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The C-polynomial of a knot

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Abstract

We derive, from the A-polynomial of a knot, a single variable polynomial for the knot, called C-polynomial, and explore topological and geometrical information about the knot encoded in the C-polynomial.

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1. Introduction

Throughout this paper W will denote a connected oriented closed 3-manifold, K a knot in W, W_K the exterior of an open regular neighborhood of K in W with the induced orientation from that of W. We give the boundary torus ∂W_K the induced orientation from that of W_K. We shall always use μ to denote an oriented essential simple closed curve in ∂W_K which is a meridian of the knot K. Fix another oriented essential simple closed curve λ in ∂W_K such that the algebraic intersection number of μ and λ in ∂W_K is $+1$ with respect to the given orientation of the torus ∂W_K. Then $\mathcal{B} = \{\mu, \lambda\}$ is a basis of $H_1(\partial W_K; \mathbb{Z}) \cong \pi_1(\partial W_K)$. Obviously $\overline{\mathcal{B}} = \{\bar{\mu}, \bar{\lambda}\}$ is also a basis of $\pi_1(\partial W_K)$ satisfying the same conditions as $\mathcal{B}$ given above, where $\bar{\mu}$ and $\bar{\lambda}$ are μ and λ with the opposite orientation. When W is a homology 3-sphere, we shall always assume that $\lambda = 0$ in $H_1(W_K; \mathbb{Z})$, i.e., λ is the canonical longitude.

With the above conventions, a two variable polynomial $A_{W,K,\mathcal{B}}(x, y) \in \mathbb{Z}[x, y]$ can be uniquely determined (up to sign) for the triple $(W, K, \mathcal{B})$. This polynomial, introduced

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by Culler et al. in [4], is called the A-polynomial of the triple $(W, K, \mathcal{B})$. Note that $A_{W,K,B}(x, y) = A_{W,K,\bar{B}}(x, y)$, up to sign. Hence when W is an oriented homology 3-sphere, we may drop the subscript $\mathcal{B}$ from the A-polynomial and consider the polynomial as a topological invariant for knots in W. When $W = S^3$, we simply write $A_K(x, y)$ for $A_{S^3,K}(x, y)$. The A-polynomial is a simplified version of the $SL_2(\mathbb{C})$-character variety of the knot exterior, yet it retains a great deal of topological and geometrical information about the knot K, its exterior W_K and the manifolds obtained by Dehn surgery on W along K [2,4–6,10,19,21].

In this paper we further explore information encoded in the A-polynomial. We derive from the A-polynomial a single variable polynomial $C_{W,K,B}(t)$ in $\mathbb{Z}[t]$ in a canonical way. We call $C_{W,K,B}(t)$ *the C-polynomial* of the triple $(W, K, \mathcal{B})$. Similarly when W is an oriented homology 3-sphere, $C_{W,K}(t) = C_{W,K,B}(t)$ can be considered as a polynomial invariant for knots in W, and when $W = S^3$, we write $C_K(t)$ for the C-polynomial. When $C_{W,K,B}(t)$ is not identically zero, we say $K \subset W$ has nontrivial C-polynomial. Note that as we will see, the nontriviality of the C-polynomial is independent of the choice of the basis $\mathcal{B}$. We shall show that the C-polynomial $C_{W,K,B}(t)$ reflects in its own way certain topological and geometrical properties of the underlying knot.

By an essential surface in a compact orientable 3-manifold, we mean an orientable properly embedded 2-dimensional submanifold each component of which is incompressible, nonboundary parallel, and does not bound a 3-ball (when the component is a 2-sphere). Note that we consider a reducing 2-sphere as an essential surface.

Theorem 1.1. *If $C_{W,K,B}(t)$ is not a monic polynomial (i.e., if its leading coefficient is not one), then either W is not a homotopy 3-sphere or W_K contains a closed essential surface or W_K is a solid torus.*

Theorem 1.1 suggests that the C-polynomial might be able to detect closed essential surfaces in knot exteriors in homotopy 3-spheres. But we have not been able to produce an example of a nontrivial knot in S^3 whose C-polynomial is not monic.

Recall that a slope in ∂W_K is an isotopy class of unoriented essential simple closed curves in the torus. The set of slopes in ∂W_K will be parameterized with respect to the fixed basis $\mathcal{B} = \{\mu, \lambda\}$ as $\{m/n;\ m, n \in \mathbb{Z},\ (m, n) = 1\}$ such that m is the μ-coordinate and n the λ-coordinate. Given a slope m/n, we use $W_K(m/n)$ to denote the manifold obtained by Dehn surgery on W along K (Dehn filling on W_K along ∂W_K) with the slope. Note that each slope m/n on ∂W_K corresponds to the pair of primitive elements $\mu^m\lambda^n$ and $\mu^{-m}\lambda^{-n}$ in $\pi_1(\partial M)$. Later on for a primitive element $\delta \in \pi_1(\partial W_K)$, we shall also use $W_K(\delta)$ to denote the surgered manifold with the slope corresponding to δ.

Actually when the C-polynomial $C_{W,K,B}(t)$ is nontrivial, it is a product of some factors $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t) \in \mathbb{Z}[t]$, where $(\varepsilon_1, \varepsilon_2) \in \{(1, 1), (-1, -1), (1, -1), (-1, 1)\}$ is a solution of the equation $A_{W,K,B}(x, y) = 0$. We call these factors the *main factors* of the C-polynomial (see Section 2). Note that a main factor may not be an irreducible polynomial over $\mathbb{Z}$. Of course there are at most four main factors in the C-polynomial of a knot.

Theorem 1.2. *Let W be an oriented homotopy 3-sphere and $K \subset W$ a knot whose exterior W_K contains no closed essential surface but is not a solid torus. Then $C_{W,K}(t)$*

is nontrivial and is of positive degree. Let $C_{W,K,(\varepsilon_1,\varepsilon_2)}(t)$ be a main factor of $C_{W,K}(t)$. If $C_{W,K,(\varepsilon_1,\varepsilon_2)}(-\varepsilon_1\varepsilon_2\varepsilon) \neq \pm 1$, where $\varepsilon \in \{\pm 1\}$, then $W_K(\varepsilon)$ has non-trivial fundamental group.

Recall that the Property-P conjecture states that for any nontrivial knot K in S^3, $S^3_K(m/n)$ has nontrivial fundamental group for every slope $m/n \neq 1/0$. The conjecture is an interesting special case of the Poincaré conjecture and remains a challenging open problem in knot theory and 3-manifold topology. See [15, Introduction] for a summary of the current status of what is known about the conjecture.

Corollary 1.3. *Let K be a nontrivial knot in the 3-sphere S^3 whose exterior S^3_K contains no essential closed surfaces. If for some main factor $C_{K,(\varepsilon_1,\varepsilon_2)}(t)$ of the C-polynomial of K we have $C_{K,(\varepsilon_1,\varepsilon_2)}(1) \neq \pm 1$ and $C_{K,(\varepsilon_1,\varepsilon_2)}(-1) \neq \pm 1$, then K has Property P.*

Proof. By [9], among all nontrivial surgeries only one of $S^3_K(1)$ and $S^3_K(-1)$ can possibly have trivial fundamental group. Now apply Theorem 1.2. □

The proofs of Theorems 1.1 and 1.2 make use of the $GL_2(\mathbb{C})$ subgroup theorem of Bass [1] and the main ideas in the part of the proof of the Smith conjecture given by Shalen [20]. These pieces of work [1,4,20] are connected together through the use of the Puiseux expansion which is a classical tool in studying singularities of plane algebraic curves (see, e.g., [3,14]). In fact each nonzero root of the main factor $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$ is the first coefficient of a Puiseux expansion at the point $(\varepsilon_1, \varepsilon_2)$ of the plane curve defined by the A-polynomial.

This paper is also related to and inspired by two other papers: [13] and [5]. We call a representation ρ of $\pi_1(W_K)$ into $SL_2(\mathbb{C})$ *peripheral unipotent* if for every peripheral element δ of $\pi_1(W_K)$ (i.e., δ can be conjugate into $\pi_1(\partial W_K)$), $\rho(\delta)$ is a unipotent element in $SL_2(\mathbb{C})$ (i.e., a trace 2 or -2 matrix). In [13], Kuga introduced a polynomial $N_{K,\rho}(t) \in \mathbb{Z}[t]$ for every knot K in S^3 which has an irreducible peripheral unipotent representation $\rho : \pi_1(S^3_K) \to SL_2(\mathbb{C})$ which is also integral, i.e., the image of ρ is contained in $SL_2(A)$ where A is the ring of algebraic integers of a number field. One can show that each irreducible factor of $t^n N_{K,\rho}(1/t)$ is a factor of the C-polynomial $C_K(t)$, where n is the degree of $N_{K,\rho}(t)$.

When W_K is hyperbolic, i.e., when the interior of W_K has a complete hyperbolic metric of finite volume, $\pi_1(W_K)$ has discrete faithful representations into $SL_2(\mathbb{C})$. Note that by the Mostow–Prasad rigidity, there are precisely $2|H_1(W_K, \mathbb{Z}_2)|$ such representations up to conjugation. Let ρ be such a representation. Then ρ is irreducible and peripheral unipotent. It also follows from the Mostow–Prasad rigidity and the Hilbert Nullstellensatz that the image of $\pi_1(W_K)$ under ρ can be assumed to be contained in $SL_2(F)$ for some number field F and $\rho(\mu) = \left(\begin{smallmatrix} \varepsilon_1 & 1 \\ 0 & \varepsilon_1 \end{smallmatrix}\right)$, $\rho(\lambda) = \left(\begin{smallmatrix} \varepsilon_2 & c \\ 0 & \varepsilon_2 \end{smallmatrix}\right)$ where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. The number c is uniquely determined, up to sign and the complex conjugation, for the hyperbolic triple $(W, K, \mathcal{B})$, and is called the cusp constant of the triple. Note that $c \neq 0$. Let $c(t) \in \mathbb{Z}[t]$ be the minimal polynomial of c, which is called the cusp polynomial of ρ. One can show that $c(t)$ is a factor of some main factor of the C-polynomial of the triple. The argument of this fact is contained in a paper of Cooper and Long [5] (although they only considered knots in S^3).

This result can be interpreted as geometric information contained in the C-polynomial. It also follows that when W_K is hyperbolic, its C-polynomial has positive degree.

Obviously the above notions of cusp constant and cusp polynomial can be similarly defined for any peripheral unipotent representation ρ of $\pi_1(W_K)$ into $SL_2(\mathbb{C})$ so long as $\rho(\mu) \neq I$ or $-I$, where I denotes the identity matrix, and the image of ρ is contained in $SL_2(F)$ for some algebraic number field F. As we will see that in many cases the C-polynomial is a product of the cusp polynomials of certain peripheral unipotent representations of $\pi_1(W_K)$, and that every root of the C-polynomial is the cusp constant of some peripheral unipotent representation of $\pi_1(W_K)$. For instance, this happens for any nontrivial knot in S^3 whose exterior contains no closed essential surface.

The rest of the paper is organized as follows. After the definition of the C-polynomial is given in Section 2, Theorems 1.1 and 1.2 are proved in Section 3. Along the way some other properties of the C-polynomial are also discussed. The paper is closed in Section 4 with some illustrating examples of C-polynomials.

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2. The definition of the C-polynomial and some nontriviality

We need to recall the definition of the A-polynomial first. For a compact manifold M, we use $R(M)$ and $X(M)$ denote the $SL_2(\mathbb{C})$ representation variety and character variety of M, respectively, and let $q: R(M) \to X(M)$ be the quotient map sending a representation ρ to its character χ_ρ (see [7] for detailed definitions). Note that q is a regular map between the two complex affine algebraic varieties. For a given knot exterior W_K and a basis $\mathcal{B} = \{\mu, \lambda\}$ of $\pi_1(\partial W_K)$, let $i^*: X(W_K) \to X(\partial W_K)$ be the regular map induced by the inclusion induced homomorphism $i_*: \pi_1(\partial W_K) \to \pi_1(W_K)$, and let Λ be the set of diagonal representations of $\pi_1(\partial W_K)$, i.e.,

$$\Lambda = \bigl\{\rho \in R(\partial W_K) \mid \rho(\mu), \rho(\lambda) \text{ are diagonal matrices}\bigr\}.$$

Then Λ is a subvariety of $R(\partial W_K)$ and $q|_\Lambda: \Lambda \to X(\partial W_K)$ is a degree 2 surjective map. We may identify Λ with $\mathbb{C}^* \times \mathbb{C}^*$ through the eigenvalue map $E: \Lambda \to \mathbb{C}^* \times \mathbb{C}^*$, which sends $\rho \in \Lambda$ to $(x, y) \in \mathbb{C}^* \times \mathbb{C}^*$ if $\rho(\mu) = \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$ and $\rho(\lambda) = \left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)$. A component of $X(W_K)$ is called trivial if it consists of only characters of reducible representations. Let $X^*(W_K)$ be the subset of $X(W_K)$ consisting of all nontrivial components of $X(W_K)$ each of which has one-dimensional image in $X(\partial W_K)$ under the map i^*. Let V be the Zariski closure of $i^*(X^*(W_K))$ in $X(\partial W_K)$, let Z be the algebraic curve $q|_\Lambda^{-1}(V)$ in Λ, and let D be the Zariski closure of $E(Z)$ in $\mathbb{C} \times \mathbb{C}$. Then $A_{W,K,B}(x, y)$ is the defining polynomial of the plane curve D with no repeated factors, normalized so that it is in $\mathbb{Z}[x, y]$, which is well defined up to sign. When $X^*(W_K)$ is an empty set, we define $A_{W,K,B}(x, y)$ to be the constant one and say that K has the trivial A-polynomial. Note that the nontriviality of $A_{W,K,B}(t)$ is independent of the choice of $\mathcal{B}$. (If the reader needs more details, see [4].)

Note that the present definition of $A_{W,K,B}(x, y)$ is a slight modification of that given in [4], that is, our $X^*(W_K)$ does not contain nontrivial components, and thus when $A_{W,K,B}(x, y)$ is nontrivial, every irreducible component of the plane curve defined by the polynomial corresponds to a nontrivial component of $X^*(W_K) \subset X(W_K)$. So the present definition is slightly more general when nontriviality is concerned.

Now we proceed to define the C-polynomial. Suppose that $(\varepsilon_1, \varepsilon_2)$ is a solution of $A_{W,K,B}(x, y) = 0$, where $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$. Consider the Taylor expansion of $A_{W,K,B}(x, y)$ at the point $(\varepsilon_1, \varepsilon_2)$:

$$A_{W,K,B}(x, y) = \sum_{k=n}^{d} \sum_{i=0}^{k} \frac{b(k,i)}{k!} \frac{\partial^k A(x,y)}{\partial x^{k-i} \partial y^i}\bigg|_{(\varepsilon_1,\varepsilon_2)} (x - \varepsilon_1)^{k-i} (y - \varepsilon_2)^i,$$

where $b(k,i)$ denotes the binomial coefficient, and the integer $n \geqslant 1$ is the lowest total degree in $(x - \varepsilon_1)$ and $(y - \varepsilon_2)$ in the expansion, i.e., there is at least one nth partial derivative of $A_{W,K,B}(x, y)$ which is nonzero valued at $(\varepsilon_1, \varepsilon_2)$. Let

$$g(x, y) = \sum_{i=0}^{n} \frac{b(n,i)}{n!} \frac{\partial^n A(x,y)}{\partial x^{n-i} \partial y^i}\bigg|_{(\varepsilon_1,\varepsilon_2)} (x - \varepsilon_1)^{n-i} (y - \varepsilon_2)^i,$$

from which we get a single variable polynomial

$$p_{(\varepsilon_1,\varepsilon_2)}(t) = \sum_{i=0}^{n} \frac{b(n,i)}{n!} \frac{\partial^n A(x,y)}{\partial x^{n-i} \partial y^i}\bigg|_{(\varepsilon_1,\varepsilon_2)} t^i.$$

Obviously $p_{(\varepsilon_1,\varepsilon_2)}(t)$ has integer coefficients since $A_{W,K,B}(x, y)$ does. Then $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$ is defined to be the polynomial $p_{(\varepsilon_1,\varepsilon_2)}(t)$ divided by the greatest common divisor of the coefficients in $p_{(\varepsilon_1,\varepsilon_2)}(t)$, and we also assume that $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$ has positive leading coefficient, which can be obviously achieved by multiplying the polynomial by -1 if needed. We note that $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$ may not be of degree n, in particular, it may be a constant. It may also be a reducible polynomial over $\mathbb{Z}$ (Example 4.4). Now the C-polynomial of the triple $(W, K, \mathcal{B})$ is defined to be the product of $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$ over all different root pairs $(\varepsilon_1, \varepsilon_2)$ of $A_{W,K,B}(x, y)$ (at most four of them), i.e.,

$$C_{W,K,B}(t) = \prod \bigl\{C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t);\ \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \text{ and } A_{W,K,B}(\varepsilon_1, \varepsilon_2) = 0\bigr\}.$$

If $A_{W,K,B}(x, y) = 0$ has no solution of the form $(\varepsilon_1, \varepsilon_2)$, then we define $C_{W,K,B}(t)$ to be the constant zero, and say that K has trivial C-polynomial in such case.

In the rest of this section, we discuss the nontriviality of the A-polynomial and C-polynomial under certain conditions. Recall from [7] that each element $\gamma \in \pi_1(W_K)$ defines a regular function τ_γ on $X(W_K)$ such that $\tau_\gamma(\chi_\rho) = \text{trace}(\rho(\gamma))$ for each character χ_ρ in $X(W_K)$. We call τ_γ the trace function on $X(W_K)$ defined by γ. We note that a nontrivial component X_0 in $X(W_K)$ belongs to $X^*(W_K)$ if and only if at least one of τ_M and τ_λ is not a constant function when restricted on X_0. It is known that when W_K is hyperbolic, any component X_0 of $X(W_K)$ which contains the character of a discrete faithful representation is a one-dimensional component in $X^*(W_K)$; in fact on such X_0, the trace function τ_δ defined by any nontrivial peripheral element δ of $\pi_1(W_K)$ is nonconstant [8]. Hence if W_K is hyperbolic, it has nontrivial A-polynomial.

If the character of a peripheral unipotent representation ρ of $\pi_1(W_K)$ (later on we shall call such character peripheral unipotent) is contained in a component of $X^*(W_K)$, then $A_{W,K,B}(x, y) = 0$ has a solution of the form $(\varepsilon_1, \varepsilon_2)$, where $2\varepsilon_1 = \text{trace}(\rho(\mu))$ and

$2\varepsilon_2 = \operatorname{trace}(\rho(\lambda))$, and thus the C-polynomial is nontrivial. Hence if W_K is hyperbolic, then $C_{W,K,B}(t)$ is nontrivial.

Note that each element ε of the group $H^1(W_K, \mathbb{Z}_2) \cong \operatorname{Hom}(\pi_1(W_K), \{\pm 1\})$ induces an isomorphism $\varepsilon_* : R(W_K) \to R(W_K)$ and an isomorphism $\varepsilon^* : X(W_K) \to X(W_K)$ as follows: $\varepsilon_*(\rho)(\gamma) = \varepsilon(\gamma)\rho(\gamma)$ for every $\gamma \in \pi_1(W_K)$ and $\varepsilon^*(\chi_\rho) = \chi_{\varepsilon_*(\rho)}$. Note that by Lefschetz duality, at least one of μ and λ, say μ, is a nontrivial element in $H_1(W_K, \mathbb{Z}_2)$, and thus there is a corresponding element $\varepsilon \in H^1(W_K, \mathbb{Z}_2)$ such that $\varepsilon(\mu) = -1$ and $\varepsilon(\lambda) = 1$ or -1 depending on whether λ is trivial or not in $H_1(W_K, \mathbb{Z}_2)$, respectively. Now if $\rho \in R(W_K)$ is a peripheral unipotent representation such that $\rho(\mu)$ has trace $2\varepsilon_1$ and $\rho(\lambda)$ has trace $2\varepsilon_2$, then $\rho' = \varepsilon^*(\rho)$ is another (nonconjugate) peripheral unipotent representation such that $\rho'(\mu)$ has trace $-2\varepsilon_1$ and $\rho(\lambda)$ has trace $2\varepsilon(\lambda)\varepsilon_2$. Hence if $C_{W,K,B}(t)$ is nontrivial, it has either two or four main factors. In summary, we have proved

Proposition 2.1. *If W_K is hyperbolic, then $C_{W,K,B}(t)$ is nontrivial of positive degree and contains either two or four main factors.*

Examples 4.2 and 4.4 give C-polynomials with two main factors, and Example 4.5 gives a C-polynomial with four main factors.

According to Thurston [22], if W_K is nonhyperbolic, then it is either Seifert fibered or contains an essential torus. Due to the simple group structure of the fundamental group of a Seifert fibered knot exterior W_K, it is not hard to determine exactly when such W_K has nontrivial C-polynomial. As an illustration, we calculate explicitly in Example 4.1 the C-polynomials for all torus knots in S^3 (their exteriors are Seifert fibered). It is conceivable that every nontrivial knot in S^3 has a nontrivial C-polynomial.

3. Proofs of Theorems 1.1 and 1.2

We retain all the notations established in the previous sections.

Lemma 3.1. *Let K be a knot in a homotopy 3-sphere W. Suppose that X_0 is a nontrivial component in $X(W_K)$. Then for any peripheral unipotent character χ_ρ in X_0, $\rho(\mu)$ is not I or $-I$, where I is the identity matrix.*

Proof. Suppose otherwise. Then ρ is one of the two trivial representations, i.e., $\rho(\pi_1(W_K)) \subset \{I, -I\}$, since $\pi_1(W_K)$ is normally generated by μ. Since X_0 is a nontrivial component, it contains an irreducible character by definition. Thus X_0 is positive dimensional by [7, Proposition 3.2.1]. Suppose that X_0 has dimension n. Then $q^{-1}(X_0)$ is an $(n+3)$-dimensional subvariety of $R(W_K)$, and $q^{-1}(\chi_\rho)$ is a 3-dimensional subvariety of $q^{-1}(X_0)$, consisting of reducible representations [7, 1.5.3 and 1.5.2]. Hence $q^{-1}(\chi_\rho)$ contains a non-Abelian reducible representation ρ' since the set of Abelian representations of $\pi_1(W_K)$ with a given character is at most two-dimensional (cf. the proof of [9, 1.5.10]). But a reducible representation with the same character as a trivial representation must be an Abelian representation. This gives a contradiction. □

Lemma 3.2. *Let W be a homotopy 3-sphere. Suppose that X_0 is a component in $X^*(W_K)$. Then any peripheral unipotent character χ_ρ in X_0 is irreducible.*

Proof. Suppose otherwise that $\chi_\rho \in X_0$ is a character which is peripheral unipotent and reducible. We first claim that the trace function τ_μ defined by the meridian μ is not constant on X_0. For otherwise it would be constantly equal to 2 or -2. Hence any character in X_0 would be peripheral unipotent by Lemma 3.1. Therefore τ_δ would be a constant function on X_0 for any $\delta \in \pi_1(\partial M)$. This contradicts the assumption that X_0 is in $X^*(W_K)$.

On the other hand, χ_ρ is also the character of a diagonal representation ρ' of $\pi_1(W_K)$ [7]. Therefore $\rho'(\lambda) = I$, and thus $\rho(\lambda)$ has trace equal to 2. Note that X_0 corresponds to a factor in the A-polynomial of (W, K). It follows that $(1, 1)$ or $(-1, 1)$ is a solution of the equation $A_{W,K}(x, y) = 0$. So by [4, Proposition 6.2] 1 or -1 is a root of the Alexander polynomial of the knot K. But this is impossible by [18, Section 8.C, Proposition 7 and Section 8.D, Corollary 3]. Note that although the above results in [18] are only stated for knots in S^3, they are still valid for knots in any homology 3-sphere. □

Lemma 3.3. *Suppose that W_K contains no closed essential surface and that $\rho \in R(W_K)$ is an irreducible peripheral unipotent representation. Then ρ is conjugate in $SL_2(\mathbb{C})$ to a representation $\rho' \in R(W_K)$ such that the image of ρ' is contained in $SL_2(A)$, where A is the ring of algebraic integers in some number field.*

Proof. The lemma essentially follows from the $GL_2(\mathbb{C})$ subgroup theorem of Bass [1]. Recall that his theorem states that if Γ is a finitely generated subgroup of $GL_2(\mathbb{C})$, then one of the following cases occurs:

(a) There is an epimorphism $f : \Gamma \to \mathbb{Z}$ such that $f(u) = 0$ for all unipotent elements $u \in \Gamma$.
(b) Γ is an amalgamated free product $\Gamma_0 *_\Lambda \Gamma_1$ with $\Gamma_0 \neq \Lambda \neq \Gamma_1$ and such that every finitely generated unipotent subgroup of Γ is contained in a conjugate of Γ_0 or of Γ_1.
(c) Γ is conjugate to a group of triangular matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with a and d roots of unity.
(d) Γ is conjugate in $GL_2(\mathbb{C})$ to a subgroup of $GL_2(A)$, where A is a ring of algebraic integers.

In our current situation, let $\Gamma = \rho(\pi_1(W_K)) \subset SL_2(\mathbb{C}) \subset GL_2(\mathbb{C})$. Then case (a) cannot occur since $\rho(\delta)$ is a unipotent element in Γ for every $\delta \in \pi_1(\partial W_K)$, and $f(\rho(\delta)) = 0$ would imply the existence of a surjective homomorphism from $\pi_1(W_K(\delta))$ to $\mathbb{Z}$ for every primitive element in $\pi_1(\partial M)$, which would imply that W_K contains a closed essential non-separating surface. Case (b) cannot occur. For otherwise there would be a closed essential surface in W_K since $\rho(\pi_1(\partial W_K))$ is a unipotent subgroup of Γ (see [20, Section 4] for more details). Case (c) is ruled out by the assumption that ρ is irreducible. So case (d) has to hold. Finally one can easily verify that two representations of a group into $SL_2(\mathbb{C})$ are conjugate in $GL_2(\mathbb{C})$ if and only if they are conjugate in $SL_2(\mathbb{C})$. □

Let F be a field with a discrete valuation v; i.e., v is a surjective homomorphism from the multiplicative group F^* to the group of integers $\mathbb{Z}$ such that $v(a+b) \geqslant \min(v(a), v(b))$

for any $a, b \in F^*$ with $a + b \neq 0$. By convention, define $v(0) = \infty$. An element a in F is called a v-integer if and only if $v(a) \geqslant 0$, and called a v-unit if and only if $v(a) = 0$. The set of v-integers in F form a subring O_v of F, called the valuation ring of v in F. The valuation ring O_v is a principal ideal domain. The set of elements in O_v with positive valuation form the unique maximal proper ideal of O_v, which is generated by any element π with $v(\pi) = 1$. Such element π is called a uniformizer of O_v. The quotient field $O_v/\pi O_v$, called the residue field of v, will be denoted by k_v. Also note that every nonzero element of O_v is of the form $\pi^n \sigma$, where $n \geqslant 0$ is an integer and σ is a v-unit.

Lemma 3.4. *Suppose that $\rho \in R(W_K)$ is an irreducible peripheral unipotent representation such that the image of ρ is contained in $SL_2(O_v)$ where O_v is the valuation ring of a discrete valuation v on some field $F \subset \mathbb{C}$. If $\rho(\alpha)$ is an upper triangular matrix for some primitive element $\alpha \in \pi_1(\partial W_K)$ and the upper right entry of the matrix is not a v-unit, then $W_K(\alpha)$ has nontrivial fundamental group.*

Proof. The proof of the lemma essentially follows that of [20, Proposition 1]. Let $\rho(\alpha) = \left(\begin{smallmatrix} \varepsilon & \eta \\ 0 & \varepsilon \end{smallmatrix}\right)$ where $\varepsilon \in \{1, -1\}$ and suppose that η is not a v-unit. Hence $v(\eta) > 0$, i.e., η belongs to the maximal ideal of O_v. Since ρ is irreducible, there is an element $\gamma \in \pi_1(W_K)$ such that $\rho(\gamma)$ is not upper triangular. Let l be the minimal nonnegative integer for which there is an element $\gamma \in \pi_1(W_K)$ such that $\rho(\gamma) = \left(\begin{smallmatrix} a & b \\ \pi^l\sigma & d \end{smallmatrix}\right)$ where σ is a v-unit and π is a uniformizer of O_v. Consider the conjugate ρ' of ρ in $GL_2(F)$, $\rho' = \left(\begin{smallmatrix} \pi^l & 0 \\ 0 & 1 \end{smallmatrix}\right)\rho\left(\begin{smallmatrix} \pi^{-l} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. One can easily check that $\rho'(\pi_1(W_K)) \subset SL_2(O_v)$, $\rho'(\gamma) = \left(\begin{smallmatrix} a & b\pi^l \\ \sigma & d \end{smallmatrix}\right)$, and $\rho'(\alpha) = \left(\begin{smallmatrix} \varepsilon & \eta\pi^l \\ 0 & \varepsilon \end{smallmatrix}\right)$. Now consider the composed homomorphism

$$\pi_1(W_K) \xrightarrow{\rho'} SL_2(O_v) \longrightarrow SL_2(k_v) \longrightarrow PSL_2(k_v).$$

Under this map, the image of α is the trivial element in $PSL_2(k_v)$ but that of γ is not. Hence the homomorphism factors through $\pi_1(W_K(\alpha))$, yielding a nontrivial representation of $\pi_1(W_K(\alpha))$ into $PSL_2(k_v)$. Therefore $W_K(\alpha)$ has nontrivial fundamental group. $\square$

We now recall some basic facts about the Puiseux expansion of a complex plane algebraic curve. We refer to [3,14] for details.

Let $B(u, v) = \sum b_{ij} u^i v^j$ be a two variable polynomial in $\mathbb{C}[u, v]$. The carrier of $B(u, v)$ is the set $\{(i, j) \in \mathbb{Z}^2;\ b_{ij} \neq 0\}$. The convex hull of the carrier of $B(u, v)$ in the real uv-plane is called the Newton polygon of $B(u, v)$. Of course the Newton polygon lies in the first quadrant of the uv-plane. We assume that $(0, 0)$ is a solution of the equation $B(u, v) = 0$ and that the Newton polygon of $B(u, v)$ has an edge e which lies in the lower left side of the polygon with a negative slope, say $-m/n$, $m > 0$, $n > 0$. Let $B_e(u, v)$ be the polynomial whose terms are those terms of $B(u, v)$ whose exponent pairs lie on the edge e. That is

$$B_e(u, v) = \sum_{(i,j) \in e} b_{ij} u^i v^j.$$

Then $B_e(u,v)$ contains at least two terms (monomials). From $B_e(u,v)$, one can define a single variable polynomial, called the edge polynomial of e, which we denote by $e(t)$, simply by replacing a term $b_{ij}u^iv^j$ in $B_e(u,v)$ by $b_{ij}t^j$. That is

$$e(t) = \sum_{b_{ij}u^iv^j \in B_e(u,v)} b_{ij}t^j.$$

Since $e(t)$ contains at least two terms, it contains at least one nonzero root.

Suppose now that $B(u,v)$ is also irreducible over $\mathbb{C}$. Then for every nonzero root c of $e(t)$, there is a series in $u^{1/k}$ of the form

$$v(u) = \sum_{i=1}^{\infty} a_i u^{i/k},$$

where k is some positive integer constant, such that

(1) the first nonzero term in the series is $cu^{n/m}$;
(2) the series is convergent for u near zero;
(3) $(u, v(u))$ satisfies the equation $B(u,v) = 0$ for u near zero.

The algorithm for producing such a series was described by Newton, and the convergence of the series was proved by Puiseux and the series was named after him, called a Puiseux expansion of the plane curve defined by $B(u,v)$ at the point $(0,0)$.

When $B(u,v)$ is reducible over $\mathbb{C}$, we factor $B(u,v)$ into irreducible factors as

$$B(u,v) = B_1(u,v)B_2(u,v)\cdots B_p(u,v).$$

Given a negative slope $-m/n$, the Newton polygon of $B(u,v)$ has an edge e with that slope if and only if the Newton polygon of some irreducible factor $B_i(u,v)$ has an edge e_i with that slope. Moreover the edge polynomial of e is the product of the edge polynomials of those e_i's. These two properties are elementary to verify, which we leave to the reader.

We now ready to prove the two theorems given in the introduction. We first prove Theorem 1.1. To get a contradiction, suppose that the leading coefficient of $C_{W,K,B}(t)$ is not 1, W is a homotopy 3-sphere, and W_K contains no closed essential surfaces but is not a solid torus. By Thurston [22], W_K is either hyperbolic or Seifert fibered. If W_K is Seifert fibered, then W is the 3-sphere and thus K is a nontrivial torus knot. By Example 4.1, the C-polynomial of every nontrivial torus knot is monic. Hence we may assume that W_K is hyperbolic. By Proposition 2.1, $C_{W,K,B}(t)$ is not trivial, i.e., is not the zero constant. It follows that the C-polynomial has a main factor, say $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$, whose leading coefficient is not equal to 1.

Recall the constructional definition of a main factor given in Section 2. In the Taylor expansion of $A_{W,K,B}(x,y)$ at the root $(\varepsilon_1,\varepsilon_2)$, let $u = x-\varepsilon_1$ and $v = y-\varepsilon_2$. Then the polynomial $A_{W,K,B}(x,y)$ can be expressed as a polynomial $B(u,v) \in \mathbb{Z}[u,v]$, i.e.,

$$B(u,v) = \sum_{k=n}^{d}\sum_{i=0}^{k} \frac{b(k,i)}{k!} \frac{\partial^k A(x,y)}{\partial x^{k-i}\partial y^i}\bigg|_{(\varepsilon_1,\varepsilon_2)} u^{k-i}v^i,$$

and the function $g(x, y)$ given in Section 2 can be expressed as a polynomial $h(u, v) \in \mathbb{Z}[u, v]$, i.e.,

$$h(u, v) = \sum_{i=0}^{n} \frac{b(n, i)}{n!} \frac{\partial^n A(x, y)}{\partial x^{n-i} \partial y^i}\bigg|_{(\varepsilon_1, \varepsilon_2)} u^{n-i} v^i.$$

Observe that if the polynomial $h(u, v)$ contains at least two terms, then the Newton polygon of $B(u, v)$ has an edge e of slope -1 and $h(u, v)$ is the polynomial $B_e(u, v)$. Hence if $h(u, v)$ contains two terms, then the polynomial $p_{(\varepsilon_1,\varepsilon_2)}(t)$ given in Section 2 is the edge polynomial $e(t)$ of the edge e for $B(u, v)$. Also the two polynomials $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$ and $p_{(\varepsilon_1,\varepsilon_2)}(t)$ have the same set of roots, including their multiplicities.

Since the leading coefficient of $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$ is not 1, $h(u, v)$ must have at least two terms by the definition of $C_{W,K,B,(\varepsilon_1,\varepsilon_2)}(t)$. Thus the main factor can be considered as the edge polynomial of the edge e. Also the polynomial has an irreducible factor $f(t)$ over $\mathbb{Z}$ whose leading coefficient is not ± 1. Let c be a root of $f(t)$. Note that c is not an algebraic integer. By the above review on Puiseux expansions, there is an irreducible factor of $B(u, v)$ over $\mathbb{C}$, which we denote by $B_0(u, v)$, such that the Newton polygon of $B_0(u, v)$ has an edge e_0 of slope -1 and its edge polynomial $e_0(t)$ has c as a root. Hence the irreducible plane curve E_0 defined by $B_0(u, v)$ has a Puiseux expansion at the point $(0, 0) \in E_0$ of the form $v = \sum_{i=1}^{\infty} a_i u^{i/k}$ whose first nonzero term is cu. Let $\{(u_j, v_j = v(u_j))\}$ be a sequence of points in $E_0 \setminus \{(0, 0)\}$ which converges to the point $(0, 0)$ (the convergence of a sequence mentioned here and later is always with respect to the classical topology of the variety involved). Note that $B_0(x - \varepsilon_1, y - \varepsilon_2)$ is an irreducible factor of $A_{W,K,B}(x, y)$, and the coordinate transformations $u = x - \varepsilon_1$ and $v = y - \varepsilon_2$ change the curve E_0 in the complex uv-plane to an irreducible curve $D_0 \subset D$ in the complex xy-plane. Therefore the sequence $\{(x_j = u_j + \varepsilon_1,\ y_j = v_j + \varepsilon_2)\} \subset D_0 \setminus \{(\varepsilon_1, \varepsilon_2)\}$ approaches the point $(\varepsilon_1, \varepsilon_2) \in D_0$. By the definition of the A-polynomial recalled in Section 2, there is a component $X_0 \subset X^*(W_K)$ such that $q^{-1}|_\Lambda(\overline{i^*(X_0)})$ contains a component Z_0 with $\overline{E(Z_0)} = D_0$ (notations from Section 2), where the overline denotes the Zariski closure. Note that X_0 is one-dimensional by [4, Proposition 2.4]. It follows that there is a sequence of points $\{\chi_j\}$ in X_0 such that $\tau_\mu(\chi_j) = x_j + x_j^{-1} \to 2\varepsilon_1$ and $\tau_\lambda(\chi_j) = y_j + y_j^{-1} \to 2\varepsilon_2$. If the sequence $\{\chi_j\}$ has no limit point in the affine curve X_0 (so the sequence provides an ideal point in the projective model of X_0), then by a fundamental result in [7], W_K contains an essential closed surface, which contradicts to our assumption on W_K. Suppose then that the sequence has a limit point χ_{ρ_*} in X_0 (we may assume that the sequence has a unique accumulation point). Then $\tau_\mu(\chi_{\rho_*}) = 2\varepsilon_1$ and $\tau_\lambda(\chi_{\rho_*}) = 2\varepsilon_2$. Thus ρ_* is a peripheral unipotent representation. By Lemma 3.2, ρ_* is irreducible. By conjugation in $SL_2(\mathbb{C})$, we may assume that $\rho_*(\mu) = \left(\begin{smallmatrix} \varepsilon_1 & 1 \\ 0 & \varepsilon_1 \end{smallmatrix}\right)$ by Lemma 3.1. Hence $\rho_*(\lambda) = \left(\begin{smallmatrix} \varepsilon_2 & c_* \\ 0 & \varepsilon_2 \end{smallmatrix}\right)$ for some number c_* uniquely associated to the irreducible peripheral unipotent character χ_{ρ_*}.

Lemma 3.5. $c_* = c$.

Proof. Still consider the sequence $\chi_j \to \chi_{\rho_*}$ in $X_0 \subset X^*(W_K)$. Let R_0 be an irreducible component in $R(W_K)$ with $q(R_0) = X_0$. Let R_0^+ be the subvariety of R_0 consisting of elements $\rho \in R_0$ with $\rho(\mu) = \left(\begin{smallmatrix} x & 1 \\ 0 & x^{-1} \end{smallmatrix}\right)$. Then $\rho_* \in R_0^+$. Since R_0 is 4-dimensional [7, Corollary 1.5.3], each component of R_0^+ is at least 2-dimensional (applying Corollary 3.14 of [16]). By varying the trace of $\rho(\mu)$ for $\rho \in R_0$ near ρ_*, we see that the map $q : R_0^{+*} \to X_0$ is locally onto near the point $\chi_{\rho_*} \in X_0$, where R_0^{+*} is a component of R_0^+ which contains ρ_*. It follows that we may get a sequence $\{\rho_j\}$ in R_0^{+*} such that $\rho_j \to \rho_*$, $\chi_{\rho_j} = \chi_j$, ρ_j is irreducible and nonperipheral unipotent for all j sufficiently large. In particular, $\rho_j(\mu) = \begin{pmatrix} x_j & 1 \\ 0 & x_j^{-1} \end{pmatrix}$, with $x_j \neq \pm 1$. Since μ and λ commute, $\rho_j(\lambda) = \begin{pmatrix} y_j & c_j \\ 0 & y_j^{-1} \end{pmatrix}$ with $c_j = (y_j - y_j^{-1})/(x_j - x_j^{-1})$. We have $x_j \to \varepsilon_1$, $y_j \to \varepsilon_2$ and $c_j \to c_*$, when $j \to \infty$. It follows that $(y_j - \varepsilon_2)/(x_j - \varepsilon_1) \to c_*$ as $j \to \infty$.

On the other hand, substitute $(x_j - \varepsilon_1, y_j - \varepsilon_2) = (u_j, v_j)$ into the Puiseux expansion $v = \sum a_i u^{i/k}$, we see that $(y_j - \varepsilon_2)/(x_j - \varepsilon_1) \to c$ as $j \to \infty$ (note again that the lowest term in the series is cu). The lemma is proved. □

From the above arguments, we have an irreducible peripheral unipotent representation $\rho_* \in R(W_K)$ such that $\rho_*(\mu) = \begin{pmatrix} \varepsilon_1 & 1 \\ 0 & \varepsilon_1 \end{pmatrix}$ and $\rho_*(\lambda) = \begin{pmatrix} \varepsilon_2 & c \\ 0 & \varepsilon_2 \end{pmatrix}$, where c is an algebraic number but is not an algebraic integer. By Lemma 3.3, ρ_* is conjugate in $SL_2(\mathbb{C})$ to a representation $\rho' \in R(W_K)$ such that the image of ρ' is contained in $SL_2(A)$ for some ring A of algebraic integers in a number field F. We may assume that $c \in F$. As c is not an algebraic integer, there is a discrete valuation v on F such that $v(c) < 0$. Let O_v be the valuation ring. Note that the ring of algebraic integers in a number field is contained in each discrete valuation ring of the field (see, for instance, [12, Theorem 10.8]). So $A \subset O_v$. Hence the image of ρ' is contained in $SL_2(O_v)$. Since O_v is a principal ideal domain and the trace of $\rho'(\mu)$ is $2\varepsilon_1$, $\rho'(\mu)$ can be conjugated in $SL_2(O_v)$ to an upper triangular matrix of the form $\begin{pmatrix} \varepsilon_1 & \eta \\ 0 & \varepsilon_1 \end{pmatrix}$ with $\eta \in O_v$. We use ρ'' to denote the representation after the conjugation. By Lemma 3.4, η is a v-unit. Hence ρ'' can be further conjugated in $SL_2(O_v)$ to a representation ρ''' such that $\rho'''(\mu) = \begin{pmatrix} \varepsilon_1 & 1 \\ 0 & \varepsilon_1 \end{pmatrix}$. Since $\rho'''(\lambda)$ commutes with $\rho'''(\mu)$, $\rho'''(\lambda) = \begin{pmatrix} \varepsilon_2 & c''' \\ 0 & \varepsilon_2 \end{pmatrix}$. But $c = c'''$ since ρ''' is conjugate to ρ_*. Hence $c \in O_v$, which gives a contradiction. The proof of Theorem 1.1 is now complete.

We now prove Theorem 1.2. Since W_K contains no closed essential surfaces, W_K is either hyperbolic or is Seifert fibered. In the latter case, W is the 3-sphere and W_K is the exterior of a torus knot. Hence the first statement of Theorem 1.2 that the C-polynomial of K has positive degree follows from Proposition 2.1 when W_K is hyperbolic and follows from Example 4.1 (which is a direct calculation) when W_K is Seifert fibered.

Now we prove the second statement of Theorem 1.2. Write $C_{W,K,(\varepsilon_1,\varepsilon_2)}(t) = t^p g(t)$, where $p \geqslant 0$ is an integer and $g(t) \in \mathbb{Z}[t]$ is not divisible by t. Suppose that $C_{W,K,(\varepsilon_1,\varepsilon_2)}(-\varepsilon_1\varepsilon_2\varepsilon) \neq \pm 1$. Then $g(-\varepsilon_1\varepsilon_2\varepsilon) \neq \pm 1$. Let $B(u, v)$ and $h(u, v)$ be defined as in the proof of Theorem 1.1. There is an irreducible factor $f(t)$ of $g(t)$ over $\mathbb{Z}$ such that $f(-\varepsilon_1\varepsilon_2\varepsilon) \neq \pm 1$. Let c be a root of $f(t)$. From the proof of Theorem 1.1 we see that there is an irreducible peripheral unipotent representation $\rho \in R(W_K)$ such that $\rho(\mu) = \begin{pmatrix} \varepsilon_1 & 1 \\ 0 & \varepsilon_1 \end{pmatrix}$, $\rho(\lambda) = \begin{pmatrix} \varepsilon_2 & c \\ 0 & \varepsilon_2 \end{pmatrix}$, and the image of ρ is contained in $SL_2(O_v)$ for every valuation ring O_v of some fixed number field F. Now $\rho(\mu^\varepsilon\lambda) = \begin{pmatrix} \varepsilon_1^\varepsilon\varepsilon_2 & \varepsilon_1^\varepsilon c + \varepsilon_1^{\varepsilon+1}\varepsilon_2\varepsilon \\ 0 & \varepsilon_1^\varepsilon\varepsilon_2 \end{pmatrix}$. Let $\eta = \varepsilon_1^\varepsilon c + \varepsilon_1^{\varepsilon+1}\varepsilon_2\varepsilon$. Then a similar proof as that of Lemma 3.4 shows that η must be a v-unit. For otherwise there will be a nontrivial homomorphism from $\pi_1(W_K(\varepsilon))$ into $PSL_2(k_v)$, and we are done. Now since η must be a v-unit for any discrete valuation v on F, η is an algebraic unit. On the other hand, $c = \varepsilon_1^{\varepsilon}\eta - \varepsilon_1\varepsilon_2\varepsilon$ is a root of $f(t)$. So η is a root of the irreducible polynomial $f_*(s) = f(\varepsilon_1^{\varepsilon}s - \varepsilon_1\varepsilon_2\varepsilon)$ in $\mathbb{Z}[s]$ and $f(-\varepsilon_1\varepsilon_2\varepsilon)$ is the constant term of $f_*(s)$, up to sign. Hence $f(-\varepsilon_1\varepsilon_2\varepsilon) \neq \pm 1$ implies that η is not an algebraic unit, giving a contradiction. This complete the proof of the theorem.

The arguments of Theorems 1.1 and 1.2, together with some remarks from the previous sections, can be also used to show the following

Theorem 3.6. *Let W be a homotopy 3-sphere and $K \subset W$ a knot whose exterior W_K contains no closed essential surface but is not a solid torus. Then the C-polynomial $C_{W,K}(t)$ is nontrivial of positive degree, and every root c of the C-polynomial is the cusp constant of some irreducible peripheral unipotent representation of $\pi_1(W_K)$ and vice versa.*

4. Examples

Example 4.1. Let K be a nontrivial torus knot in S^3 of type (p, q). We may assume that $|p| > q \geqslant 2$. Note that S^3_K is a Seifert fibered space whose base orbifold is a disk with two cone points of indices $|p|$ and q. Hence S^3_K contains no closed essential surfaces. Also note that a fiber in ∂S^3_K of the Seifert fiberation represents the element $\mu^{pq}\lambda$ in $\pi_1(\partial S^3_K)$. From these conditions, one can deduce (cf. [4, Proposition 2.7]) that when $q = 2$,

$$A_K(x, y) = \begin{cases} 1 + x^{2p}y, & \text{if } p > 2, \\ x^{-2p} + y, & \text{if } p < -2; \end{cases}$$

and when $q \neq 2$,

$$A_K(x, y) = \begin{cases} l - 1 + x^{2pq}y^2, & \text{if } p > q, \\ -x^{-2pq} + y^2, & \text{if } p < -q. \end{cases}$$

So when $q = 2$, $C_K(t)$ has two main factors: $C_{K,(1,-1)}(t) = t - 2p$ and $C_{K,(-1,-1)}(t) = t + 2p$, and when $q \neq 2$, $C_K(t)$ has four main factors: $C_{K,(1,-1)}(t) = t - pq$, $C_{K,(-1,-1)}(t) = t + pq$, $C_{K,(1,1)}(t) = t + pq$ and $C_{K,(-1,1)}(t) = t - pq$. When $q = 2$, $C_{K,(1,-1)}(1) = 1 - 2p \neq \pm 1$ and $C_{K,(1,-1)}(-1) = -1 - 2p \neq \pm 1$, and thus K has Property P by Corollary 3. Similarly when $q \neq 2$, K also has Property P.

Example 4.2. Let K be the figure-eight knot in S^3. Then S^3_K is hyperbolic and contains no closed essential surfaces [23]. The A-polynomial of the knot is $A_K(x, y) = -x^4 + (1 - x^2 - 2x^4 - x^6 + x^8)y - x^4y^2$ [4, Appendix]. For this knot $C_K(t)$ has two main factors:

$$C_{K,(1,-1)}(t) = t^2 + 12,$$
$$C_{K,(-1,-1)}(t) = t^2 + 12.$$

Now $C_{K,(1,-1)}(\varepsilon) \geqslant 12$ for both $\varepsilon = 1$ and $\varepsilon = -1$. Hence the knot has Property P.

Example 4.3. Let W be the manifold obtained by Dehn surgery on S^3 along the figure-eight knot with slope $-1/2$. Then W is hyperbolic [23] and is a homology 3-sphere. Let

$K \subset W$ be the core of the sewn solid torus. Then W_K is homeomorphic to the exterior of the figure-eight knot in S^3. The A-polynomial $A_{W,K}(x, y)$ can be obtained from that of figure-eight knot (the previous example) and we have $A_{W,K}(x, y) = 1 - x^2y^4 - x^4y^7 - 2x^4y^8 - x^4y^9 - x^6y^{12} + x^8y^{16}$. In this case $C_{W,K}(t)$ has two main factors:

$$C_{K,(1,-1)}(t) = 49t^2 - 48t + 12,$$
$$C_{K,(-1,-1)}(t) = 49t^2 + 48t + 12.$$

They are not monic polynomials.

Example 4.4. Let K be the 7_4 knot in S^3. Then S^3_K is hyperbolic and contains no closed essential surfaces [11]. Also $A_K(x, y) = 1 + (-3 + 7x^2 + 4x^4 - 6x^6 + x^8 + 3x^{10} - 2x^{12} + x^{14})y + (3 - 10x^2 + 3x^4 + 21x^6 - 3x^8 - 17x^{10} + 6x^{12} + 10x^{14} - 2x^{16} - 3x^{18} + 3x^{20} - x^{22})y^2 + (-1 + 3x^2 - 3x^4 - 2x^6 + 10x^8 + 6x^{10} - 17x^{12} - 3x^{14} + 21x^{16} + 3x^{18} - 10x^{20} + 3x^{22})y^3 + (x^8 - 2x^{10} + 3x^{12} + x^{14} - 6x^{16} + 4x^{18} + 7x^{20} - 3x^{22})y^4 + x^{22}y^5$ [4, Appendix]. The A-polynomial is reducible over $\mathbb{Z}$; $A_K(x, y) = [1 + (-1 + x^2 + 2x^4 + x^6 - x^8)y + x^8y^2][1 + (-2 + 6x^2 + 2x^4 - 7x^6 + 2x^8 + 3x^{10} - 2x^{12} + x^{14})y + (1 - 2x^2 + 3x^4 - 2x^6 - 7x^8 + 2x^{10} + 6x^{12} - 2x^{14})y^2 + x^{14}y^3]$. In this case $C_K(t)$ has two main factors:

$$C_{K,(1,-1)}(t) = \big(t^2 - 8t + 28\big)\big(t^3 - 14t^2 + 28t - 136\big),$$
$$C_{K,(-1,-1)}(t) = \big(t^2 + 8t + 28\big)\big(t^3 + 14t^2 + 28t + 136\big),$$

each being reducible over $\mathbb{Z}$. It is easy to check that $C_{K,(1,-1)}(\varepsilon) \neq \pm 1$ for both $\varepsilon = 1$ and $\varepsilon = -1$. Hence the knot has Property P.

Example 4.5. Let K be the $(-2, 3, 7)$-pretzel knot in S^3. Then S^3_K is hyperbolic and contains no closed essential surfaces [17]. Also $A_K(x, y) = -1 + (x^{16} - 2x^{18} + x^{20})y + (2x^{36} + x^{38})y^2 + (-x^{72} - 2x^{74})y^4 + (-x^{90} + 2x^{92} - x^{94})y^5 + x^{110}y^6$ [4, Appendix]. In this case, $C_K(t)$ has four main factors:

$$C_{K,(1,1)}(t) = t^3 + 55t^2 + 1006t + 6119,$$
$$C_{K,(-1,1)}(t) = t^3 - 55t^2 + 1006t - 6119,$$
$$C_{K,(1,-1)}(t) = t^3 - 55t^2 + 1010t - 6193,$$
$$C_{K,(-1,-1)}(t) = t^3 + 55t^2 + 1010t + 6193.$$

It is known that each of the surgeries on K with slopes 18 and 19 produces a manifold with cyclic fundamental group (due to Fintushel and Stern). We have

$$\begin{aligned} &C_{K,(1,1)}(-18) = -1, && C_{K,(-1,1)}(18) = 1,\\ &C_{K,(1,-1)}(18) = -1, && C_{K,(-1,-1)}(18) = 1;\\ &C_{K,(1,1)}(-19) = 1, && C_{K,(-1,1)}(19) = -1,\\ &C_{K,(1,-1)}(19) = 1, && C_{K,(-1,-1)}(-19) = -1. \end{aligned}$$

The calculations of this example suggest that the following statement might be true:

Let W be an oriented homotopy 3-sphere and $K \subset W$ a knot whose exterior W_K contains no closed essential surface but is not a solid torus. Let $C_{W,K,(\varepsilon_1,\varepsilon_2)}(t)$ be a main factor of $C_{W,K}(t)$. If $C_{W,K,(\varepsilon_1,\varepsilon_2)}(-\varepsilon_1\varepsilon_2 n) \neq \pm 1$, where n is an integer, then $W_K(n)$ has noncyclic fundamental group.

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