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Detection of essential surfaces in 3-manifolds with SL_2-trees

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1. Introduction

According to Bass-Serre theory [Se], given a field F with a discrete valuation $v : F^* \to \mathbb{Z}$, there is a canonical way to construct a simplicial tree $T = T_{F,v}$ on which $SL_2(F)$ acts simplicially without inversion. When a group π has a representation into the group $SL_2(F)$, then there is an induced action of π on the tree via the representation. If further π is the fundamental group of a compact 3-manifold M and the action of π on the tree T is nontrivial (meaning that there is no point on T fixed by every element of π), then the action induces a splitting of M along an essential surface in the following way: let $\widetilde{M} \xrightarrow{p} M$ be the universal covering, there is a $\pi_1(M)$-equivariant map $f : \widetilde{M} \to T$ which is transverse to the set E of midpoints of edges in T such that $p(f^{-1}(E))$ is an essential surface in M. We say such an essential surface is associated to (or detected by) an SL_2-tree. A natural question is: which essential surfaces in a compact 3-manifold can be associated to SL_2-trees and how do they depend on the choice of the field and discrete valuation? This is the main issue we are going to address in this paper.

Any 3-manifold mentioned in this paper is automatically assumed orientable and connected. By an essential surface in a compact 3-manifold M we mean an orientable, properly embedded, incompressible surface each component of which is neither boundary parallel in M nor bounds a 3-ball (when it is a 2-sphere). Also recall that a discrete valuation on a field F is a homomorphism v from the multiplicative group F^* onto the group of integers $\mathbb{Z}$ such that

$$v(a+b) \geq min(v(a), v(b))$$

for any $a, b \in F^*$ with $a + b \neq 0$. Let $v : F^* \to \mathbb{Z}$ be a discrete valuation. The set

$$O_v = \{0\} \cup \{a \in F^*; v(a) \geq 0\}$$

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forms a subring of F, called the valuation ring of v. If T is the SL_2-tree defined by (F, v), then the action of an element $A \in SL_2(F)$ has a fixed point if and only if A is conjugate in $GL_2(F)$ to an element in $SL_2(O_v)$ (see [Se] for details). The latter condition is equivalent to the condition that the trace of A belongs to O_v. We summarize the above discussion in the following lemma.

Lemma 1 *Let v be a discrete valuation on a field F and let T be the associated SL_2-tree. Suppose that M is a compact 3-manifold with a representation $\phi : \pi_1(M) \to SL_2(F)$ such that $v((\phi(\gamma))) < 0$ for some $\gamma \in \pi_1(M)$. Then M contains an essential surface associated to the SL_2-tree T.*

Lemma 1 exhibits an attractive connection between the topology of 3-manifolds and the theory of SL_2-trees. This connection was first explored in the proof of the Smith conjecture [MB] and further developed in work of Culler-Shalen and others [CS 1-2] [CGLS] [CCGLS]. In particular the following lemma follows from [CS1] (although they only considered a special case there).

Lemma 2 *Let M be a compact 3-manifold with nonempty boundary such that $\pi_1(M)$ acts nontrivially on the SL_2-tree T defined by a discrete valuation v of a field F via a representation $\phi : \pi_1(M) \to SL_2(F)$.*
(1) Let $C \subset \partial M$ be a connected complex such that $tr(\phi(\gamma)) \in O_v$ for every loop in C, then there exists an essential surface S in M associated to the tree such that S is disjoint from C.
(2) If $\gamma \subset M$ is a loop such that $tr(\phi(\gamma)) \notin O_v$, then γ intersects every essential surface associated to the tree.

Note that in Lemma 2, any loop γ in M can be considered as an element in $\pi_1(M)$ up to conjugation and taking inverse, and thus $tr(\phi(\gamma))$ is well defined.

Proof. (1) Let $i^* : \pi_1(C) \to \pi_1(M)$ be the inclusion induced homomorphism. Then $i^*(\pi_1(C))$ is a finitely generated subgroup of $\pi_1(M)$ (well defined up to conjugation). The condition of (1) implies that every element in this subgroup has a fixed point in T. Hence by [Se, Corollary 3 to Proposition 26], the subgroup $i^*(\pi_1(C))$ is contained in a vertex stabilizer. Now one can apply [CS1, Proposition 2.3.1] to get the conclusion of (1).

(2) Let S be an essential surface in M associated to the action. Then for each component D of $M - S$, $i^*(\pi_1(D)) \subset \pi_1(M)$ is contained in a vertex stabilizer. Hence the given loop γ cannot be contained in such component D since γ does not fix any vertex of T. So γ must intersect the surface S. □

A compact 3-manifold is called a knot exterior if it has boundary which is a torus. Recall that for a knot exterior M, a slope in the torus ∂M is called a boundary slope if there is an essential surface S in M such that ∂S is a non-empty set of parallel simple essential loops in ∂M of the given slope.

Corollary 3 *Let M be an irreducible knot exterior such that $\pi_1(M)$ acts non-trivially on the SL_2-tree defined by a discrete valuation v of a field F via a representation $\phi : \pi_1(M) \to SL_2(F)$.*
(1) If $tr(\phi(\gamma)) \in O_v$ for two different slopes γ in ∂M, then M contains an essential closed surface associated to the tree.
(2) If $tr(\phi(\delta)) \in O_v$ for one slope δ in ∂M and $tr(\phi(\gamma)) \notin O_v$ for another slope γ in ∂M, then M contains an essential surface S associated to the tree such that the boundary of S is nonempty and is of the slope δ, i.e. δ is a boundary slope.

We call a boundary slope which arises as in part (2) of Corollary 3 strongly associated to (or strongly detected by) an SL_2-tree, following Cooper-Long [CL2].

Often used fields with discrete valuations for detecting essential surfaces in 3-manifolds are the following two types:

Type 1 (from an ANI-representation): Suppose that M is a compact 3-manifold which has a representation $\rho : \pi_1(M) \to SL_2(F)$, where F is an algebraic number field, i.e. a finite extension field of $\mathbb{Q}$, such that $tr(\rho(\delta))$ is not an algebraic integer for some element γ in $\pi_1(M)$. We shall call such representation ρ an ANI-representation of $\pi_1(M)$ in $SL(2, \mathbb{C})$ (here ANI stands for algebraic non-integral). Now let $\overline{\mathbb{Z}}_{\mathrm{F}}$ be the integral closure of $\mathbb{Z}$ in F. Then $\overline{\mathbb{Z}}_{\mathrm{F}}$ is the ring of algebraic integers in F and it is well known that

$$\overline{\mathbb{Z}}_{\mathrm{F}} = \cap\{O_v;\, v \text{ discrete valuation of } F\}.$$

Since $tr(\rho(\gamma))$ is not an algebraic integer, there is a discrete valuation v of F such that $tr(\rho(\gamma)) \notin O_v$. Hence by Lemma 1, M contains an essential surface associated to the SL_2-tree defined by the pair (F, v). Henceforth we shall call an essential surface which arises this way associated to (or detected by) an ANI-representation of $\pi_1(M)$ in $SL(2, \mathbb{C})$. The interplay between essential surfaces and ANI-representations played a crucial role in the final resolution of the Smith conjecture [MB]. Note also that every discrete valuation v of a number field F is a P-adic valuation defined by a prime ideal P of $\overline{\mathbb{Z}}_{\mathrm{F}}$. So we may also say that an essential surface associated to an ANI-representation is detected by a P-adic valuation.

Type 2 (from an ideal point): Let M be a compact 3-manifold. Now instead of considering a single representation of $\pi_1(M)$ into $SL(2, \mathbb{C})$, we consider a certain family of them. Let $R(M)$ be the set of all $SL(2, \mathbb{C})$-representations and let $X(M)$ be the set of the characters of elements in $R(M)$ (see [CS1] for definitions). Then both $R(M)$ and $X(M)$ have naturally the structure of algebraic set and are thus usually called the $SL(2, \mathbb{C})$ representation variety and character variety of M respectively. Note that the natural surjective map $t : R(M) \to X(M)$ which sends a representation to its character is a regular map between the two algebraic sets. Suppose now that $X(M)$ is positive dimensional as an algebraic set and let $X_0 \subset X(M)$ be any algebraic curve. Let $\tilde{X}_0$ be the smooth projective completion of X_0. Any ideal point $x \in \tilde{X}_0$ defines a discrete valuation v_x on the function field $K = \mathbb{C}(\tilde{X}_0) \cong \mathbb{C}(X_0)$ as follows: for any rational function $f \in K = \mathbb{C}(\tilde{X}_0)$, $v(f) = n$ if x a zero of f of order n, and $v(f) = -n$ if x is a pole of f of order n. Let R_0 be an irreducible component of $t^{-1}(X_0)$ such that $t(R_0) = X_0$ and let $F = \mathbb{C}(R_0)$ be the function field on R_0. Then F is a finitely generated extension of K and the valuation v_x extends to a discrete valuation, which we still denote by v_x for simplicity, on F ([MS]). On the other hand there is a tautological representation $P : \pi_1(M) \to SL_2(F)$, which induces a nontrivial action of $\pi_1(M)$ on the SL_2-tree defined by the pair (F, v_x). In fact the non-triviality is guaranteed by the existence of an element $\gamma \in \pi_1(M)$ with $v_x(tr(P(\gamma))) < 0$ (see [CS1] [CGLS] for details). Thus by Lemma 1, M has an essential surface associated to the tree defined by (F, v_x). Henceforth we shall call an essential surface which arises this way associated to (or detected by) an ideal point of a curve of the character variety of M. The interplay between essential surfaces in 3-manifolds and ideal points of character varieties has led to significant progresses in understanding 3-manifold topology (see [Sh] for a comprehensive exposition).

Our first result says that in strongly detecting boundary slopes, ANI-representations are at least as effective as ideal points.

Theorem 4 *Let M be an irreducible knot exterior. If a boundary slope of M is strongly detected by an ideal point of a curve in $X(M)$, then it is strongly detected by an ANI-representation of $\pi_1(M)$ into $SL(2, \mathbb{C})$.*

In most cases, the converse of Theorem 4 also holds, which has been discussed and proved in [CL3]. We note that [CL3, Corollary 10] did not give the whole converse of Theorem 4; there was a condition implicitly assumed in its proof. See Remark 15 for more detailed explanation.

Actually something more general is true. Our next result indicates that in strongly detecting boundary slopes with SL_2-trees, we only need to use ANI-representations, provided that the relevant field F with discrete valuation has characteristic zero.

Theorem 5 *Let M be an irreducible knot exterior. If a boundary slope of M is strongly detected by a $SL_2(F)$-tree where the field F has characteristic zero, then the boundary slope is also strongly detected by an ANI-representation.*

There are closed graph manifolds with only finite cyclic $SL(2, \mathbb{C})$-representations [Mi]. Hence an essential torus in such a manifold can not be detected by either an ideal point or an ANI-representation. In [BZ1] closed hyperbolic Haken 3-manifolds were found whose $SL(2, \mathbb{C})$ character varieties are zero dimensional and thus essential surfaces in such manifolds can not be detected by ideal points of $SL(2, \mathbb{C})$-character varieties. The following theorem seems to suggest that ANI-representations are better detectors of essential surfaces than ideal points.

Theorem 6 *There exist closed hyperbolic 3-manifolds which have essential surfaces associated to ANI-representations but not associated to ideal points.*

The first effort to answer the question of whether every boundary slope of a knot exterior is strongly detected by an ideal point was made in [CL1], unfortunately with an error found later, as mentioned in [CL2]. Here we give a negative answer to this question by proving the following theorem.

Theorem 7 *There exist infinitely many irreducible knot exteriors M each of which has a boundary slope which cannot be strongly detected by either an ideal point of a curve of $X(M)$ or an ANI-representation of $\pi_1(M)$.*

Note that if T is a SL_2-tree defined by a field F with discrete valuation v, the identity element and its negative in $SL_2(F)$ act trivially on T. Hence the $SL_2(F)$ action on T factors through a $PSL_2(F)$ action on T. We remark that all our above results still hold if we consider everything in the more general setting of PSL-representations. For simplicity we do not pursue this generality here, except that at a few convenient occasions we make some relevant notes. For instance, our Example 18 shows that there are closed Haken 3-manifolds (non-hyperbolic) which have no ANI-representations into $SL(2,\mathbb{C})$ but have ANI-representations into $PSL(2,\mathbb{C})$, which means that such manifolds contain essential surfaces which are not detected by SL_2-trees but by PSL_2-trees.

The following questions remain open.

Question 8 Let M be a hyperbolic knot exterior. Is every boundary slope of M strongly detected by an ideal point of a curve of $X(M)$?

Question 9 Is there a closed hyperbolic Haken 3-manifold which has no ANI representation into $PSL(2,\mathbb{C})$?

We remark that very often, for a Haken hyperbolic 3-manifold, a discrete faithful representation of its fundamental group into $SL(2,\mathbb{C})$ or $PSL(2,\mathbb{C})$ may not be an ANI-representation. For instance, this occurs for any finite Haken cover (if such exists) of a non-Haken hyperbolic 3-manifold. The following question is more difficult but more attractive; a positive answer would imply the well known virtual Haken conjecture.

Question 10 Let π be the fundamental group of an irreducible 3-manifold. If π is not a finite group, does it have a finite index subgroup $\tilde{\pi}$ which has an ANI-representation into $PSL(2,\mathbb{C})$?

The main tool used in proving Theorems 4-6 is the A-polynomial introduced in [CCGLS]. Several feature properties of the A-polynomial found in [CCGLS] [CL3] [BZ2] will be applied. Theorem 7 is proved by a careful gluing technique of representations along an essential torus. After some preparation in Sect. 2, Theorems 4-7 will be proven in Sect. 3-6, which constitute the rest of the paper.

2. Some preparation

Let $\overline{\mathbb{Z}}$ denote the ring of all algebraic integers and $\overline{\mathbb{Q}}$ the field of all algebraic numbers. We first recall the following form of the Hilbert Nullstellensatz in algebraic geometry.

Theorem 11 *Suppose that $V \subset \mathbb{C}^n$ is a nonempty affine algebraic set defined by polynomials with coefficients in $\overline{\mathbb{Q}}$. Then V contains at least one point whose coordinates are all in $\overline{\mathbb{Q}}$.*

In particular if v is an isolated point in V, then all coordinates of v are in $\overline{\mathbb{Q}}$.

For a finitely generated group π, we use $R(\pi)$ and $X(\pi)$ to denote the $SL(2, \mathbb{C})$ representation variety and character variety of π.

Lemma 12 *Let π be a finitely generated group and let $\gamma_1, ..., \gamma_k$ be some given elements in π. If π has a representation $\rho_0 \in R(\pi)$ such that $tr(\rho_0(\gamma_i)) \in \overline{\mathbb{Q}}$ for $i = 1, ..., k$, then π has a representation $\rho_1 \in R(\pi)$ such that $tr(\rho_1(\gamma_i)) = tr(\rho_0(\gamma_i))$ for $i = 1, ..., k$ and such that $\rho_1(\pi)$ is contained in $SL_2(F)$ for some number field F.*

Proof. It is well known that the affine variety $R(\pi)$ is the zero set of the set of polynomials over $\mathbb{Q}$ coming from a presentation of $\pi_1(M)$. We may assume that γ_i, $i = 1, ..., k$, are among the generators of the presentation. Now add the equations $tr(\rho(\gamma_i)) = tr(\rho_0(\gamma_i))$, $i = 1, ..., k$, to the set of defining equations of $R(M)$, we get a subvariety which is non-empty (since at least ρ_0 is contained in it) and is certainly defined over $\overline{\mathbb{Q}}$. Now we apply Hilbert Nullstelensatz to this subvariety. □

Lemma 13 *Suppose that $A(X, Y) = A_0(X) + A_1(X)Y + ... + A_n(X)Y^n \in \overline{\mathbb{Z}}[X, Y]$ is a two variable polynomial such that $n > 0$, $A_n(X) = \epsilon X^k f(X)$ where $\epsilon = 1$ or -1, $k \geq 0$, $f(X)$ is monic of positive degree, $f(0) \neq 0$, and no root of $f(X)$ is also a root of all $A_i(X)$. Then apart from finitely many primes p, for every p-th root of unity $\xi \neq 1$ and every root α of $f(X)$, the polynomial $A(\xi\alpha, Y)$ in Y has a root y in $\overline{\mathbb{Q}}$ but not in $\overline{\mathbb{Z}}$.*

Proof. First note that if $\xi \neq 1$ is a p-th root of unity, then $\xi - 1$ is a root of

$$\frac{(X+1)^p - 1}{(X+1) - 1} = X^{p-1} + pX^{p-2} + ... + p,$$

which is irreducible by Eisenstein's criterion. From this we only need two facts: that $\xi - 1$ divides p in $\overline{\mathbb{Z}}$, and that $\xi - 1$ is not a unit of $\overline{\mathbb{Z}}$.

We say that x divides y in a commutative ring if there exists z in that ring with $xz = y$; note that if x and y are in $\mathbb{Z}$ and x divides y in $\overline{\mathbb{Z}}$, then x divides y

in $\mathbb{Z}$. Now since $f(X)$ is monic and $f(0) \neq 0$, each root α of $f(X)$ is a non-zero algebraic integer. Choose some i such that α is not a root of $A_i(X)$, and since $\overline{\mathbb{Z}}$ is a ring, we may write

$$A_i(X) = (X - \alpha)g(X) + c$$

with $g(X) \in \overline{\mathbb{Z}}[X]$ and $0 \neq c \in \overline{\mathbb{Z}}$. Let d be the product of all the conjugates of c. If p does not divide d, then

$$A_i(\xi\alpha) = (\xi\alpha - \alpha)g(\xi\alpha) + c$$

is not divisible by $\xi - 1$ in $\overline{\mathbb{Z}}$. For otherwise $\xi - 1$ would divide c, hence divide d; but $\xi - 1$ divides p, hence $\xi - 1$ would divide 1, contradicting the fact that $\xi - 1$ is not a unit.

Suppose in addition that $\xi\alpha$ is not a root of $A_n(X) = X^k f(X)$; this gives finitely many additional primes to avoid. Now since $A_n(X) = X^k(X - \alpha)h(X)$ with $h(X) \in \overline{\mathbb{Z}}[X]$,

$$A_n(\xi\alpha) = (\xi\alpha)^k(\xi\alpha - \alpha)h(\xi\alpha)$$

is divisible by $\xi - 1$. It follows that $A_n(\xi\alpha)$ does not divide $A_i(\xi\alpha)$ in $\overline{\mathbb{Z}}$ and thus

$$A(\xi\alpha, Y) = A_n(\xi\alpha)\Pi(Y - y_j)$$

has a root y_j which is not in $\overline{\mathbb{Z}}$.

Each root α of $f(X)$ thus gives only finitely many primes p to avoid, so the lemma is proved. □

Corollary 14 *Suppose that* $A(X, Y) = A_0(X) + A_1(X)Y + ... + A_n(X)Y^n \in \overline{\mathbb{Z}}[X, Y]$ *is a two variable polynomial such that* $n > 0$, $A_n(X) = \epsilon X^k f(X)$ *where* $\epsilon = 1$ *or* -1, $k \geq 0$, $f(X)$ *is monic of positive degree,* $f(0) = 1$ *or* -1. *Then we have either*

(1) $A(X, Y) = 0$ *has infinitely many solutions* (x, y) *such that* x *is an algebraic unit and* y *is an algebraic number but is not an algebraic integer; or*

(2) $f(X)$ *has an irreducible non-constant factor* $h(X)$ *over* $\mathbb{Z}$ *such that* $h(X)$ *divides* $A(X, Y)$ *in* $\mathbb{Z}[X, Y]$.

Proof. Since $f(X)$ is monic and $f(0) = \pm 1$, every root α of $f(X)$ is an algebraic unit. If α is not a root of $A_i(X)$ for some i, then by Lemma 13, $A(X, Y) = 0$ has infinitely many solutions of the form $(\xi\alpha, y)$ where ξ is root of unity and y is not an algebraic integer. Thus we have part (1) of the corollary.

So suppose that a root α of $f(X)$ is also a root of $A_i(X)$ for all i. Let $h(X)$ be the minimal polynomial of α over $\mathbb{Z}$. Then $h(X)$ is not a constant and divides $f(X)$ and every $A_i(X)$. This gives part (2) of the corollary. □

We now recall and establish some notations. For a compact manifold W, we already used $R(W)$ and $X(W)$ to denote the $SL(2, \mathbb{C})$ representation variety and character variety of $\pi_1(W)$ respectively, and $t : R(W) \to X(W)$ the canonical map. We shall use $\overline{R}(W)$ and $\overline{X}(W)$ to denote the $PSL(2,\mathbb{C})$ representation variety and character variety of $\pi_1(W)$ respectively (cf. [BZ1] for discussion regarding $PSL(2,\mathbb{C})$ representation theory of 3-manifolds). Recall that each element χ_ρ is the character of an element $\rho \in R(W)$, which is a complex valued function defined on $\pi_1(W)$ as $\chi_\rho(\gamma) = tr(\rho(\gamma))$ for each $\gamma \in \pi_1(W)$ and that similarly each element $\overline{\chi}_{\bar\rho}$ is the character of an element $\bar\rho \in R(W)$, which is a complex valued function defined on $\pi_1(W)$ as $\overline{\chi}_{\bar\rho}(\gamma) = [tr(B)]^2$ for each $\gamma \in \pi_1(W)$, where $B \in SL(2,\mathbb{C})$ is an inverse of $\bar\rho(\gamma) \in PSL(2,\mathbb{C})$ under the canonical map from $SL(2,\mathbb{C})$ to $PSL(2,\mathbb{C})$. Obviously $\overline{\chi}_{\bar\rho}$ is well defined. Also recall that each element $\gamma \in \pi_1(W)$ defines a regular function τ_γ on $X(W)$ as $\tau_\gamma(\chi_\rho) = \chi_\rho(\gamma)$ for each $\chi_\rho \in X(W)$ (called the trace function of γ on $X(W)$) and also defines a regular function $\bar\tau_\gamma$ on $\overline{X}(W)$ as $\bar\tau_\gamma(\overline{\chi}_{\bar\rho}) = \overline{\chi}_{\bar\rho}(\gamma)$ for each $\overline{\chi}_{\bar\rho} \in \overline{X}(W)$ (called the trace function of γ on $\overline{X}(W)$).

With respect to a fixed meridian-longitude basis $\{\mu, \lambda\}$ in a torus T, the set of slopes in T can be parameterized by $\mathbb{Q} \cup \{1/0\}$. By convention, we use p/q to denote the slope whose meridian coordinate is p and longitude coordinate q. If M is a 3-manifold with a torus boundary component T, then $M(T, p/q)$ denotes the manifold obtained by Dehn filling M along T with slope p/q. If M is a knot exterior, then we simply use $M(p/q)$ to denote $M(\partial M, p/q)$.

For a knot exterior M and a fixed basis $\{\mu, \lambda\}$ for $\pi_1(\partial M)$, we use $A(X, Y)$ to denote the A-polynomial of M with respect to the basis. For convenience of arguments in later sections, we briefly recall the construction of $A(X, Y)$. We refer the reader to [CCGLS] for details. Let $i^* : X(M) \to X(\partial M)$ be the regular map induced by the inclusion induced homomorphism $i_* : \pi_1(\partial M) \to \pi_1(M)$. Let Λ be the set of diagonal representations of $\pi_1(\partial M)$, i.e.

$$\Lambda = \{\rho \in R(\partial M) \mid \rho(\mu), \rho(\lambda) \text{ are diagonal matrices}\}.$$

Then Λ is a subvariety of $R(\partial M)$ and $t|_\Lambda : \Lambda \to X(\partial M)$ is a degree 2 surjective map. We may identify Λ with $\mathbb{C}^* \times \mathbb{C}^*$ through the eigenvalue map $P : \Lambda \to \mathbb{C}^* \times \mathbb{C}^*$, which sends $\rho \in \Lambda$ to $(x, y) \in \mathbb{C}^* \times \mathbb{C}^*$ if $\rho(\mu) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ and $\rho(\lambda) = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$. Let V be the Zariski closure of $i^*(X(M))$ in $X(\partial M)$. Then it is known that V is at most one dimensional. Let V_1 be the set of one dimensional components in V, let Z be the algebraic curve $t|_\Lambda^{-1}(V_1)$ in Λ, and let D to be the Zariski closure of $P(Z)$ in $\mathbb{C} \times \mathbb{C}$. Then $A(X, Y)$ is the defining polynomial of the plane curve D with no repeated factors. Note that as in [CCGLS], $A(X, Y)$ is normalized so that it is in $\mathbb{Z}[X, Y]$, well defined up to sign.

3. Proof of Theorem 4

The main tool we shall use in proving this theorem and next two is the A-polynomial introduced in [CCGLS]. The proof involves some applications of some powerful results found in [CCGLS], [CL3] and [BZ2], together with the number theoretical results given in the previous section.

Let μ be a boundary slope strongly detected by an ideal point x of some curve $X_0 \subset X(M)$. This means, by definition, that there is a sequence of representations $\rho_j \in t^{-1}(X_0)$ such that $\{tr(\rho_j(\mu))\}$ are bounded in $\mathbb{C}$, but $\{tr(\rho_j(\lambda))\}$ are not for some element λ in $\pi_1(\partial M)$. We may assume by Corollary 3 that (μ, λ) form a basis for $\pi_1(\partial M)$. Let $A(X, Y)$ be the A-polynomial of M defined with respect to this basis as recalled in Sect. 2. Since $\{tr(\rho_j(\lambda))\}$ are not bounded, X_0 will contribute a component in the plane curve defined by the A-polynomial of M. Obviously the trace function τ_λ is not constant on X_0. By [BZ1], X_0 is a curve which provides a Culler-Shalen semi-norm or norm on the real 2-plane $H_1(\partial M; \mathbb{R})$, according as τ_μ is a constant or not on X_0.

If τ_μ is constant, say $\tau_\mu \equiv \xi$, on X_0, then by the construction of $A(X, Y)$, we see that $X - \xi$ is a factor of $A(X, Y)$ and ξ must be a root of unity. Since the trace function of λ is not constant, there exist plenty of representations ρ in $t^{-1}(X_0)$ with $tr(\rho(\lambda))$ in $\overline{\mathbb{Q}}$ but not in $\overline{\mathbb{Z}}$. Hence by Lemma 12, we may find such $\rho \in R(M)$ which is an ANI-representation of $\pi_1(M)$ with $tr(\rho(\mu)) = \xi$ and $tr(\rho(\lambda)) \notin \overline{\mathbb{Z}}$. So μ is a boundary slope strongly detected by an ANI-representation.

If τ_μ is not constant on X_0, then X_0 provides a Culler-Shalen norm on $H_1(\partial M; \mathbb{R})$ whose norm polygon, denoted B_0, has a pair of corners of slope 0 in the XY-plane (corresponding to the slope μ) [BZ1]. Let $A_0(X, Y)$ be the factor of $A(X, Y)$ corresponding to the curve X_0 and let N_0 be the Newton polygon of $A_0(X, Y)$. Then it is proved in [BZ2] that the norm polygon B_0 and the Newton polygon N_0 are dual to each other, meaning that the norm polygon B_0 has a pair of corners of slope p/q if and only if the Newton polygon N_0 has a pair of sides of the same slope, and vice versa. So in particular N_0 has a pair of sides of slope 0 in the (XY)-plane. This implies that the Newton polygon N of the whole A-polynomial $A(X, Y)$ must have a pair of sides of slope 0. Note that any coefficient of $A(X, Y)$ which corresponds to a corner of its Newton polygon N is equal to 1 or -1 [CL3]. Hence the polynomial $A(X, Y) \in \mathbb{Z}[X, Y]$ must have the form

$$A(X, Y) = A_0(X) + A_1(X)Y + ... + A_n(X)Y^n$$

such that $A_i(X) \in \mathbb{Z}[X, Y]$, $A_0 = \epsilon_1 X^j f(X)$ and $A_n(X) = \epsilon_2 X^k f(X)$, where $n > 0$, $\epsilon_1 = 1$ or -1, $\epsilon_2 = 1$ or -1, j and k are non-negative integers, $f(X)$ is monic with positive degree, $f(0) = 1$ or -1. In fact $f(X)$ is the edge polynomial of the pair of sides of N with the slope 0. Hence by Corollary 14, we either have that $A(X, Y) = 0$ has infinitely many solutions (x, y) such that x is an algebraic unit, y is not an algebraic integer but is an algebraic number; or that $A(X, Y)$ contains an irreducible factor of the form $h(X) \in \mathbb{Z}[X]$ which divides $f(X)$.

We first consider the former case. Note that except for finitely many points in the plane curve defined by $A(X, Y)$, a solution (x, y) of $A(X, Y) = 0$ corresponds to a representation $\rho \in R(M)$ such that x and x^{-1} are eigenvalues of $\rho(\mu)$, and y and y^{-1} are the eigenvalue of $\rho(\lambda)$. It follows that we have a representation $\rho_0 \in R(M)$ such that $tr(\rho_0(\mu))$ is an algebraic integer and $tr(\rho_0(\lambda))$ is not an algebraic integer but is an algebraic number. Here we used the fact that a number c is an algebraic unit if and only if $c + c^{-1}$ is an algebraic integer. Now applying Lemma 12, we get an ANI-representation $\rho_1 \in R(M)$ such that $tr(\rho_1(\mu))$ is an algebraic integer but $tr(\rho_1(\lambda))$ is not. Thus μ is a boundary slope also strongly detected by an ANI-representation.

We now consider the latter case when $A(X, Y)$ has an irreducible non-constant factor $h(X) \in \mathbb{Z}[X]$. Note that the plane curve defined by $h(X)$ corresponds to some curve of $X(M)$, on each irreducible component of which the trace function of μ is constant. Thus we may simply divide $h(X)$ from $A(X, Y)$ and apply the above argument to the remaining polynomial $A'(X, Y)$ since X_0 will still contribute a nontrivial factor in $A'(X, Y) \in \mathbb{Z}[X, Y]$. That is, we may assume that $f(X)$ is irreducible and does not divide $A(X, Y)$. This completes the proof of Theorem 4.

4. Proof of Theorem 5

Suppose that F is a field whose characteristic is zero, v is a discrete valuation on F, and μ is a boundary slope strongly detected by the SL_2-tree defined by the pair (F, v) via some representation $\phi : \pi_1(M) \to SL_2(F)$. So we have $v(tr(\phi(\mu))) \geq 0$ but $v(tr(\phi(\lambda))) < 0$ for another slope λ in ∂M (cf. Corollary 3 (2)). Note that μ is the unique slope on ∂M whose image under the map ϕ has trace with non-negative valuation under v. So we may assume that $\{\mu, \lambda\}$ form a basis of $\pi_1(\partial M)$.

Now let K be the subfield of F generated over the ground field $\mathbb{Q}$ by all the entries of $\phi(\gamma)$ for all $\gamma \in \pi_1(M)$. Since $\pi_1(M)$ is finitely generated, K is a finitely generated extension of $\mathbb{Q}$. We may also assume that K contains a uniformizer σ of the valuation ring O_v, i.e. σ is an element of F with $v(\sigma) = 1$. Hence v restricts to a discrete valuation on K, which we still denote by v. So μ is a boundary slope strongly detected by the SL_2-tree defined by the pair (K, v) via the representation $\phi : \pi_1(M) \to SL_2(K)$. On the other hand we may now embed K in $\mathbb{C}$ as a subfield. Therefore ϕ may be considered as a representation of $\pi_1(M)$ into $SL_2(\mathbb{C})$. Recall from Sect. 2, the Zariski closure V of $i^*(X(M))$ in $X(\partial M)$ is at most one dimensional and is defined over $\mathbb{Q}$. Now consider the image of the character $\chi_\phi \in X(M)$ under the map i^* in V. We divide the proof into two cases, depending on whether the point $i^*(\chi_\phi)$ is isolated or not in V.

Case 1. Suppose that $i^*(\chi_\phi)$ is an isolated point of V.
Then by the Hilbert Nullstellensatz, both $tr(\phi(\mu))$ and $tr(\phi(\lambda))$ are algebraic numbers. Hence by Lemma 12, there is an ANI-representation $\rho_1 \in R(M)$ such that $tr(\rho_1(\mu)) = tr(\phi(\mu))$ and $tr(\rho_1(\lambda)) = tr(\phi(\lambda))$. Since ρ_1 is ANI, there is a number field J such that $\rho_1(\pi_1(M)) \subset SL_2(J)$. Let $w = tr(\rho_1(\mu)) = tr(\phi(\mu))$ and $z = tr(\rho_1(\lambda)) = tr(\phi(\lambda))$. Since we have $v(w) \geq 0$ and $v(z) < 0$, z cannot be integral over the subring $\mathbb{Z}[w]$ of J. Now by a theorem of Krull ([Jn, Page 255, Theorem 14]), the integral closure of $\mathbb{Z}[w]$ in J is the intersection of all valuation rings in J which contain $\mathbb{Z}[w]$. It follows that the integral closure of $\mathbb{Z}[w]$ is contained in some valuation ring of J which misses z. But for a number field J, any valuation ring of J, different from J, is the valuation ring of a discrete valuation of J. Hence J has a discrete valuation v' such that $w \in O_{v'}$ but $z \notin O_{v'}$. This means that μ is also a boundary slope strongly detected by the ANI-representation $\rho_1 : \pi_1(M) \to SL_2(J)$.

Case 2. Suppose that $i^*(\chi_\phi)$ is not an isolated point of V.
Therefore we may assume that $i^*(\chi_\phi)$ is contained in a one dimensional component of V. This implies that if $A(X, Y)$ is the A-polynomial of M with respect to the basis $\{\mu, \lambda\}$, then $A(X, Y)$ has a solution (x_0, y_0) such that x_0 is an eigenvalue of $\phi(\mu)$ and y_0 is the eigenvalue of $\phi(\lambda)$. Now by Theorem 4, it is sufficient to show that μ is also detected by some ideal point of some curve X_0 in $X(M)$. Suppose otherwise. Then the A-polynomial must have the form

$$A(X, Y) = \epsilon_1 X^j + A_1(X)Y + ... + A_{n-1}(X)Y^{n-1} + \epsilon_2 X^k Y^n \in \mathbb{Z}[X, Y],$$

where $\epsilon_1 = 1$ or -1, $\epsilon_2 = 1$ or -1 (i.e. the Newton polygon of $A(X, Y)$ cannot have sides of slope 0). Since $A(X, Y) = 0$ has the solution (x_0, y_0) such that x_0 is an v-unit but y_0 is not (again we used the fact that a number c is an v-unit if and only if $c + c^{-1}$ is a v-integer; i.e. $v(c) = 0$ if and only if $v(c + c^{-1}) \geq 0$). But obviously when x_0 is a v-unit, all the roots of $A(x_0, Y)$ for the variable Y are v-units. This gives a contradiction. The proof of Theorem 5 is now complete.

Remark 15 From the proof above we see that if μ is a boundary slope of an irreducible knot exterior M strongly detected by an representation $\phi : \pi_1(M) \to SL_2(K)$ where K is a subfield of $\mathbb{C}$ with a discrete valuation, then μ is also strongly detected by an ideal point of a curve in $X(M)$ provided that $i^*(\chi_\phi)$ is not an isolated point in $\overline{i^*(X(M))}$. In the proof of [CL3, Corollary 10] it was implicitly assumed that for an ANI-representation $\phi \in R(M)$, $i^*(\chi_\phi)$ is not an isolated point of $\overline{i^*(X(M))} \subset X(\partial M)$. But that seems not to be known. It does not follow directly from the proof of [T1, Theorem 5.6].

5. Proof of Theorem 6

Recall that a knot exterior is called small if it contains no closed essential surfaces. We have noted that if $A(X, Y)$ is the A-polynomial of a knot exterior M with respect to a basis $\{\mu, \lambda\}$ of $\pi_1(\partial M)$, then except for finitely many, a solution (x, y) of $A(X, Y) = 0$ corresponds to a representation $\rho \in R(M)$ such that x is an eigenvalue of $\rho(\mu)$ and y an eigenvalue of $\rho(\lambda)$. We can be more specific about this in case M is a small knot exterior, as explained in the following lemma.

Lemma 16 *Let M be a small knot exterior. Let $A(X, Y)$ be the A-polynomial of M with respect to a basis $\{\mu, \lambda\}$ of $\pi_1(\partial M)$. If (x, y) is a solution of $A(X, Y) = 0$ such that $x \neq 0$ and $y \neq 0$, then there is a representation $\rho \in R(M)$ such that $tr(\rho(\mu)) = x + x^{-1}$ and $tr(\rho(\lambda)) = y + y^{-1}$.*

Proof. Let D_0 be a component of the plane curve D defined by $A(X, Y)$ which contains the point (x, y). From the construction of the A-polynomial (recalled in Sect. 2), we see that there is an irreducible curve X_0 in $X(M)$ and an irreducible curve Z_0 in Λ (notations from Sect. 2) such that D_0 is the Zariski closure of $P(Z_0)$ in $\mathbb{C} \times \mathbb{C}$, that the Zariski closure Y_0 of $i^*(X_0)$ in $X(\partial M)$ is an irreducible curve, and that the regular map $i^*|_{X_0} : X_0 \to Y_0$ and the regular map $t|_{Z_0} : Z_0 \to Y_0$ are dominating maps. Passing to smooth projective completion, we get rational maps $\tilde{P} : \tilde{Z}_0 \to \tilde{D}_0$, $\tilde{i}^* : \tilde{X}_0 \to \tilde{Y}_0$ and $\tilde{t} : \tilde{Z}_0 \to \tilde{Y}_0$, all surjective. Let d be a point in $\tilde{D}_0$ which maps to the point (x, y) in D_0 under the birational isomorphism $\tilde{D}_0 \to D$, let z be a point in $\tilde{Z}_0$ such that $\tilde{P}(z) = d$, let $v = \tilde{t}(z) \in \tilde{Y}_0$ and finally let u be a point in $\tilde{X}_0$ such that $\tilde{i}^*(u) = v$. We claim that u is not an ideal point of $\tilde{X}_0$. Suppose it were, then since $\tilde{\tau}_\mu(u) = x + x^{-1}$ and $\tilde{\tau}_\lambda(u) = y + y^{-1}$ are both finite, it follows from Corollary 3 (1) that M contains a closed essential surface. But this contradicts our assumption that M is small. Therefore u is mapped to a point χ_ρ in X_0 under the birational isomorphism $\tilde{X}_0 \to X_0$. It follows from the construction of D_0 that x is an eigenvalue of $\rho(\mu)$ and y an eigenvalue of $\rho(\lambda)$. $\square$

For a knot exterior M in the 3-sphere S^3, $\{\mu, \lambda\}$ will always be the standard meridian-longitude basis of $\pi_1(\partial M)$ and slopes on ∂M will be parameterized by rational numbers with respect to this basis.

Example 17 Let M be the exterior of the 5_2 knot in S^3. Then the manifold $M(10)$ is a hyperbolic Haken 3-manifold with 0-dimensional $SL(2, \mathbb{C})$ character variety but with an ANI-representation. Hence $M(10)$ has an essential closed surface which cannot be detected by ideal points of $X(M(10))$ but can be detected by an ANI-representation.

Proof. By [HT], we know that the knot exterior M is small and hyperbolic, the slope 10 is a boundary slope but is not a boundary slope of a punctured essential torus or a punctured 2-sphere in M. Hence $M(10)$ is irreducible and contains no essential torus. Now applying [CGLS,Theorem 2.0.3], we see that $M(10)$ must be a Haken 3-manifold. Hence according to Thurston [T2], $M(10)$ is a hyperbolic 3-manifold.

The A-polynomial of M with respect to the standard meridian-longitude basis $\{\mu, \lambda\}$ is given in the appendix of [CCGLS]:

$$A(X, Y) = (Y - 1)[1 + (-1 + 2X^2 + 2X^4 - X^8 + X^{10})Y \\ +(X^4 - X^6 + 2X^{10} + 2X^{12} - X^{14})Y^2 + X^{14}Y^3]$$

Now let $YX^{10} = 1$, then $A(X, X^{-10})$ has an irreducible factor

$$(2 - X^2 + X^6 + 4X^8 + X^{10} - X^{14} + 2X^{16})/X^{16}.$$

Hence $A(X, Y) = 0$ has a solution (x, y) such that x is not an algebraic unit and $yx^{10} = 1$. By Lemma 16, there is a representation $\rho \in R(M)$ such that $tr(\rho(\mu)) = x + x^{-1}$ and $tr(\rho(\lambda)) = y + y^{-1}$. Hence $tr(\rho(\mu^{10}\lambda)) = 2$. It follows that $\rho(\mu^{10}\lambda) = I$ (since $\rho(\mu)$ is not parabolic). Therefore $\rho \in R(M(10))$. It also follows that $X(M(10))$ is zero dimensional. For otherwise the A-polynomial $A(X, Y)$ of M would contain a factor $X^{10}Y - 1$. Hence by the Hilbert Nullstellensatz, the representation $\rho \in X(M(10))$ is an ANI-representation. □

Example 18 Let M be the exterior of the Fig. 8 knot in S^3. Then the manifold $M(-4)$ is a Haken 3-manifold with zero dimensional $PSL(2, \mathbb{C})$ character variety, with no ANI-representation into $SL(2, \mathbb{C})$, but with an ANI-representation into $PSL(2, \mathbb{C})$.

Proof. Again by [HT], the knot exterior M is small and hyperbolic, the slope -4 is a boundary slope of a punctured essential torus in M. It is also known that $M(-4)$ is a Haken 3-manifold with essential torus. The A-polynomial of M with respect to the standard meridian-longitude basis $\{\mu, \lambda\}$ is:

$$A(X, Y) = (Y - 1)[-X^4 + (1 - X^2 - 2X^4 - X^6 + X^8)Y - X^4Y^2].$$

Now let $YX^{-4} = 1$, then $A(X, X^4) = -X^6(X^4 - 1)(1 + X^2)^2$. Hence every $SL(2, \mathbb{C})$-representation $\rho \in R(M(-4))$ has $tr(\rho(\mu))$ an algebraic integer. Hence $R(M(-4)) \subset R(M)$ has no ANI representation by Corollary 3.

But if we set $YX^{-4} = -1$, then

$$A(X, X^4) = -X^4(-X^4 - 1)(X - 1)^2(X + 1)^2(2 + 3X^2 + 2X^4).$$

Hence $A(X, Y) = 0$ has a solution (x, y) such that x is not an algebraic unit and $yx^{-4} = -1$. By Lemma 16, there is a representation $\rho \in R(M)$ such that $tr(\rho(\mu)) = x + x^{-1}$ and $tr(\rho(\lambda)) = y + y^{-1}$. Also $\rho(\mu^{-4}\lambda) = -I$. Therefore there is an $PSL(2, \mathbb{C})$-representation $\overline{\rho} \in \overline{R}(M(-4))$ which is ANI.

It also follows that the $PSL(2,\mathbb{C})$-character variety $\overline{X}(M(-4))$ of $M(-4)$ is zero dimensional. For every $PSL(2,\mathbb{C})$-representation of M lifts to a $SL(2,\mathbb{C})$-representation of M. So if $\overline{X}(M(-4)) \subset \overline{X}(M)$ were positive dimensional, then $X(M)$ would have a curve X_0 such that for every $\rho \in t^{-1}(X_0)$, $\rho(\mu^{-4}\lambda) = I$ or $-I$. It follows that the A-polynomial $A(X,Y)$ of M would contain a factor $X^4Y - 1$ or $X^4Y + 1$. □

A similar argument applies to boundary slopes of other 2-bridge knots whose A-polynomials are given in [CCGLS]. All the cases we have checked worked out the same way as Example 17 if the boundary slope is not a boundary slope of an essential punctured torus. Therefore Theorem 6 follows.

6. Proof of Theorem 7

We proceed to construct the examples. Each example is obtained by gluing a torus knot exterior and a cabled space along a torus in such a way that the resulting knot exterior M has the property that there are two slopes α and β on ∂M such that for each component X_0 of $\overline{X}(M)$, exactly one of the trace functions $\bar{\tau}_\alpha$ and $\bar{\tau}_\beta$ is a constant function on X_0 and that on the other hand there is a boundary slope δ on ∂M which is neither α nor β. Hence the slope δ cannot be strongly detected by any ideal point of any curve in $X(M)$. We now give the details.

Let M_0 be the knot complement of the (s,t)-torus knot, $s,t \geq 2$, $(s,t) = 1$. Let μ_0 be the meridian and λ_0 the longitude in ∂M_0. Let $C_{p,q}$ be the standard cabled space of type (p,q), $p,q \geq 2, (p,q) = 1$ ([GL]). Recall that $C_{p,q}$ is obtained as follows: let V be an unknotted trivial solid torus in S^3, V_1 a concentric solid torus in the interior of V, $K_{p,q}$ a curve on ∂V_1 of slope p/q in the standard meridian-longitude coordinates determined by V_1, $N(K_{p,q})$ a regular neighborhood of $K_{p,q}$ in V, then $C_{p,q} = V - int N(K_{p,q})$. Let $T_1 = \partial N(K_{p,q})$ and $T_2 = \partial V$. Then $\partial C_{p,q} = T_1 \cup T_2$, consisting of two tori. Let $\mu_1, \lambda_1 \subset T_1$ be the standard meridian and longitude of $K_{p,q}$ (when considered as a knot in $V \subset S^3$). Let $\mu_2, \lambda_2 \subset T_2$ be the standard meridian and longitude determined by the solid torus V. Slopes on each of T_0, T_1 and T_2 will be parametrized by the meridian-longitude pairs mentioned above. For convenience, we shall also consider a slope in T_i as an element in $\pi_1(T_i)$ or $H_1(T_i,\mathbb{Z})$.

Now we glue M_0 and $C_{p,q}$ together by an orientation reversing homeomorphism h from T_0 to T_1 satisfying $h(\mu_0) = \mu_1^{pq}\lambda_1$ and $h(\lambda_0) = \mu_1^{(1-stpq)}\lambda_1^{-st}$. The resulting 3-manifold is denoted by M. M is neither hyperbolic nor Seifert fibred. There is an essential annulus in $C_{p,q}$ with both its boundary component in T_2 with the slope p/q ([GL, Lemma 3.1]). This annulus certainly remains essential in M. Thus p/q is a boundary slope of M. Theorem 7 is equivalent to

Theorem 19 *The boundary slope p/q in $\partial M = T_2$ cannot be strongly detected by an ideal point of a curve in $X(M)$.*

Proof. We use $M(m/n)$ to denote the manifold obtained by Dehn filling M along ∂M with slope m/n, $C_{p,q}(T_2, m/n)$ the manifold obtained by Dehn filling $C_{p,q}$ along T_2 with slope m/n, and $M_0(m/n)$ similarly defined.

Claim 20 $M(p/q)$ is the lens space $L(q, p)$.

Note that $M(p/q) = C_{p,q}(T_2, p/q) \cup_h M_0$. Also $C_{p,q}(T_2, p/q) = L(q, p)\#V_*$, where $L(q, p)$ is a lens space whose fundamental group has order q and V_* is a solid torus whose meridian slope is the slope pq on $T_1 = \partial C_{p,q}(T_2, p/q) = \partial V$. This follows easily by considering the natural Seifert fibration on $C_{p,q}$ (cf. [GL, Lemma 3.1]). Therefore V_* is filled in M_0 along $T_0 = \partial M_0$ with the slope μ_0 of $T_0 = \partial M_0$ which produces the 3-sphere S^3. The claim follows.

Recall that $\overline{R}(W)$ denote the $PSL(2, \mathbb{C})$ representation variety of a compact 3-manifold W (Sect. 2).

Claim 21 $\overline{R}(M_0) = \overline{R}(M_0(st)) \cup \overline{R}(M_0(0))$.

Obviously $\overline{R}(M_0)$ contains $\overline{R}(M_0(st)) \cup \overline{R}(M_0(0))$. Let $\bar{\rho} \in \overline{R}(M_0)$. We need to show that $\bar{\rho}$ is either in $\overline{R}(M_0(st))$ or in $\overline{R}(M_0(0))$. Since $H_1(M_0, \mathbb{Z}_2) = \mathbb{Z}_2$, the representation $\bar{\rho} : \pi_1(M_0) \to PSL(2, \mathbb{C})$ lifts to a representation $\rho : \pi_1(M_0) \to SL(2, \mathbb{C})$; i.e. if $\phi : SL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$ is the canonical quotient map, then $\bar{\rho} = \phi \circ \rho$. Note that M_0 is a Seifert fibered space whose base orbifold is a disk with two singular points of indices s and t [Mr]. If ρ is a non-abelian representation, then ρ must send the fiber of the Seifert fibration of M_0 (considered as an element of $\pi_1(M_0)$) to the identity matrix I or $-I$. Hence $\bar{\rho} \in \overline{R}(M_0(st))$. If ρ is an abelian representation, then $\bar{\rho} \in \overline{R}(M_0(0))$. The claim is proved.

Claim 22 $\overline{R}(M) = \overline{R}(M(1/0)) \cup \overline{R}\left(M\left(\dfrac{1-stpq}{-stq^2}\right)\right)$.

Obviously $\overline{R}(M)$ contains $\overline{R}(M(1/0)) \cup \overline{R}(M(\frac{1-stpq}{-stq^2}))$. Let $\bar{\rho} \in \overline{R}(M)$. We need to show that $\bar{\rho}$ is either in $\overline{R}(M(1/0))$ or in $\overline{R}(M(\frac{1-stpq}{-stq^2}))$. Let $\bar{\rho}_0$ be the restriction of $\bar{\rho}$ on $\pi_1(M_0)$ and $\bar{\rho}_1$ the restriction of $\bar{\rho}$ on $\pi_1(C_{p,q})$ (note that both $\pi_1(M_0)$) and $\pi_1(C_{p,q})$ can be considered as subgroups of $\pi_1(M)$). By Claim 21, $\bar{\rho}_0$ is either in $\overline{R}(M_0(st))$ or in $\overline{R}(M_0(0))$.

If $\bar{\rho}_0 \in \overline{R}(M_0(st))$, then $\bar{\rho}(\mu_0^{st}\lambda_0) = \bar{\rho}_0(\mu_0^{st}\lambda_0) = 1$ and so $\bar{\rho}(\mu_1) = \bar{\rho}_1(\mu_1) = 1$. Hence $\bar{\rho}_1 \in \overline{R}(C_{p,q}(T_1, 1/0)))$. But $C_{p,q}(T_1, 1/0)$ is a solid torus whose meridian slope is the slope $1/0$ on T_2. Therefore $\bar{\rho}_1(\mu_2) = \bar{\rho}(\mu_2) = 1$ and thus $\bar{\rho} \in \overline{R}(M(1/0))$.

If $\bar{\rho}_0 \in \overline{R}(M_0(0))$, then $\bar{\rho}(\lambda_0) = \bar{\rho}_0(\lambda_0) = 1$ and so $\bar{\rho}(\mu_1^{1-stpq}\lambda_1^{-st}) = \bar{\rho}_1(\mu_1^{1-stpq}\lambda_1^{-st}) = 1$. Hence $\bar{\rho}_1 \in \overline{R}(C_{p,q}(T_1, \frac{1-stpq}{-st}))$. But $C_{p,q}(T_1, \frac{1-stpq}{-st})$ is a solid torus whose meridian slope is the slope $\frac{1-stpq}{-stq^2}$ on T_2. Therefore $\bar{\rho}_1(\mu_2^{1-stpq}\lambda_2^{-stq^2}) = \bar{\rho}(\mu_2^{1-stpq}\lambda_2^{-stq^2}) = 1$, and thus $\bar{\rho} \in \overline{R}(M(\frac{1-stpq}{-stq^2}))$. The claim is proven.

The first Betti number of M is one. In fact $H_1(M;\mathbb{Z}) = \mathbb{Z}$ with $\mu_2^p\lambda_2$ as a generator. The slope $(1 - stpq)/-stq^2$ in $\partial M = T_2$ is null homologous in M (bounds a Seifert surface in M), i.e. $H_1(M(\frac{1-stpq}{-stq^2}), \mathbb{Z}) = \mathbb{Z}$. It follows that $\overline{X}(M)$ consists of several curves, exactly one of the curves comes from $\overline{X}(H_1(M,\mathbb{Z}))$ which consists entirely of reducible characters, and all other curves come from $\overline{X}(M(1/0))$, each of which contains irreducible characters.

Now it is clear that the boundary slope p/q in ∂M can not be strongly detected by an ideal point of the $PSL(2,\mathbb{C})$-character variety of M since at such ideal point, either the trace function $\bar{\tau}_{\mu_2}$ (notation from Sect. 2) has limit value equal to 4 or the trace function $\bar{\tau}_{\mu_2^{1-stpq}\lambda_2^{-stq^2}}$ has limit value equal to 4 (cf. Corollary 3 (2)).

Remark 23 Note that by [BZ1], each curve in $\overline{X}(M(1/0))$ provides a Culler-Shalen semi-norm on $H_1(\partial M;\mathbb{R})$ such that the slope $1/0$ has zero semi-norm and all the integer slopes n have the same minimal nonzero semi-norm value s among all classes in $H_1(\partial M;\mathbb{Z})$. It is proven in [BZ1] that for any knot exterior M with a Culler-Shalen semi-norm, if r is not a boundary slope in ∂M and $M(r)$ is a manifold without non-cyclic $PSL(2,\mathbb{C})$-representation, then r has the zero semi-norm value or the minimal non-zero semi-norm value s. The above examples show that the assumption that r is a non-boundary slope is necessary. This is because $M(p/q)$ is a lens space and p/q has non-zero semi-norm bigger than s. It follows that if $X_0 \subset \overline{X}(M(1/0)) \subset \overline{X}(M)$ is the curve which provided the semi-norm, then the function $\bar{\tau}_{\mu_2^p\lambda_2^q} - 4$ must have zero limit value at some ideal point of X_0.

It is also interesting to point out that for each integer slope n, the manifold $M(n)$ is obtained by gluing the $(p-nq, q)$-torus knot exterior and the (s,t)-torus knot exterior together along their torus boundary such that the fiber slope of one side is identified to the meridian slope of the other side and vice versa, and thus $M(n)$ has no non-cyclic representations into $PSL(2,\mathbb{C})$ [Mi] but has non-cyclic fundamental group. By the way we record that $H_1(M(n);\mathbb{Z}) = \mathbb{Z}_{|1-(p-nq)qst|}$.

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