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Reducible and finite Dehn fillings

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Abstract

We show that the distance between a finite filling slope and a reducible filling slope on the boundary of a hyperbolic knot manifold is one.

Introduction

Let M be a knot manifold, that is, a connected, compact, orientable 3-manifold whose boundary is a torus. A knot manifold is said to be *hyperbolic* if its interior admits a complete hyperbolic metric of finite volume. Let $M(\alpha)$ denote the manifold obtained by Dehn filling M with slope α, and let $\Delta(\alpha, \beta)$ denote the distance between two slopes α and β on ∂M. When M is hyperbolic but $M(\alpha)$ is not, we call the corresponding filling (slope) an *exceptional* filling (slope). Perelman's recent proof of Thurston's geometrization conjecture implies that a filling is exceptional if and only if it is either reducible, toroidal, or Seifert-fibered. These include all manifolds whose fundamental groups are either cyclic, finite, or very small (that is, they contain no non-abelian free subgroup). Sharp upper bounds on the distance between exceptional filling slopes of various types have been established in many cases, including:

- $\Delta(\alpha, \beta) \leqslant 1$ if both α and β are reducible filling slopes (see [**11**]);
- $\Delta(\alpha, \beta) \leqslant 1$ if both α and β are cyclic filling slopes (see [**8**]);
- $\Delta(\alpha, \beta) \leqslant 1$ if α is a cyclic filling slope and β is a reducible filling slope (see [**4**]);
- $\Delta(\alpha, \beta) \leqslant 2$ if α is a cyclic filling slope and β is a finite filling slope (see [**3**]);
- $\Delta(\alpha, \beta) \leqslant 2$ if α is a reducible filling slope and β is a very small filling slope (see [**1**]);
- $\Delta(\alpha, \beta) \leqslant 3$ if both α and β are finite filling slopes (see [**5**]);
- $\Delta(\alpha, \beta) \leqslant 3$ if α is a reducible filling slope and β is a toroidal filling slope (see [**13**, **14**]);
- $\Delta(\alpha, \beta) \leqslant 8$ if both α and β are toroidal filling slopes (see [**9**]).

In this paper, we give the sharp upper bound on the distance between a reducible filling slope and a finite filling slope.

Theorem 1. *Let M be a hyperbolic knot manifold. If $M(\alpha)$ has a finite fundamental group and $M(\beta)$ is a reducible manifold, then $\Delta(\alpha, \beta) \leqslant 1$.*

Example 7.8 of [**4**] describes a hyperbolic knot manifold M and slopes $\alpha_1, \alpha_2, \beta$ on ∂M such that $M(\beta)$ is reducible, $\pi_1(M(\alpha_1))$ is finite cyclic, $\pi_1(M(\alpha_2))$ is finite non-cyclic, and $\Delta(\alpha_1, \beta) = \Delta(\alpha_2, \beta) = 1$. In fact there are hyperbolic knot manifolds with reducible and finite fillings for every finite type: cyclic, dihedral, tetrahedral, octahedral, and icosahedral in the terminology of [**3**]; see [**15**].

A significant reduction of Theorem 1 was obtained in [**1**]. Before describing this work, we need to introduce some notation and terminology.

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Denote the octahedral group by O, the binary octahedral group by O^*, and let $\varphi : O^* \to O$ be the usual surjection. We say that α is an *$O(k)$-type* filling slope if $\pi_1(M(\alpha)) \cong O^* \times \mathbb{Z}/j$ for some integer j coprime to 6 and the image of $\pi_1(\partial M)$ under the composition $\pi_1(M) \to \pi_1(M(\alpha)) \xrightarrow{\cong} O^* \times \mathbb{Z}/j \xrightarrow{\text{proj}} O^* \xrightarrow{\varphi} O$ is $\mathbb{Z}/k$. Clearly $k \in \{1, 2, 3, 4\}$. It is shown in [**5**, Section 3] that k is independent of the choice of isomorphism $\pi_1(M(\alpha)) \xrightarrow{\cong} O^* \times \mathbb{Z}/j$.

A lens space whose fundamental group has order $p \geqslant 2$ will be denoted by L_p.

THEOREM 2. *Let M be a hyperbolic knot manifold. If $M(\alpha)$ has a finite fundamental group and $M(\beta)$ is a reducible manifold, then $\Delta(\alpha, \beta) \leqslant 2$. Further, if $\Delta(\alpha, \beta) = 2$, then $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, $M(\beta) \cong L_2 \# L_3$, and α is an $O(k)$-type filling slope for some $k \in \{1, 2, 3\}$.*

Proof. This is Theorem 1.1 of [**1**] except that that theorem only claimed that α is an $O(k)$-type filling slope for some $k \in \{1, 2, 3, 4\}$. Since $H_1(M)$ contains 2-torsion, the argument in the last paragraph of the proof of [**3**, Theorem 2.3, p. 1026] shows that $k \in \{1, 2, 3\}$. □

Thus, in order to prove Theorem 1, we are reduced to considering the case where $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, $M(\beta) \cong L_2 \# L_3$, and α is an $O(k)$-type filling slope for some $k \in \{1, 2, 3\}$. We do this below. We also assume that $\Delta(\alpha, \beta) = 2$ in order to derive a contradiction.

An *essential surface* in M is a compact, connected, orientable, incompressible, and non-boundary parallel, properly embedded 2-submanifold of M. A slope β on ∂M is called a *boundary slope* if there is an essential surface F in M with non-empty boundary of the given slope β. A boundary slope β is called *strict* if there is an essential surface F in M of boundary slope β such that F is neither a fiber nor a semi-fiber. The latter means that F does not split M into the union of two twisted I-bundles. When M has a closed essential surface S, let $\mathcal{C}(S)$ be the set of slopes γ on ∂M such that S is compressible in $M(\gamma)$. A slope η is called a *singular slope* for S if $\eta \in \mathcal{C}(S)$ and $\Delta(\eta, \gamma) \leqslant 1$ for each $\gamma \in \mathcal{C}(S)$.

Since $\pi_1(M(\alpha))$ is finite, the first Betti number of M is 1, $M(\alpha)$ is irreducible by [**11**], and neither α nor β is a singular slope by [**2**, Theorem 1.5]. As $M(\beta)$ is reducible, β is a boundary slope. Further, by [**1**, Proposition 3.3] we may assume that up to isotopy, there is a unique essential surface P in M with boundary slope β. This surface is necessarily planar. It is also separating as $M(\beta)$ is a rational homology 3-sphere, and so has an even number of boundary components. This number is at least 4 since M is hyperbolic.

LEMMA 3. *If $\Delta(\alpha, \beta) = 2$, then α is of type $O(2)$.*

Proof. According to Theorem 2, we must show that α does not have type $O(k)$ for $k = 1, 3$.

Let $X_0 \subset X(M(\beta)) \subset X(M)$ be the unique non-trivial curve. (We refer the reader to [**1**, Section 6] for notation, background results, and further references on $\mathrm{PSL}_2(\mathbb{C})$ character varieties.) Since β is not a singular slope, [**4**, Proposition 4.10] implies that the regular function $f_\alpha : X_0 \to \mathbb{C}, \chi_\rho \mapsto (\mathrm{trace}(\rho(\alpha)))^2 - 4$, has a pole at each ideal point of X_0. (We have identified $\alpha \in H_1(\partial M)$ with its image in $\pi_1(\partial M) \subset \pi_1(M)$ under the Hurewicz homomorphism.) In particular, the Culler–Shalen seminorm $\| \cdot \|_{X_0} : H_1(\partial M; \mathbb{R}) \to [0, \infty)$ is non-zero. Hence there is a non-zero integer s_0 such that for all $\gamma \in H_1(\partial M)$ we have

$$\|\gamma\|_{X_0} = |\gamma \cdot \beta| s_0,$$

where $\gamma \cdot \beta$ is the algebraic intersection number of the two classes (cf [**1**, Identity 6.1.2]). Fix a class $\beta^* \in H_1(\partial M)$ satisfying $\beta \cdot \beta^* = \pm 1$, so in particular $\|\beta^*\|_{X_0} = s_0$. We can always find such a β^* so that

$$\alpha = \beta + 2\beta^*.$$

According to [**1**, Proposition 8.1], if $\pm\beta \neq \gamma \in H_1(\partial M)$ is a slope satisfying $\Delta(\alpha,\gamma) \equiv 0 \pmod{k}$, then $2s_0 = \|\alpha\|_{X_0} \leqslant \|\gamma\|_{X_0} = \Delta(\gamma,\beta)s_0$. Hence $\Delta(\gamma,\beta) \geqslant 2$. Consideration of $\gamma = \beta^*$ and $\gamma = \beta - \beta^*$ then shows that $k \neq 1, 3$. □

Lemma 4. *If $\Delta(\alpha,\beta) = 2$, then P has exactly four boundary components.*

Proof. We continue to use the notation developed in the proof of the previous lemma.

By [**1**, Case 1, Section 8] we have $2 \leqslant 1 + 3/s_0 < 3$, and so s_0 is either 2 or 3. We claim that $s_0 = 2$. To prove this, we shall suppose that $s_0 = 3$ and derive a contradiction.

It follows from the method of proof of [**3**, Lemma 5.6] that $\pi_1(M(\alpha)) \cong O^* \times \mathbb{Z}/j$ has exactly two irreducible characters with values in $\mathrm{PSL}_2(\mathbb{C})$ corresponding to a representation ρ_1 with image O and a representation ρ_2 with image D_3 (the dihedral group of order 6). Further, ρ_2 is the composition of ρ_1 with the quotient of O by its unique normal subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. It follows from [**1**, Proposition 7.6] that if $s_0 = 3$, the characters of ρ_1 and ρ_2 lie on X_0 and provide jumps in the multiplicity of zero of f_α over f_{β^*}. Lemma 4.1 of [**3**] then implies that both $\rho_1(\beta^*)$ and $\rho_2(\beta^*)$ are non-trivial. By the previous lemma, α is a slope of type $O(2)$. Thus $\rho_1(\beta^*)$ has order 2. Since $\rho_2(\beta^*) \neq \pm I$ and ρ_2 factors through ρ_1, it follows that $\rho_2(\beta^*)$ also has order 2.

Next, we claim that β^* lies in the kernel of the composition of ρ_2 with the abelianization $D_3 \to H_1(D_3;\mathbb{Z}/2)$. To see this, first note that β is non-zero in $H_1(M;\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ since $H_1(M(\beta);\mathbb{Z}/2) = H_1(L_2 \# L_3;\mathbb{Z}/2) = \mathbb{Z}/2$. Thus exactly one of β^* and $\beta^* + \beta$ is zero in $H_1(M;\mathbb{Z}/2)$. (Recall that duality implies that the image of $H_1(\partial M;\mathbb{Z}/2)$ in $H_1(M;\mathbb{Z}/2)$ is $\mathbb{Z}/2$.) Since β lies in the kernel of ρ_2, it follows that $\rho_2(\beta^*)$ is sent to zero in $H_1(D_3;\mathbb{Z}/2)$. But then $\rho_2(\beta^*)$ has order 3 in D_3, contrary to what we deduced in the previous paragraph. Thus $s_0 = 2$. Now apply the argument at the end of the proof of [**1**, Proposition 6.6] to see that $4 = 2s_0 \geqslant |\partial P| \geqslant 4$. Hence P has four boundary components. □

The four-punctured 2-sphere P cuts M into two components X_1 and X_2. If P_i denotes the copy of P in ∂X_i then M is the union of X_1 and X_2 with P_1 and P_2 identified by a homeomorphism $f : P_1 \to P_2$. The boundary of P cuts ∂M into four annuli $A_{11}, A_{21}, A_{12}, A_{22}$ listed in the order they appear around ∂M, where A_{11}, A_{12} are contained in X_1 and A_{21}, A_{22} are contained in X_2. The arguments given in the proof of [**1**, Lemma 4.5] show that for each i, the two annuli A_{i1} and A_{i2} in X_i are unknotted and unlinked. This means that there is a neighborhood of $A_{i1} \cup A_{i2}$ in X_i that is homeomorphic to $E_i \times I$, where E_i is a three-punctured 2-sphere and I is the interval $[0,1]$, such that $(E_i \times I) \cap P_i = (E_i \times \partial I)$, and the exterior of $E_i \times I$ in X_i is a solid torus V_i. We label the boundary components of E_i as $\partial_j E_i$ $(j = 1,2,3)$ so that $\partial_j E_i \times I = A_{ij}$ for $j = 1,2$.

Let $\hat{P}$ be the 2-sphere in $M(\beta)$ obtained from P by capping off ∂P with four meridian disks from the filling solid torus V_β. These disks cut V_β into four 2-handles H_{ij} $(i,j = 1,2)$ such that the attaching annulus of H_{ij} is A_{ij} for each i,j. Let $X_i(\beta)$ be the manifold obtained by attaching H_{ij} to X_i along A_{ij} $(j = 1,2)$. Then $X_1(\beta)$ is a once-punctured $L(2,1)$ and $X_2(\beta)$ a once-punctured $L(3,1)$.

It follows from the description above that X_1 is obtained from $E_1 \times I$ and V_1 by identifying $\partial_3 E_1 \times I$ with an annulus A_1 in ∂V_1 whose core curve is a $(2,1)$ curve in ∂V_1. We can assume that A_1 is invariant under the standard involution of V_1 whose fixed-point set is a pair of arcs contained in disjoint meridian disks of V_1. Note that the two boundary components of A_1 are interchanged under this map. Similarly, X_2 is obtained from $E_2 \times I$ and V_2 by identifying

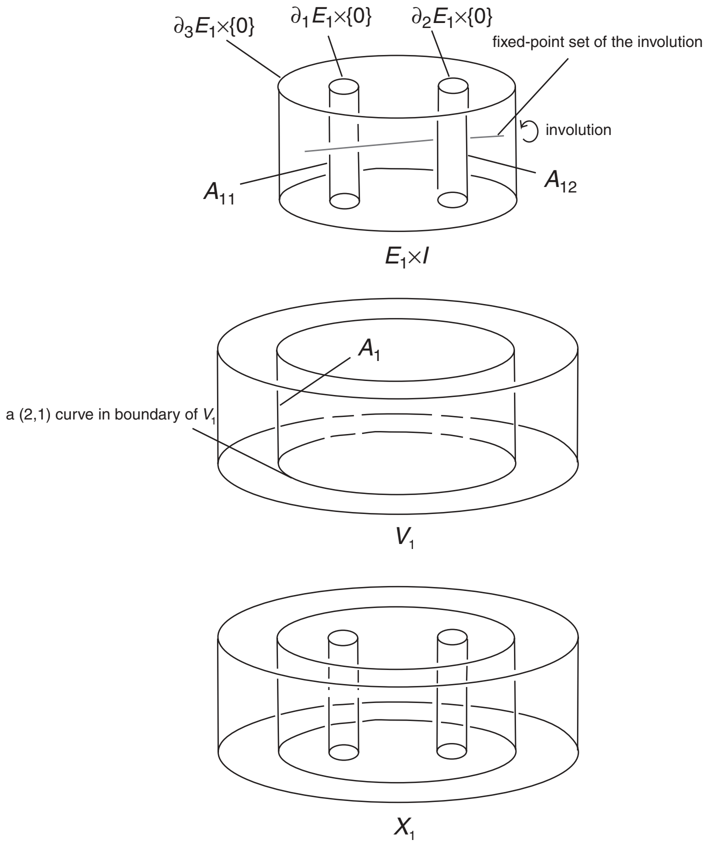

FIGURE 1. *The component X_1 and the involution τ_1. (A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.)*

$\partial_3 E_2 \times I$ with an annulus A_2 in ∂V_2 whose core curve is a (3,1) curve in ∂V_2. Again we can suppose that A_2 is invariant under the standard involution of V_2, which interchanges the two boundary components of A_2. See Figures 1 and 2.

The map $f| : \partial P_1 \to \partial P_2$ is constrained in several ways by our hypotheses. For instance, the fact that ∂M is connected implies that $f(\partial_1 E_1 \times \{i\}) = \partial_2 E_2 \times \{j\}$ for some i, j. Other conditions are imposed by the homology of M.

LEMMA 5. *We can assume that either*

(a) $f(\partial_1 E_1 \times \{0\}) = \partial_1 E_2 \times \{0\}, f(\partial_2 E_1 \times \{0\}) = \partial_2 E_2 \times \{0\}, f(\partial_1 E_1 \times \{1\}) = \partial_2 E_2 \times \{1\}$, *and* $f(\partial_2 E_1 \times \{1\}) = \partial_1 E_2 \times \{1\}$, *or*

(b) $f(\partial_1 E_1 \times \{0\}) = \partial_1 E_2 \times \{0\}, f(\partial_2 E_1 \times \{0\}) = \partial_1 E_2 \times \{1\}, f(\partial_1 E_1 \times \{1\}) = \partial_2 E_2 \times \{1\}$, *and* $f(\partial_2 E_1 \times \{1\}) = \partial_2 E_2 \times \{0\}$.

Proof. Without loss of generality, we can suppose that $f(\partial_1 E_1 \times \{0\}) = \partial_1 E_2 \times \{0\}$. Hence, as ∂M is connected, one of the following four possibilities arises:

(a) $f(\partial_2 E_1 \times \{0\}) = \partial_2 E_2 \times \{0\}, f(\partial_1 E_1 \times \{1\}) = \partial_2 E_2 \times \{1\}$, and $f(\partial_2 E_1 \times \{1\}) = \partial_1 E_2 \times \{1\}$;

(b) $f(\partial_2 E_1 \times \{0\}) = \partial_1 E_2 \times \{1\}, f(\partial_1 E_1 \times \{1\}) = \partial_2 E_2 \times \{1\}$, and $f(\partial_2 E_1 \times \{1\}) = \partial_2 E_2 \times \{0\}$;

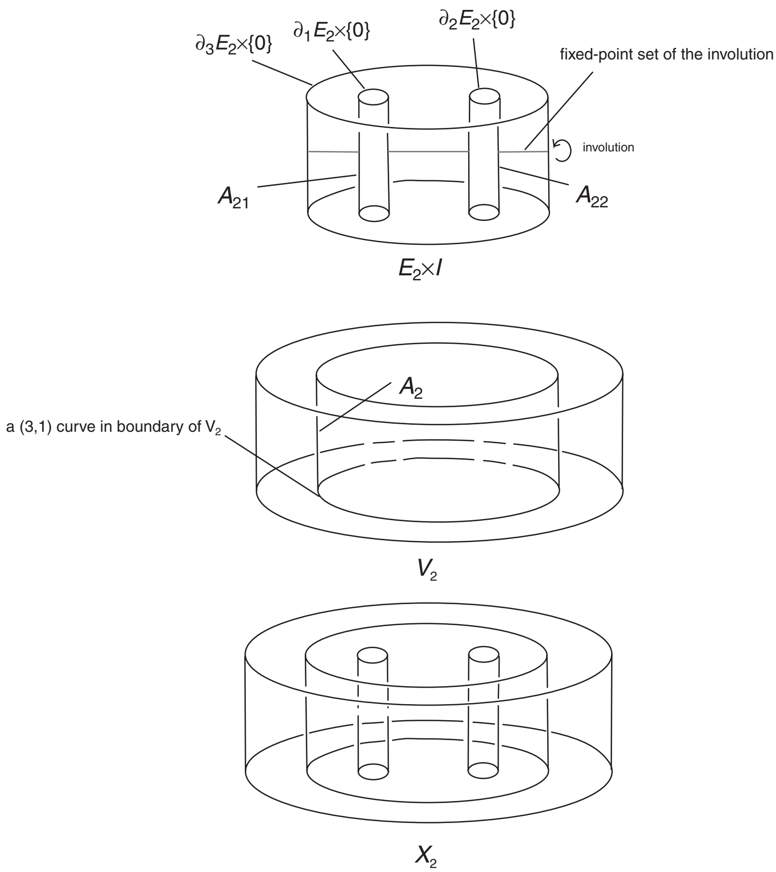

FIGURE 2. *The component X_2 and the involution τ_2. (A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.)*

(c) $f(\partial_2 E_1 \times \{0\}) = \partial_1 E_2 \times \{1\}, f(\partial_1 E_1 \times \{1\}) = \partial_2 E_2 \times \{0\}$, and $f(\partial_2 E_1 \times \{1\}) = \partial_2 E_2 \times \{1\}$; or

(d) $f(\partial_2 E_1 \times \{0\}) = \partial_2 E_2 \times \{1\}, f(\partial_1 E_1 \times \{1\}) = \partial_2 E_2 \times \{0\}$, and $f(\partial_2 E_1 \times \{1\}) = \partial_1 E_2 \times \{1\}$.

Let $a_i, b_i, x_i \in H_1(X_i)$ be represented, respectively, by $\partial_1 E_i$, $\partial_2 E_i$, and a core of V_i, $i = 1, 2$. Then $H_1(X_i)$ is the abelian group generated by a_i, b_i, x_i, subject to the relation

$$2x_1 = a_1 + b_1, \quad i = 1, \tag{1}$$

$$3x_2 = a_2 + b_2, \quad i = 2. \tag{2}$$

Since $f(\partial_1 E_1 \times \{0\}) = \partial_1 E_2 \times \{0\}$, we may orient $\partial E_1, \partial E_2$ so that in $H_1(M)$ we have $a_1 = a_2$. Then $H_1(M)$ is the quotient of $H_1(X_1) \oplus H_1(X_2)$ by this relation together with the additional relations corresponding to the four possible gluings:

(a) $b_1 = b_2$, $b_1 = a_2$;
(b) $b_1 = -a_2$, $b_1 = -b_2$;
(c) $b_1 = -a_2$, $b_1 = b_2$;
(d) $b_1 = -b_2$, $b_1 = a_2$.

Taking $\mathbb{Z}/3$ coefficients, equation (1) allows us to eliminate x_1, while (2) gives $a_2 + b_2 = 0$. Hence $H_1(M; \mathbb{Z}/3) \cong \mathbb{Z}/3 \oplus A$, where the $\mathbb{Z}/3$ summand is generated by x_2 and A is defined by generators b_1, a_2, b_2, and relations $a_2 + b_2 = 0$ plus those listed in (a)–(d) above. Thus $A = 0$

in cases (a) and (b), and $A \cong \mathbb{Z}/3$ in cases (c) and (d). Since $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, we conclude that cases (c) and (d) are impossible. $\square$

1. *The proof of Theorem* 1 *when case* (a) *of Lemma* 5 *arises*

Figure 1 depicts an involution τ_1 on $E_1 \times I$ under which $\partial_3 E_1 \times I$ is invariant, has its boundary components interchanged, and $\tau_1(A_{11}) = A_{12}$. Then τ_1 extends to an involution of X_1 since its restriction to $\partial_3 E_1 \times I = A_1$ coincides with the restriction to A_1 of the standard involution of V_1. Evidently $\tau_1(\partial_1 E_1 \times \{0\}) = \partial_2 E_1 \times \{1\}$ and $\tau_1(\partial_2 E_1 \times \{0\}) = \partial_1 E_1 \times \{1\}$.

Figure 2 depicts an involution τ_2 on $E_2 \times I$ under which each of the annuli $\partial_3 E_2 \times I, A_{21}$, and A_{22} is invariant. Further, it interchanges the components of $E_2 \times \partial I$ and as in the previous paragraph, τ_2 extends to an involution of X_2. Note that $\tau_2(\partial_j E_2 \times \{0\}) = \partial_j E_2 \times \{1\}$ for $j = 1, 2$.

Next consider the orientation-preserving involution $\tau_2' = f(\tau_1|P_1)f^{-1}$ on P_2. By construction we have $\tau_2'(\partial_j E_2 \times \{0\}) = \partial_j E_2 \times \{1\}$ for $j = 1, 2$, and therefore $\tau_2' = g(\tau_2|P_2)g^{-1}$, where $g : P_2 \to P_2$ is a homeomorphism whose restriction to ∂P_2 is isotopic to $1_{\partial P_2}$. The latter fact implies that g is isotopic to a homeomorphism $g' : P_2 \to P_2$ which commutes with $\tau_2|P_2$. Hence τ_2' is isotopic to $\tau_2|P_2$ through orientation-preserving involutions whose fixed-point sets consist of two points. In particular, τ_1 and τ_2 can be pieced together to form an orientation-preserving involution $\tau : M \to M$.

For each slope γ on ∂M, we find that τ extends to an involution τ_γ of the associated Dehn filling $M(\gamma) = M \cup V_\gamma$, where V_γ is the filling solid torus. Thurston's orbifold theorem applies to our situation and implies that $M(\gamma)$ has a geometric decomposition. In particular, $M(\alpha)$ is a Seifert fibered manifold whose base orbifold is of the form $S^2(2, 3, 4)$, a 2-sphere with three cone points of orders 2,3,4, respectively.

It follows immediately from our constructions that $X_1(\beta)/\tau_\beta$ and $X_2(\beta)/\tau_\beta$ are 3-balls. Thus $M(\beta)/\tau_\beta = (X_1(\beta)/\tau_\beta) \cup (X_2(\beta)/\tau_\beta) \cong S^3$ and since $\partial M/\tau \cong S^2$, it follows that M/τ is a 3-ball. More precisely, M/τ is an orbifold (N, L^0), where N is a 3-ball, L^0 is a properly embedded 1-manifold in N that meets ∂N in four points, and M is the double-branched cover of (N, L^0). We will call (N, L^0) a *tangle*, and if we choose some identification of $(\partial N, \partial L^0)$ with a standard model of (S^2, *four points*), then (N, L^0) becomes a *marked* tangle. Capping off ∂N with a 3-ball B gives $N \cup_\partial B \cong S^3$. Then, if γ is a slope on ∂M, we have $V_\gamma/\tau_\gamma \cong (B, T_\gamma)$, where T_γ is the rational tangle in B corresponding to the slope γ. Hence

$$\begin{aligned} M(\gamma)/\tau_\gamma &= (M/\tau) \cup (V_\gamma/\tau_\gamma) \\ &= (N, L^0) \cup (B, T_\gamma) \\ &= (S^3, L^0(\gamma)), \end{aligned}$$

where $L^0(\gamma)$ is the link in S^3 obtained by capping off L^0 with the rational tangle T_γ.

We now give a more detailed description of the tangle (N, L^0). For $i = 1, 2$, let $B_i = V_i/\tau_i$, $W_i = E_i \times I/\tau_i$, $Y_i = X_i/\tau_i$, and $Q_i = P_i/\tau_i$. Figure 3 gives a detailed description of the branch sets in B_i, W_i, Y_i with respect to the corresponding branched covering maps. Note that N is the union of Y_1, Y_2, and a product region $R \cong Q_1 \times I$ from Q_1 to Q_2 that intersects the branch set L^0 of the cover $M \to N$ in a 2-braid. In fact, it is clear from our constructions that we can think of the union $(L^0 \cap R) \cup (\partial N \cap R)$ as a '4-braid' in R with two 'fat strands' formed by $\partial N \cap R$; see Figure 4(a). By an isotopy of R fixing Q_2, and which keeps R, Q_1, and Y_1 invariant, we may untwist the crossings between the two fat strands in Figure 4(a) so that the pair (N, L^0) is as depicted in Figure 4(b).

The slope β is the boundary slope of the planar surface P, and hence the rational tangle T_β appears in Figure 4(b) as two short horizontal arcs in B lying entirely in $Y_2(\beta) = X_2(\beta)/\tau_\beta$. Since $\Delta(\alpha, \beta) = 2$, we have that T_α is a tangle of the form shown in Figure 5(a). Recall that

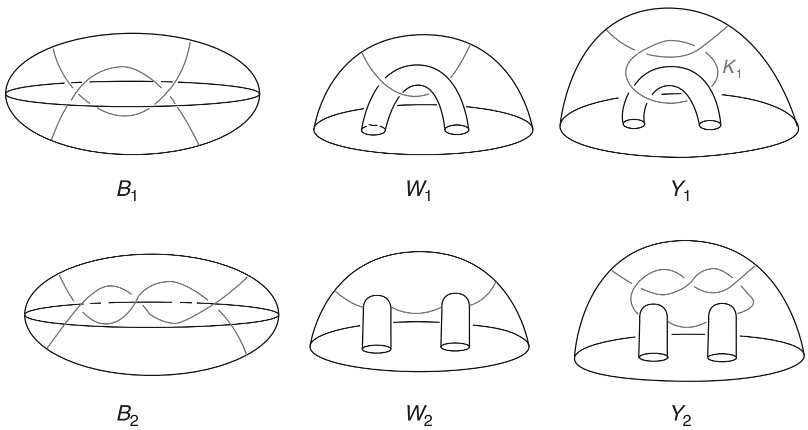

Figure 3. *The branch sets in B_i, W_i, and Y_i. (A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.)*

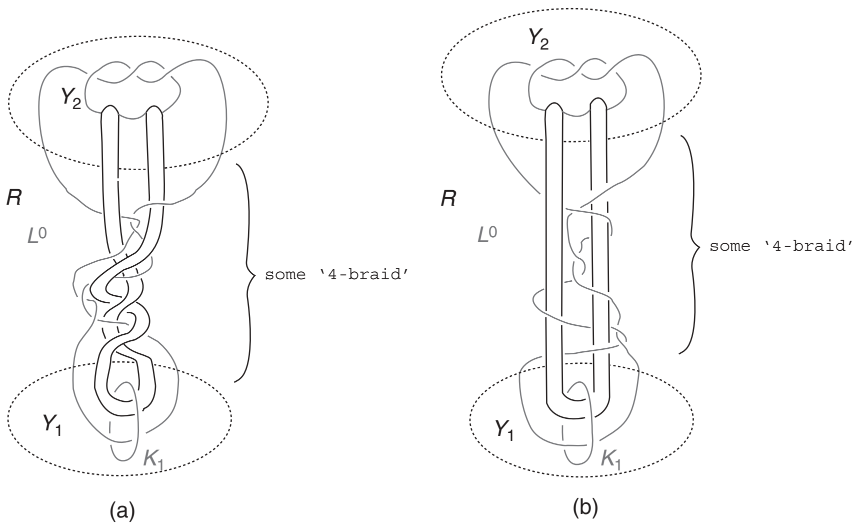

Figure 4. *The branch set L^0 in N. (A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.)*

$M(\alpha)$ is a Seifert fibered manifold with base orbifold of type $S^2(2, 3, 4)$, and is the double-branched cover of $(S^3, L^0(\alpha))$. Write $L = L^0(\alpha)$.

Lemma 6. *The link L is a Montesinos link of type $(p/2, q/3, r/4)$.*

Proof. By Thurston's orbifold theorem, the Seifert-fibering of $M(\alpha)$ can be isotoped to be invariant under τ_α. Hence the quotient orbifold is Seifert-fibered in the sense of Bonahon–Siebenmann, and so either L is a Montesinos link or $S^3 \setminus L$ is Seifert-fibered. From Figure 5(a) we see that L is a 2-component link with an unknotted component and linking number ± 1. However, the only link L with this property such that $S^3 \setminus L$ is Seifert-fibered is the Hopf link (see [**6**]), whose 2-fold cover is P^3. Thus L must be a Montesinos link. Since the base orbifold of $M(\alpha)$ is $S^2(2, 3, 4)$, it follows that L has type $(p/2, q/3, r/4)$ (cf. [**7**, Section 12.D]). $\square$

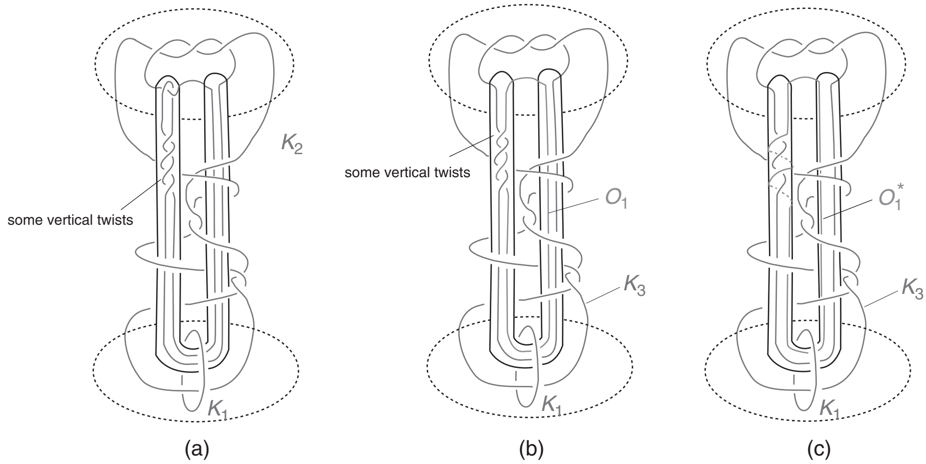

FIGURE 5. *The tangle fillings $N(\alpha)$ and $N(\lambda)$. (A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.)*

It is easy to check that any Montesinos link L of the type described in Lemma 6 has two components, one of which, say K_1, is a trivial knot, and the other, K_2, a trefoil knot. Our aim is to use the particular nature of our situation to show that the branch set L cannot be a Montesinos link of type $(p/2, q/3, r/4)$, and thus derive a contradiction.

From Figure 4, we see that L^0 has a closed, unknotted component, which must be the component K_1 of the Montesinos link of type $(p/2, q/3, r/4)$ described above. Then $L^0 \setminus K_1 = K_2 \cap N$, which we denote by K_2^0.

Now delete K_1 from N and let U be the double-branched cover of N branched over K_2^0. Then U is a compact, connected, orientable 3-manifold with boundary a torus that can be identified with ∂M. In particular, if we consider α and β as slopes on ∂U, then both $U(\alpha)$ and $U(\beta)$ are the lens space $L(3,1)$, since they are 2-fold covers of S^3 branched over a trefoil knot. Hence the cyclic surgery theorem of [**8**] implies that U is either a Seifert-fibered space or a reducible manifold.

LEMMA 7. *The double-branched cover U is not a Seifert-fibered space.*

Proof. Suppose that U is a Seifert-fibered space, with base surface F and $n \geqslant 0$ exceptional fibers. If F is non-orientable then U contains a Klein bottle, and hence $U(\alpha) \cong L(3,1)$ does too. However, since non-orientable surfaces in $L(3,1)$ are non-separating, this implies that $H_1(L(3,1); \mathbb{Z}/2) \ncong 0$, which is clearly false. Thus F is orientable.

If U is a solid torus, then clearly $U(\alpha) \cong U(\beta) \cong L(3,1)$ implies $\Delta(\alpha,\beta) \equiv 0 \pmod 3$, contradicting the fact that $\Delta(\alpha,\beta) = 2$. Thus we assume that U is not a solid torus, and take $\phi \in H_1(\partial U)$ to be the slope on ∂U of a Seifert fiber. Then $U(\phi)$ is reducible [**12**] so $d = \Delta(\alpha,\phi) > 0$, and $U(\alpha)$ is a Seifert-fibered space with base surface F capped off with a disk, and n or $n+1$ exceptional fibers, according to $d = 1$ or $d > 1$. Since $U(\alpha)$ is a lens space and U is not a solid torus, we must have that F is a disk, $n = 2$, and $d = 1$. Similarly $\Delta(\beta,\phi) = 1$. In particular, without loss of generality $\beta = \alpha + 2\phi$ in $H_1(\partial U)$.

The base orbifold of U is of the form $D^2(p,q)$, with $p, q > 1$. Then $H_1(U)$ is the abelian group defined by generators x, y and the single relation $px + qy = 0$. Suppose that $\alpha \mapsto ax + by$ in $H_1(U)$. Then $H_1(U(\alpha))$ is presented by the matrix $\left(\begin{smallmatrix} p & a \\ q & b \end{smallmatrix}\right)$. Similarly, since $\phi \mapsto px$ in $H_1(U)$, it

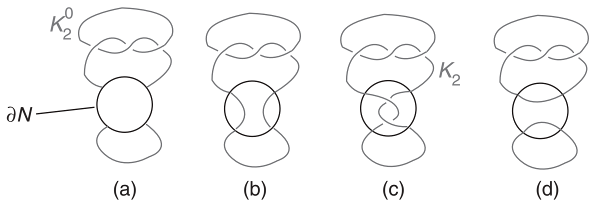

Figure 6. *The tangle* (N, K_2^0) *and its* β, α, *and* λ *fillings.* (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.*)

follows that $H_1(U(\beta))$ is presented by $\begin{pmatrix} p & a+2p \\ q & b \end{pmatrix}$. However, the determinants of these matrices differ by $2pq \geqslant 8$, so they cannot both be 3 in absolute value. This completes the proof of the lemma. $\square$

Thus U is reducible, say $U \cong V \# W$, where $\partial V = \partial U$ and $W \ncong S^3$ is closed. Consideration of $U(\alpha)$ and $U(\beta)$ shows that $W \cong L(3,1)$ and $V(\alpha) \cong V(\beta) \cong S^3$, and so Theorem 2 of [**10**] implies that $V \cong S^1 \times D^2$. It follows that any simple closed curve in ∂U which represents either α or β is isotopic to the core curve of V. Let $\lambda \in H_1(\partial U)$ denote the meridional slope of V. Then $\{\beta, \lambda\}$ is a basis of $H_1(\partial U)$ and up to changing the sign of α we have $\alpha = \beta \pm 2\lambda$.

Since $U \cong (S^1 \times D^2) \# L(3,1)$, we can find a homeomorphism between the pair (N, K_2^0) and the tangle shown in Figure 6(a), with the β, α, and λ fillings shown in Figures 6(b), (c) and (d), respectively. (We show the case $\alpha = \beta + 2\lambda$; the other possibility can be handled similarly.)

Recall that in Figure 4(b), the slope β corresponds to the rational tangle consisting of two short 'horizontal' arcs in the filling ball B. It follows that under the homeomorphism from the tangle shown in Figure 6(a) to (N, K_2^0) shown in Figure 4(b), the tangles T_α and T_λ are sent to rational tangles of the forms shown in Figure 5(a) and (b), respectively. From Figure 6(d) we see that $L^0(\lambda)$ is a link of three components $K_1 \cup O_1 \cup K_3$, where O_1 is a trivial knot which bounds a disk D disjoint from K_3 and which intersects ∂N in a single arc; see Figure 5(b). Push the arc $O_1 \cap B$ with its two endpoints fixed into ∂B along D, and let O_1^* be the resulting knot (see Figure 5(c)). Then there is a disk D_* (which is a subdisk of D) satisfying the following conditions:

(1) $\partial D_* = O_1^*$;
(2) D_* is disjoint from K_3;
(3) the interior of D_* is disjoint from B.

Perusal of Figure 5(c) shows that the following condition is also achievable.

(4) $D_* \cap Q_2$ has a single arc component, and this arc component connects the two boundary components of Q_2 and is outermost in D_* amongst the components of $D_* \cap Q_2$.

Among all disks in S^3 which satisfy conditions (1)–(4), we may assume that D_* has been chosen so that

(5) $D_* \cap Q_2$ has the minimal number of components.

Claim 8. *Suppose that* $D_* \cap Q_2$ *has circle components. Then each such circle separates* $K_3 \cap Q_2$ *from* ∂Q_2 *in* Q_2.

Proof. Let δ be a circle component of $D_* \cap Q_2$. Then δ is essential in $Q_2 \setminus (Q_2 \cap K_3)$, for if it bounds a disk D_0 in $Q_2 \setminus (Q_2 \cap K_3)$, then an innermost component of $D_* \cap D_0 \subset D_* \cap Q_2$ will bound a disk $D_1 \subset D_0$. We can surger D_* using D_1 to get a new disk satisfying

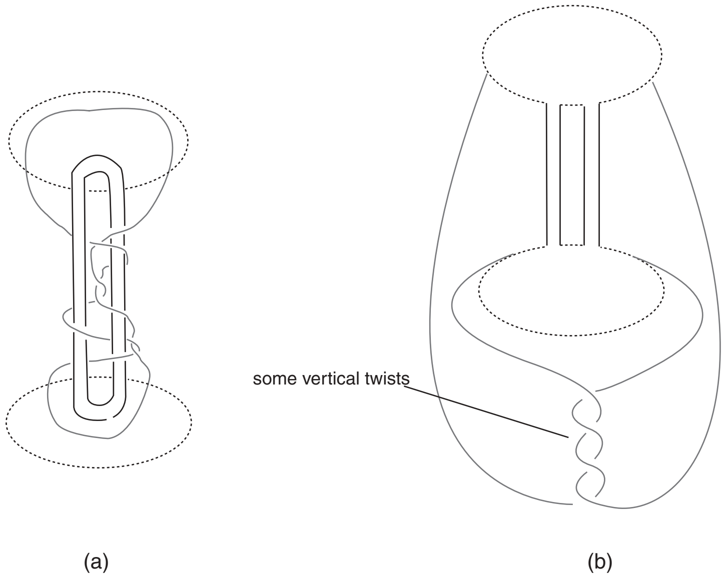

FIGURE 7. *Capping off the* 4*-braid to obtain a trivial link.* (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.*)

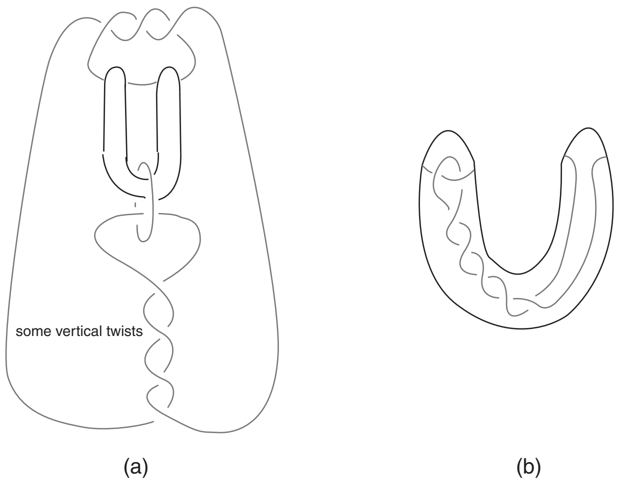

FIGURE 8. *The pair* (N, L^0) *and the filling tangle* T_α. (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.*)

conditions (1)–(4) above, but with fewer components of intersection with Q_2 than D_*, contrary to assumption (5).

Next since the arc component of $D_* \cap Q_2$ connects the two boundary components of Q_2, δ cannot separate the two boundary components of Q_2 from each other.

Lastly, suppose that δ separates the two points of $Q_2 \cap K_3$. Then δ is isotopic to a meridian curve of K_3 in S^3. However, this is impossible since δ also bounds a disk in D_* and is therefore null-homologous in $S^3 \setminus K_3$. The claim follows. □

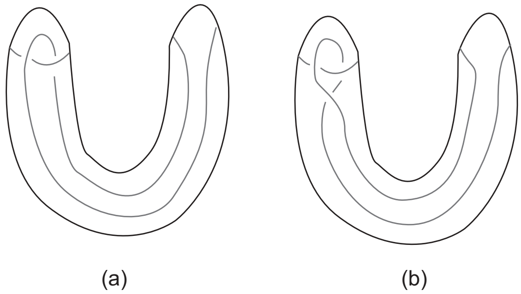

FIGURE 9. *The two possible* T_α. (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.*)

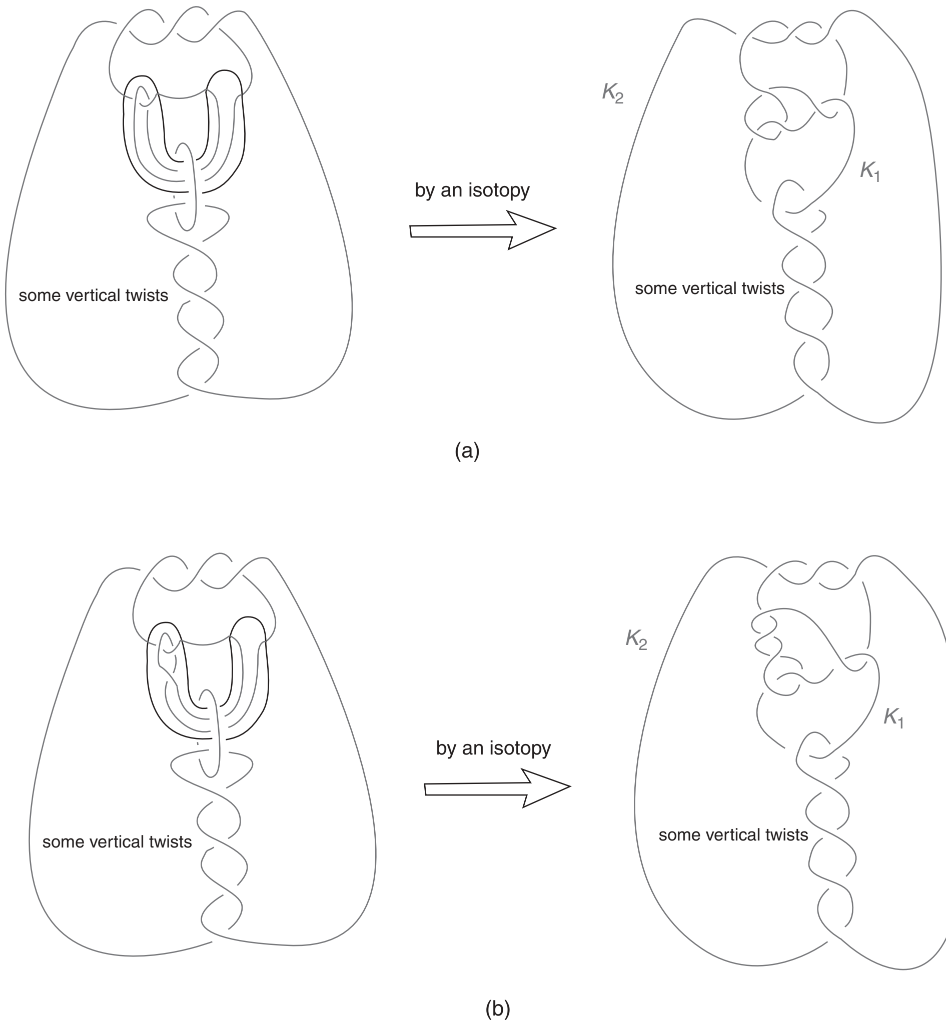

FIGURE 10. *Representation of* L *as a Montesinos link of the type* $(1/3, -3/8, m/2)$ *or* $(1/3, -5/8, m/2)$. (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.*)

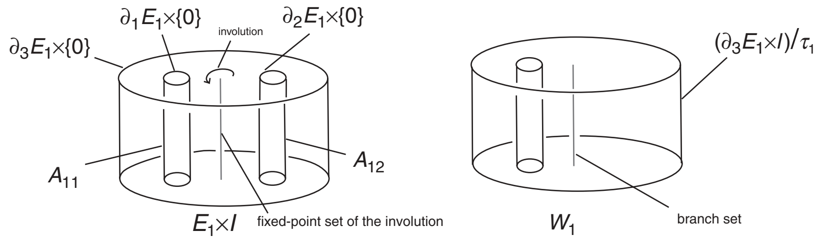

FIGURE 11. *Involution on* $E_1 \times I$. (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online*.)

It follows from Claim 8 that there are disjoint arcs in Q_2: one, say σ_1, which connects the two points of $Q_2 \cap K_3$ and is disjoint from D_*, and $\sigma_2 = D_* \cap Q_2$ the other. Hence we obtain a '2-bridge link' of two components — one fat and one thin — in S^3 by capping off the '4-braid' in R with σ_1 and σ_2 in $Y_2 \subset Y_2(\beta)$ and with $K_3 \cap Y_1 \subset Y_1$ and $\partial N \cap Y_1 \subset Y_1$ in the 3-ball $Y_1(\beta)$ (see Figure 7(a)). Furthermore, since the disk D_* gives a disk bounded by the 'fat knot' which is disjoint from the 'thin knot', the link is a trivial link.

Now it follows from the standard presentation of a 2-bridge link as a 4-plat (see [**7**, Section 12.B]), that there is an isotopy of R, fixed on the ends Q_1, Q_2 and on the two fat strands, taking the '4-braid' to one of the form shown in Figure 7(b). Hence (N, L^0) has the form shown in Figure 8(a). The filling rational tangle T_α is of the form shown in Figure 8(b). Since the component $K_2^0(\alpha)$ of $L^0(\alpha) = L$ has to be a trefoil, there are only two possibilities for the number of twists in T_α; see Figure 9. The two corresponding possibilities for L are shown in Figure 10. However, these are Montesinos links of the form $(1/3, -3/8, m/2)$ and $(1/3, -5/8, m/2)$, respectively.

This final contradiction completes the proof of Theorem 1 under the assumptions of case (a) of Lemma 5.

2. *The proof of Theorem* 1 *when case* (b) *of Lemma* 5 *arises*

In this case we choose an involution τ_1 on $E_1 \times I$ as shown in Figure 11. Then $\tau_1(\partial_3 E_1 \times \{j\}) = \partial_3 E_1 \times \{j\}$, $\tau_1(\partial_1 E_1 \times \{j\}) = \partial_2 E_1 \times \{j\}$ $(j = 0, 1)$, and the restriction of τ_1 on $\partial_3 E_1 \times I$ extends to an involution of V_1 whose fixed-point set is a core circle of this solid torus. Thus we obtain an involution τ_1 on X_1. The quotient of V_1 by τ_1 is a solid torus B_1 whose core circle is the branch set. Further, A_1/τ_1 is a longitudinal annulus of B_1. The quotient of $E_1 \times I$ by τ_1 is also solid torus W_1, in which $(\partial_3 E_1 \times I)/\tau_1$ is a longitudinal annulus. Figure 11 depicts W_1 and its branch set. It follows that the pair ($Y_1 = X_1/\tau_1$, branch set of τ_1) is identical to the analogous pair in Section 1 (see Figure 3).

Next we take τ_2 to be the same involution on X_2 as that used in Section 1. An argument similar to the one used in that section shows that τ_1 and τ_2 can be pieced together to form an involution τ on M. From the previous paragraph we see that the quotient $N = M/\tau$ and its branch set are the same as those in Section 1. Hence the argument of that section can be used from here on to obtain a contradiction. This completes the proof of Theorem 1 in case (b).

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