so assertion (1) of the theorem holds.

By (b), K has no finite surgery slopes. Thus if $M_K(p/q)$ is small Seifert with base orbifold $S^2(a,b,c)$, then $p \neq 0$ and (a,b,c) is either a Euclidean or hyperbolic triple, so $|p| \leq 3$ by Theorem 1.3. Consideration of $H_1(S^2(a,b,c))$ shows that (a,b,c) is a hyperbolic triple. Hence assertion (2) of the theorem holds.

Theorem 1.3 combines with (b) and assertion (2) to show that if $M_K(p/q)$ is small Seifert then $0 < |p| \leq 3$. Thus assertion (3) of the theorem holds.

Since K is not a Eudave-Muñoz knot, each toroidal slope of K is integral. It follows from [Go1] and [Te1] that no genus one knot in the 3-sphere admits a toroidal filling slope of distance 5 or more from the longitude. Such knots with toroidal slopes of distance 4 are determined in [GW, Theorem 24.4]. In particular, all such knots are twist knots and the non-longitudinal slope is ± 4. This proves assertion (4). □

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Département de Mathématiques, Université du Québec à Montréal, 201 avenue du Président-Kennedy, Montréal, Québec, Canada H2X 3Y7
E-mail address: boyer.steven@uqam.ca
URL: http://www.cirget.uqam.ca/boyer/boyer.html

Department of Mathematics, University of Texas at Austin, 1 University Station, Austin, Texas 78712
E-mail address: gordon@math.utexas.edu
URL: http://www.ma.utexas.edu/text/webpages/gordon.html

Department of Mathematics, SUNY at Buffalo, Buffalo, New York 14214-3093
E-mail address: xinzhang@buffalo.edu
URL: http://www.math.buffalo.edu/~xinzhang