ELSEVIER

Topology and its Applications 99 (1999) 99–110

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Dehn filling with non-degenerate boundary slope rows

Xingru Zhang [1]

Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14214-3093, USA

Received 22 September 1997; received in revised form 6 March 1998

Abstract

We prove that Dehn filling a small link exterior with a non-degenerate boundary slope row produces a 3-manifold which is either Haken and ∂-irreducible or one of very restricted typies of reducible manifolds (Theorem 2), generalizing a result of Culler, Gordon, Luecke and Shalen in the case of a knot exterior (Theorem 1). The result provides some interesting applications on exceptional Dehn fillings (Corollaries 3 and 4) and on telling if a link is small (Corollaries 5 and 6). © 1999 Published by Elsevier Science B.V. All rights reserved.

Keywords: 3-manifolds; Dehn filling; Essential surfaces; Non-degenerate boundary slope rows

AMS classification: 57M25; 57M99

Throughout this paper when a manifold is mentioned, it is assumed to be orientable, compact and smooth, and when a 3-manifold is mentioned, it is also assumed to be connected. By a surface we mean a 2-manifold. For a surface in a 3-manifold, it is assumed to be properly embedded unless otherwise specified. A surface in a 3-manifold is said to be essential if each component of it is incompressible and non-boundary parallel. Recall that a slope in a 2-torus T is the isotopy class of an unoriented simple closed essential curve in T. We use $\Delta(r,s)$ to denote the minimal geometric intersection number between two slopes r and s in T. Now consider an irreducible 3-manifold M whose boundary is a torus. For each slope r in ∂M, one can construct a closed 3-manifold $M(r)$ by the so called Dehn filling operation, i.e., one attaches a solid torus V to M by a gluing homeomorphism of their boundary tori so that a curve of slope r in ∂M bounds a meridian disk of V. A slope r in the torus ∂M is called a *boundary slope* if there exists an essential surface F in M such that ∂F is a non-empty set of parallel essential simple closed curves in ∂M of the slope r. Concerning Dehn filling M with a boundary slope, the following important result was obtained in [2, Theorem 2.0.3].

[1] E-mail: xinzhang@math.buffalo.edu.

PII: S0166-8641(98)00081-9

Theorem 1. *Let M be an irreducible* 3*-manifold M whose boundary is a torus and whose first Betti number is one. If r is a boundary slope of M, then*

(i) *$M(r)$ is a Haken manifold*; *or*

(ii) *$M(r)$ is a connected sum of two nontrivial lens spaces*; *or*

(iii) *M contains an essential closed surface which remains essential in $M(s)$ whenever $\Delta(r,s) > 1$*; *or*

(iv) *M fibers over S^1 with fiber an essential planar surface having boundary slope r. In particular, $M(r) = S^2 \times S^1$.*

In item (i), by a *Haken* 3-manifold, we mean an irreducible 3-manifold which contains an essential surface. In item (ii), by a *nontrivial* lens space we mean a lens space which is neither S^3 nor $S^2 \times S^1$. In item (iv), by a *planar* surface, we mean a 2-sphere with punctures. We remark that one of useful consequences of Theorem 1 is that under the assumptions of Theorem 1, if $M(r)$ is an irreducible 3-manifold which is not Haken, then M contains an essential closed surface. For instance, if M is the exterior of a knot in S^3 whose meridian slope is a boundary slope, then M contains an essential closed surface.

The purpose of this paper is to generalize the above theorem to the situation where M may have more than one torus boundary components. Let M be an irreducible 3-manifold such that $\partial M = T_1 \cup \cdots \cup T_n$ is a set of $n \geqslant 1$ tori. Let $(T_{i_1}, \ldots, T_{i_k})$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, be a sub-collection of $(T_1, \ldots, T_n)$. Let r_{i_j} be a slope on T_{i_j}. A slope row $(r_{i_1}, \ldots, r_{i_k})$ is called a *boundary slope row* of M if there exists an essential surface $(F, \partial F) \subset (M, T_{i_1} \cup \cdots \cup T_{i_k})$ such that $\partial F \cap T_{i_j}$ is a non-empty set of parallel curves in T_{i_j} of the given slope r_{i_j} for each of $j = 1, \ldots, k$. Here we call the number k the *length* of the slope row. A boundary slope row $(r_{i_1}, \ldots, r_{i_k})$ of M is said to be *non-degenerate* if any proper sub-row of $(r_{i_1}, \ldots, r_{i_k})$ is not a boundary slope row of M. Note that by definition, a boundary slope row of length one (i.e., $k = 1$) is automatically non-degenerate.

For a slope row $(r_{i_1}, \ldots, r_{i_k})$, we shall use $M(r_{i_1}, \ldots, r_{i_k})$ to denote the manifold obtained by Dehn filling M along T_{i_j} with slope r_{i_j}, $j = 1, \ldots, k$ and leave the rest of torus components of ∂M untouched. So $M(r_{i_1}, \ldots, r_{i_k})$ is a 3-manifold with $n - k$ torus boundary components $\{T_i;\ i = 1, \ldots, n\} - \{T_{i_j};\ j = 1, \ldots, k\}$. The main result of this paper is the following theorem.

Theorem 2. *Let M be an irreducible* 3*-manifold with $\partial M = T_1 \cup \cdots \cup T_n$ a set of n tori, $n \geqslant 1$. Suppose that M does not contain any closed essential surface. Let $(r_{i_1}, \ldots, r_{i_k})$ $(1 \leqslant i_1 < \cdots < i_k \leqslant n)$ be a non-degenerate boundary slope row of M. Then $M(r_{i_1}, \ldots, r_{i_k})$ is either*

(1) *a Haken* 3*-manifold and ∂-irreducible, or*

(2) *a connected sum of two nontrivial lens spaces* (*thus $k = n$*)*, or*

(3) *a connected sum of a nontrivial lens space and a solid torus* (*thus $k = n - 1$*)*, or*

(4) *a connected sum of two solid tori* (*thus $k = n - 2$*)*, or*

(5) $S^2 \times S^1$ (*thus $k = n$*).

The following two corollaries of Theorem 2 are somewhat surprising.

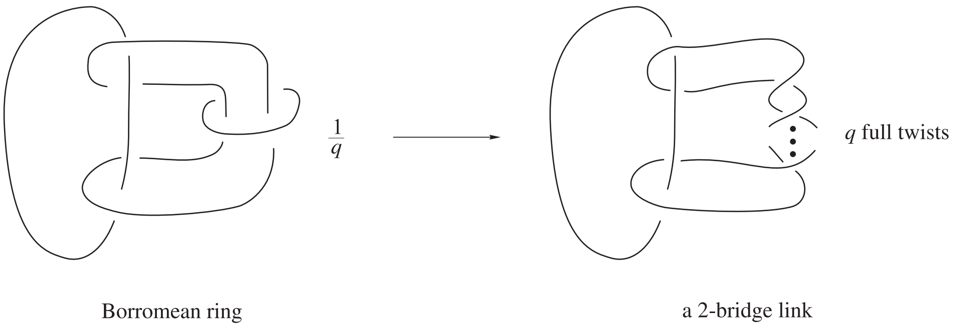

Fig. 1.

Corollary 3. *Let M be an irreducible 3-manifold with ∂M consisting of $n \geqslant 4$ tori. Suppose that M does not contain any closed essential surface. Then Dehn filling M along any one of the boundary components of M with any slope produces an irreducible and ∂-irreducible 3-manifold.*

Proof. Let T_i be any boundary component of M and r_i a slope in T_i. We first show that $M(r_i)$ is irreducible. Suppose otherwise that $M(r_i)$ is reducible. Then (r_i) is a non-degenerate boundary slope row of M (of length one) since there is at least an essential planar surface P in M such that all boundary components of P are contained in T_i. Now applying Theorem 2 (with $k = 1$), we see that we must have $n \leqslant 3$ to obtain a reducible 3-manifold $M(r_i)$. This contradicts to our assumption that $n \geqslant 4$.

Now if $M(r_i)$ is ∂-reducible, then since we have proved that $M(r_i)$ is irreducible, $M(r_i)$ must be a solid torus. Thus $n = 2$. Again we get a contradiction to our assumption that $n \geqslant 4$. □

We remark that Corollary 3 is no longer true if $n \leqslant 3$. For example, let M be the exterior of the Borromean ring in S^3. Then M has no closed essential surfaces but Dehn filling M along any one of the three components of ∂M with the meridian slope produces a reducible manifold (here is an argument that M does not contain any closed essential surfaces). Suppose otherwise that S is a closed essential surface in M. Since the Borromean ring is an alternating link in S^3, there is an embedded annulus in M such that the interior of A is disjoint from S and that one component of ∂A is an essential curve in S and the other component of ∂A is a meridian curve in one of the boundary components, say T_1, of M [10]. Let $M(p/q, \emptyset, \emptyset)$ denotes the manifold obtained by Dehn filling M along T_1 with the slope p/q, where p/q is respect to the standard meridian-longitude coordinates on T_1 such that p is the meridian coordinate and q the longitude coordinate. The symbol $\emptyset$ here denotes the empty set which means leave the corresponding torus component untouched. By [2, Theorem 2.3.4] if S is compressible in $M(p/q, \emptyset, \emptyset)$ then $q = 1$ or -1. But on the other hand we have, by Rolfsen's surgery formula [12, p. 267], that for any integer $q \neq 0$, $M(1/q, \emptyset, \emptyset)$ is the exterior of a 2-bridge link in S^3 (see Fig. 1). Therefore

$M(1/q, \emptyset, \emptyset)$ does not contain any closed essential surfaces [4, Corollary 1.2] and thus S must be compressible in $M(1/q, \emptyset, \emptyset)$. This contradiction completes the argument. For $n = 2$ or 1, one can easily find many examples against Corollary 3.

Corollary 4. *Let M be an irreducible 3-manifold with ∂M consisting of $n \geqslant 6$ tori. Suppose that M does not contain any closed essential surface. Then Dehn filling M along any one of the boundary components of M with any slope produces an irreducible, ∂-irreducible and non-Seifert fibred 3-manifold.*

Since the proof of Corollary 4 needs some notations and machinery used in the proof of Theorem 2, it will be postponed until Theorem 2 is proved.

We shall call a 3-manifold *big* if it is irreducible and contains a closed essential surface. In general, it is a hard problem to determine whether a torally bounded 3-manifold is big. Theorem 2 and Corollary 3 have some interesting applications on this problem. For instance, we have

Corollary 5. *Let L be a non-split link in S^3 of more than three components with the property that there is a component K in L such that $L - K$ is a split link. Then the exterior M of L in S^3 is a big 3-manifold.*

Proof. Since L is non-split, M is irreducible. Let T_i be the torus boundary component of M corresponding to K. Then the Dehn filling M along T with the meridian slope of T gives the link exterior of $L - K$ in S^3 and thus is a reducible 3-manifold by our assumption. Hence by Corollary 3, M must contain a closed essential surface. □

In particular, any Brunnian link in S^3 of more than three components has a big exterior. As Corollary 3, Corollary 5 does not hold if $n \leqslant 3$ (for example, the Borromean ring).

Corollary 6. *Let $L = K_1 \cup K_2 \cup K_3$ be a non-split link in S^3 of three components such that $K_2 \cup K_3$ is a split link and K_2 is a nontrivial knot in S^3. Then the link exterior M of L in S^3 is big.*

Proof. Certainly M is irreducible. Let T_1 be the torus boundary of M corresponding to K_1 and μ_1 be the meridian slope on T_1. Then $M(\mu_1)$ is the exterior of $K_2 \cup K_3$ in S^3 and thus is a reducible 3-manifold. Since K_2 is a nontrivial knot, $M(\mu_1)$ is not a connected sum of two solid tori. Note that (μ_1) is a non-degenerate boundary slope row of M. Hence by Theorem 2, we see that M must contain a closed essential surface. □

The proof of Theorem 2 follows essentially the approach used in [2] for proving Theorem 1, making use of compression bodies and the Handle Addition Lemma (Lemma 9 below). Here is a rough outline of the proof. We start with an essential separating surface F in M realizing the non-degenerate boundary slope row $(r_{i_1}, \ldots, r_{i_k})$ and having the minimal number of boundary components. By studying the handle decompositions of certain associated submanifolds of M and of $M(r_{i_1}, \ldots, r_{i_k})$, we prove that

(a) if F is a non-planar surface then Theorem 2(1) holds;
(b) if F is a connected planar surface, then one of (2)–(4) of Theorem 2 holds;
(c) if F is a disconnected planar surface, then Theorem 2(5) holds.

Note that it follows from a result of Hatcher [5] that a torally bounded 3-manifold M with $n \geqslant 1$ boundary tori can have only finite many length one boundary slope rows. But in general, M may have infinitely many non-degenerate boundary slope rows of length $k,\ 1 < k \leqslant n$.

Example 7. Let M be the exterior of the Whitehead link in S^3. Then M has infinitely many non-degenerated boundary slope rows of length 2.

Proof. Let T_1 and T_2 be the two torus boundary components of M with the standard meridian-longitude coordinates. Note that $M(1/n, \emptyset)$ is the exterior of the twisted knot in S^3 shown by Fig. 2. Note that by Rolfsen's link surgery formula, the standard meridian-longitude coordinates for $M(1/n, \emptyset)$ as a knot exterior in S^3 is the same meridian-longitude coordinates on T_2 of M as a link exterior in S^3. Now according to [6], the slope $4n$ when $n \leqslant 1$ or the slope $-4n-2$ when $n \geqslant 1$, is a boundary slope of $M(1/n, \emptyset)$.

By [5], there are infinite many choices for n such that none of $(1/n, \emptyset)$, $(\emptyset, 4n)$ and $(\emptyset, -4n-2)$ are boundary slope rows of M (of length one). For such n, any incompressible surface S_n in $M(1/n, \emptyset)$ with the slope $4n$ (when $n \leqslant -1$) or with the slope $-4n-2$ (when $n \geqslant 1$) must intersect T_1 as well. Now arrange S_n by isotopy so that the number of the components of $S_n \cap T_1$ is minimal. Then the surface $F_n = M \cap S_n$ is essential surface in M with boundary slope row $(1/n, 4n)$ (when $n \leqslant -1$) or $(1/n, -4n-2)$ (when $n \geqslant 1$), by standard cut-paste argument. Thus all but finitely many of the slope rows of the form $(1/n, 4n)$ $(n \leqslant -1)$ and $(1/n, -4n-2) (n \geqslant 1)$ are non-degenerate boundary slope rows of M of length two. $\square$

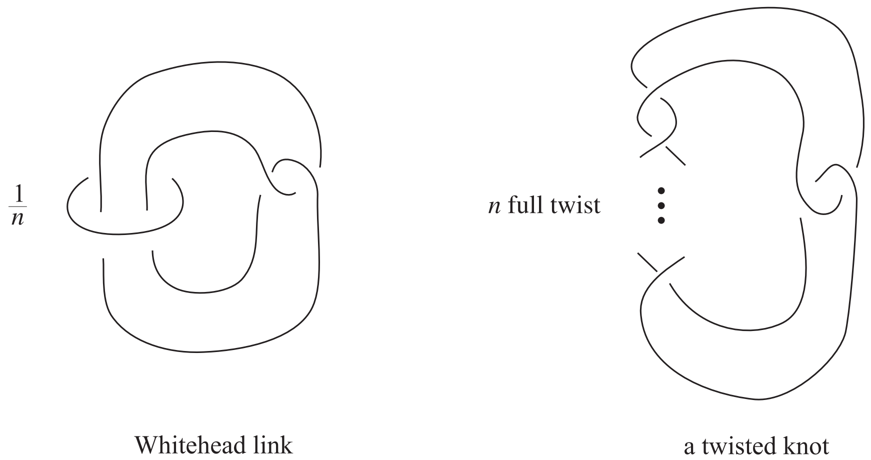

Fig. 2.

Obviously each of $(0, \emptyset)$ and $(\emptyset, 0)$ is a non-degenerate boundary slope (row) of M of length one.

Question 8. Let M be an irreducible 3-manifold whose boundary is a set of $n > 1$ tori and whose interior admits a complete hyperbolic structure of finite volume. Is it true that for each k, $1 < k \leqslant n$, M has infinitely many non-degenerate boundary slope rows of length k?

Proof of Theorem 2. If S is a surface and $c_1, \ldots, c_p$ are disjoint simple closed curves in S, then $\sigma(S; \bigcup c_i)$ will denote the surface resulting from surgery along $c_1, \ldots, c_p$. If Y is a 3-manifold and $c_1, \ldots, c_p$ are disjoint simple closed curves in ∂Y, then $\tau(Y; \bigcup c_i)$ will denote the 3-manifold obtained by attaching 2-handles to Y along disjoint regular neighborhood of $c_1, \ldots, c_p$ in ∂Y. Note that if $c_1, \ldots, c_p \subset S \subset \partial Y$, then $\sigma(S; \bigcup c_i) \subset \partial\tau(Y; \bigcup c_i)$.

Lemma 9 (Handle Addition Lemma). *Let Y be an irreducible* 3*-manifold, S a surface* (*may not be connected*) *in ∂Y which is compressible in Y, and c a simple closed curve in S such that $S - c$ is incompressible in Y. Suppose that $\sigma(S; c)$ has no* 2*-sphere components. Then $\tau(Y; c)$ is irreducible and $\sigma(S; c)$ is incompressible in $\tau(Y; c)$.*

There are several versions of Handle Addition Lemma. The first one was due to Przytycki [11] and subsequently generalized and used in various forms in [1,3,7–9,13,14]. The version stated above is from [1].

A *compression body* is a cobordism W (rel ∂) between surfaces $\partial_+ W$ and $\partial_- W$ such that $W \cong \partial_+ W \times I \cup$ 2-handles $\cup$ 3-handles and $\partial_- W$ has no 2-sphere components. It follows that W is irreducible and $\partial_- W$ is incompressible in W. If Y is an irreducible 3-manifold and $S \subset \partial Y$ is a surface, then there exists a maximal compression body $W \subset Y$ with $\partial_+ W = S$, which is unique up to isotopy. The *inner boundary* of W is $S^- = \partial_- W \cup \partial S \times I$. Thus $\partial S^- = \partial S$ and S^- is incompressible in Y (since W is maximal). Note that the 2-handles may be assumed to be disjoint.

Since M does not contain, in particular, non-separating closed surfaces, we have that $H_1(M; \mathbb{Q}) = \mathbb{Q}^n$ and $H_2(M; \mathbb{Q}) = 0$. It follows that if $(S, \partial S) \subset (M, \partial M)$ is a non-separating surface, then $[\partial S] \neq 0$ in $H_1(\partial M)$, where each component of ∂S is given the induced orientation from some orientation of S. Hence, by successively tubing adjacent oppositely oriented components of $\partial S \cap T_i$ in each of T_i and then compressing, we obtain an essential non-separating surface $(S', \partial S') \subset (M, \partial M)$ such that $\partial S'$ is not empty and its components in each torus T_i are homologous when given the orientation induced by some orientation of S'.

For the given non-degenerate boundary slope row $(r_{i_1}, \ldots, r_{i_k})$, we may assume, without loss of generality, that $i_j = j$, $j = 1, \ldots, k$ (after possibly re-ordering the components of ∂M). Among all essential separating, not necessarily connected, surfaces in $(M, T_1 \cup \cdots \cup T_k)$ with slope row $(r_1, \ldots, r_k)$, let F be one such that $|\partial F|$ (the number of components of ∂F) is minimal. Since the boundary slope row is non-degenerate of length k, each component of F has non-empty intersection with each of T_i, $i = 1, \ldots, k$.

By the minimality of $|\partial F|$, F is either connected or has exactly two components each of which is non-separating. In the latter case, we may assume that F consists of two parallel copies of a non-separating surface G which has all its boundary components oriented coherently on each component of T_i, $i = 1, \ldots, k$. When F is disconnected, we denote its two parallel components by G and G'. Since F is separating, it may be oriented so that $[(F, \partial F)] = 0$ in $H_2(M, T_1 \cup \cdots \cup T_k)$ and $|\partial F \cap T_i|$ is a positive even integer which we denote by $2m_i$ for $i = 1, \ldots, k$. Let $m = m_1 + \cdots + m_k$.

Let X and X' be the two components into which F separates M. So $M = X \cup_F X'$. In the case that F is disconnected, we may assume that X' is the component $G \times I$. Let $S = \partial X - (T_{k+1} \cup \cdots \cup T_n)$ and $S' = \partial X' - (T_{k+1} \cup \cdots \cup T_n)$. The boundary of F cuts each torus T_i, $i = 1, \ldots, k$, into $2m_i$ annuli, m_i each contained in S and S'. We order all these annuli as A_j, A'_j, $j = 1, 2, \ldots, m$, so that $A_j \subset S$, $A'_j \subset S'$, $T_1 = A_1 \cup A'_1 \cup \cdots \cup A_{m_1} \cup A'_{m_1}$, $T_2 = A_{m_1+1} \cup A'_{m_1+1} \cup \cdots \cup A_{m_1+m_2} \cup A'_{m_1+m_2}, \ldots, T_k = A_{m_1+\cdots+m_{k-1}+1} \cup A'_{m_1+\cdots+m_{k-1}+1} \cup \cdots \cup A_{m_1+\cdots+m_k} \cup A'_{m_1+\cdots+m_k}$. So we have

$$S = F \cup \left(\bigcup_{j=1}^{m} A_j \right) \quad \text{and} \quad S' = F \cup \left(\bigcup_{j=1}^{m} A'_j \right).$$

When F is connected, each of S and S' has genus

$$f + \sum_{i=1}^{k} m_i = f + m,$$

where f is the genus of F. When F is disconnected, each of S and S' has genus

$$2g + \left(\sum_{i=1}^{k} m_i \right) - 1 = 2g + m - 1,$$

where g is the genus of G. Note that in the latter case, each of A_j (and each of A'_j) connects G and G'.

Let J_i be the solid torus attached to T_i with slope r_i, $i = 1, \ldots, k$, in forming $M(r_1, \ldots, r_k)$. Then J_i can be considered as the union of $2m_i$ 2-handles with attaching regions

$$A_{m_1+\cdots+m_{i-1}+1}, A'_{m_1+\cdots+m_{i-1}+1}, \ldots, A_{m_1+\cdots+m_i}, A'_{m_1+\cdots+m_i}.$$

Let c_j be the center curve of A_j and c'_j the center curve of A'_j, $j = 1, \ldots, m$. Then

$$M(r_1, \ldots, r_k) = \tau\left(X; \bigcup_{j=1}^{m} c_j \right) \cup_{\widehat{F}} \tau\left(X'; \bigcup_{j=1}^{m} c'_j \right) = \widehat{X} \cup_{\widehat{F}} \widehat{X}',$$

where $\widehat{F}$ is a closed surface obtained by capping off the boundary components of F in T_i with meridian disks of J_i, $i = 1, \ldots, k$. Note that

$$\widehat{F} = \partial \widehat{X} - \left(\bigcup_{i=k+1}^{n} T_i \right) = \partial \widehat{X}' - \left(\bigcup_{i=k+1}^{n} T_i \right).$$

We first need to consider two special cases, that is, when $F = G \cup G'$ is disconnected and G is either a disk or an annulus. If G is a disk, then M is boundary reducible. Since M is irreducible, M must be a solid torus. Thus $n = k = 1$ and $M(r_1) = S^2 \times S^1$, i.e., Theorem 2(5) holds. If G is an annulus, then $k = 1$ or 2. In case $k = 1$, we have $m = m_1 = 2$ and $F \cup A_1$ is a separating annulus in M and thus must be boundary parallel by the minimality assumption on $|\partial F|$. One can now easily see that M is a twisted interval bundle over the Klein bottle and $M(r_1) = S^2 \times S^1$. So Theorem 2(5) holds in this case. Now assume $k = 2$. Then $m_1 = m_2 = 1$ and $F \cup A_1$ is a separating annulus in M and thus must be boundary parallel since the boundary slope row (r_1, r_2) is non-degenerate of length 2. One can now easily see that M is a trivial interval bundle over a torus and $M(r_1, r_2) = S^2 \times S^1$. So again Theorem 2(5) holds.

So we may assume now that G is not a disk or an annulus. Let $F_j = F \cup A_j \subset \partial X$. Note that F_j is connected and $\partial F_j = \bigcup_{p \neq j} \partial A_p$. Also note that F_j is neither a 2-sphere nor an annulus. Since $|\partial F_j| = |\partial F| - 2$, F_j is compressible in M (since M is assumed to contain no closed essential surfaces and since F_j is not an annulus) and thus is compressible in X (since $F_j \subset X$ and F is incompressible in $M = X \cup_F X'$). Let V_j be a maximal compression body for F_j in X. The inner boundary of V_j, F_j^-, is incompressible in X and thus is also incompressible in M. Since M does not contain closed essential surfaces, if a component of F_j^- is a closed surface, it must be a torus parallel to one of $T_{k+1}, \ldots, T_n$.

We now show that F_j^- does not contain non-separating components. Note that $[F_j^-, \partial F_j^-] = [F_j, \partial F_j] = [F, \partial F] = 0$ in $H_2(M, T_1 \cup \cdots \cup T_n)$. It follows that the number of non-separating components of F_j^- is even. Also each of such components must have boundary. Therefore if F_j^- has non-separating components, it must have one, denoted G_1, with at most $m - 1$ boundary components. But then the surface which consists of two parallel copies of G_1 is an essential surface in $(M, T_1 \cup \cdots \cup T_k)$ with at most $2m - 2$ boundary components, contradicting to the minimality assumption on $|\partial F| = 2m$.

So F_j^- has no non-separating components. Therefore other than some (possibly empty) set of boundary parallel tori, F_j^- contains exactly $m - 1$ annuli, each being parallel into ∂M. Let B be one of these annuli. Since B separates M and $\mathrm{int}(B) \cap F_j = \emptyset$, $\partial B = \partial A_p$ for some $p \neq j$. Thus we may number the annuli as B_j^p so that $\partial B_j^p = \partial A_p$ for all $p \neq j$. Let U_p be the solid torus realizing the parallelism from B_j^p to A_p. Then since F cannot be contained in U_p, $U_p \subset X$, $\partial U_p = B_j^p \cup A_p$ and all the U_p's are mutually disjoint.

Lemma 10. *For each fixed j $(1 \leqslant j \leqslant m)$, there exist mutually disjoint* (*properly embedded*) *disks E_j^p in X, $p \neq j$, such that E_j^p meets c_p transversely in a single point and is disjoint from c_q if $q \neq j$ or p.*

Proof. Let D_p be a meridian disk of U_p which is a boundary compressing disk for B_j^p. So $\partial D_p = \alpha_p \cup \beta_p$ with α_p, β_p being spanning arcs of the annuli A_p, B_j^p, respectively. Now $V_j \cong F_j \times I \cup$ 2-handles $\cup$ 3-handles; dually, $V_j \cong \partial_- V_j \times I \cup$ 0-handles $\cup$ 1-handles. An isotopy of β_p (rel ∂) in B_j^p will move it off the disks that constitute the attaching regions of the 1-handles, and then a further isotopy (rel ∂) in V_j (using the product structure of

$\partial_- V_j \times I$) will take it to an arc β'_p in $F_j \subset S \subset \partial X$. A corresponding isotopy and extension of D_p gives a disk E_j^p in X with $\partial E_j^p = \alpha_p \cup \beta'_p$. These disks E_j^p, $p \neq j$, satisfy the conditions stated. $\square$

Let W_j be the (possibly punctured) compression body in X with $\partial_+ W_j = S$ defined by

$$W_j = S \times I \cup \left(\bigcup_{p, p \neq j} E_j^p \times I \right).$$

Then $\partial_- W_j$ is a closed connected surface of genus $f+1$ in the case that F is connected, and of genus $2g$ in the case that F is disconnected.

Lemma 11. *For $1 \leqslant j \leqslant m$, $\tau(W_j, \bigcup_{p,p\neq j} c_p) \cong \partial_- W_j \times I$.*

Proof. This follows by canceling the 2-handle corresponding to c_p with E_j^p, $p \neq j$. $\square$

Let $X_0 = X$, and let $X_q = \tau(X; \bigcup_{j=1}^{q} c_j)$ for $1 \leqslant q \leqslant m$. Thus $X_m = \widehat{X}$.

Lemma 12. *If F is either non-planar or connected, then, for $0 \leqslant q \leqslant m-1$, X_q is irreducible and each component of $\partial X_q - \bigcup_{j=q+1}^{m} c_j$ is incompressible in X_q.*

Proof. We prove this by induction on q, using the handle addition lemma. The assertion holds obviously for $q = 0$. So suppose that $1 \leqslant q \leqslant m-1$, and that the assertion holds for $q-1$. Thus X_{q-1} is irreducible and each component of

$$\partial X_{q-1} - \bigcup_{j=q}^{m} c_j = \left(\partial X_{q-1} - \bigcup_{j=q+1}^{m} c_j \right) - c_q$$

is incompressible in X_{q-1}. Let

$$S_q = \partial X_{q-1} - \bigcup_{j=q+1}^{m} c_j - \bigcup_{i=k+1}^{n} T_i.$$

Then S_q is connected and $S_q - c_q$ is incompressible in X_{q-1}. Certainly each of $T_{k+1}, \ldots, T_n$ is incompressible in X_{q-1}. So the lemma will follow from the handle addition lemma if we can show that S_q is compressible in X_{q-1}. To do this, let $(D_q^*, \partial D_q^*) \subset (X, F_q)$ be a disjoint union of disks such that the maximal compression body V_q for F_q in X can be expressed as $F_q \times I \cup D_q^* \times I \cup$ 3-handles. Since $\partial D_q^* \cap c_j = \emptyset$ for $j \neq q$, we have $(D_q^*, \partial D_q^*) \subset (X_{q-1}, S_q)$. We claim that some component of ∂D_q^* is essential in $\partial X_{q-1} - (\bigcup_{i=k+1}^{n} T_i)$ and thus in S_q. For if not, then $\sigma(\partial X_{q-1} - (\bigcup_{i=k+1}^{n} T_i); \partial D_q^*)$ would be homeomorphic to the disjoint union of $\partial X_{q-1} - (\bigcup_{i=k+1}^{n} T_i)$ with some 2-spheres. However $\partial X_{q-1} - \bigcup_{i=k+1}^{n} T_i$ is a connected surface which has, in the case that F is connected, genus $f + m - (q-1) \geqslant f + 2 \geqslant 2$, and, in the case that F is disconnected (and non-planar), genus $2g + m - 1 - (q-1) \geqslant 2g + 1 \geqslant 3$. On the other hand,

$$\sigma\left(\partial X - \left(\bigcup_{i=k+1}^{n} T_i \right); \partial D_q^* \right) = \sigma(S; \partial D_q^*)$$

contains $m-1$ tori, corresponding to the tori $A_p \cup B_q^p$, $p \neq q$, and some other (possibly empty) components. Therefore

$$\sigma\left(\partial X_{q-1} - \left(\bigcup_{i=k+1}^{n} T_i\right); \partial D_q^*\right) = \sigma\left(\sigma\left(\partial X - \left(\bigcup_{i=k+1}^{n} T_i\right); \partial D_q^*\right); \bigcup_{j=1}^{q-1} c_j\right)$$

contains $m-q \geqslant 1$ tori. This contradiction completes the proof of the lemma. □

Lemma 13. *Suppose that F is non-planar. Then $M(r_1,\dots,r_k)$ is irreducible and ∂-irreducible, and $\widehat{F}$ is an incompressible surface in $M(r_1,\dots,r_k)$.*

Proof. Note that $\partial_- W_m$ is connected, closed surface of genus $\geqslant 2$. Hence by our assumption, $\partial_- W_m$ must be compressible in M and thus in X. But by Lemma 11, $\tau(W_m; \bigcup_{j=1}^{m-1} c_j) \cong \partial_- W_m \times I$. So $\partial X_{m-1} - (\bigcup_{i=k+1}^{n} T_i)$ is compressible in X_{m-1}. By the previous lemma, each component of $\partial X_{m-1} - c_m$ is incompressible in X_{m-1}. Hence by the handle addition lemma, $\widehat{X} = \tau(X_{m-1}; c_m)$ is irreducible and each component of $\partial\widehat{X}$ is incompressible in $\widehat{X}$. Similar discussion works for X'. The lemma now follows since $M(r_1,\dots,r_k) = \widehat{X} \cup_{\widehat{F}} \widehat{X}'$. □

We remark that if the genus of F is larger than one when it is connected or if the genus of G is larger than zero when F is disconnected, then $\widehat{F}$ is also essential in $M(r_1,\dots,r_k)$.

Lemma 14. *Suppose that F is planar.*

(a) *In the case that F is connected, we have $k \geqslant n-2$. Further when $k = n-2$, $M(r_1,\dots,r_{n-2})$ is a connected sum of two solid tori*; *when $k = n-1$, $M(r_1,\dots,r_{n-1})$ is a connected sum of a nontrivial lens space and a solid torus*; *when $k = n$, $M(r_1,\dots,r_n)$ is a connected sum of two nontrivial lens spaces.*

(b) *In the case that F is disconnected, we have $k = n$ and $M(r_1,\dots,r_n) \cong S^2 \times S^1$.*

Proof. *Case* (a): F is connected. In this case, the surface $\partial_- W_m$ is a torus. Hence by our assumption, $\partial_- W_m$ is either compressible in X or is parallel to one of the tori $T_{k+1},\dots,T_n$.

Subcase (a1): $\partial_- W_m$ is compressible. Then $\partial_- W_m$ bounds a solid torus in X since X is irreducible. Thus X is a handlebody of genus m. If $m = 1$, then $k = 1$ and F is an essential annulus in M with $\partial F \subset T_1$ and X is a solid torus ($\partial X = F \cup A_1$). Since F is not parallel to A_1, the minimal geometric intersection number in ∂X between a component of ∂F and the boundary of a meridian disk of X is at least two. Therefore $\widehat{X}$ is a punctured nontrivial lens space. So suppose that $m > 1$. By Lemma 10, X_{m-2} is a handlebody of genus two (canceling the 2-handle correspond to c_p with the disk E_m^p, $p = 1,\dots,m-2$). Consider the disjoint simple closed curves $c_{m-1}, c_m \subset \partial X_{m-2}$. Since the boundaries of E_m^{m-1} and E_{m-1}^m are disjoint from $c_1,\dots,c_{m-2}$, we have E_m^{m-1} and E_{m-1}^m properly embedded in X_{m-2}. Further ∂E_m^{m-1} (respectively ∂E_{m-1}^m) intersects c_{m-1} (respectively c_m) transversely in a single point. Also $\partial X_{m-2} - c_{m-1} \cup c_m$ is incompressible in X_{m-2}, by Lemma 12. We can now apply [2, Lemma 2.3.2] to see that $\widehat{X} = \tau(X_{m-2}; c_{m-1} \cup c_m)$ is a punctured nontrivial lens space.

Subcase (a2): $\partial_- W_m$ is parallel to one of $T_{k+1}, \ldots, T_n$. Then $X = W_m$ and by Lemma 11,

$$X_{m-1} = \tau\left(W_m; \bigcup_{p,\, p \neq m} c_p\right) \cong \partial_- W_m \times I.$$

Therefore $\widehat{X} = X_m = \sigma(X_{m-1}; c_m)$ is a punctured solid torus.

A similar discussion works for X'. Therefore $k \geqslant n-2$, and $M(r_1, \ldots, r_k)$ is either a connected sum of two nontrivial lens spaces (when $k = n$) or a connected sum of a nontrivial lens space and a solid torus (when $k = n-1$) or a connected sum of two solid tori (when $k = n-2$). Hence the lemma holds in case (a).

Case (b): F is disconnected. We already knew that Lemma 14 holds when $m \leqslant 2$. So suppose that $m > 2$. Recall that $\partial_- W_m$ is a 2-sphere in this case and thus X is a handlebody of genus $m-1$ since X is irreducible. Also by Lemma 11,

$$\tau\left(W_m; \bigcup_{p,\, p \neq m} c_p\right) \cong \partial_- W_m \times I.$$

Therefore X_{m-1} is a 3-ball and $\widehat{X}$ is thus a punctured 3-ball. On the other hand $\widehat{X}'$ is obviously a copy of $S^2 \times I$ since $X' = G \times I$. Therefore $k = n$ and $M(r_1, \ldots, r_n) = S^2 \times S^1$. □

Theorem 2 is a combination of the last two lemmas. □

Proof of Corollary 4. By Corollary 3, we only need to show that for any torus component T_i of ∂M and for any slope r_i on T_i, $M(r_i)$ is not a Seifert fibred space. Suppose otherwise that $M(r_i)$ is a Seifert fibred space. Then since $M(r_i)$ has at least five torus boundary components, it contains an essential torus. Since M does not contain closed essential surface, any essential torus in $M(r_i)$ must intersect and only intersect T_i. It follows that (r_i) is a non-degenerate boundary slope row of M (of length one). Now we adopt the notations used in the proof of Theorem 2. Among all essential separating surfaces in M realizing the boundary slope row (r_i), let $(F, \partial F) \subset (M, T_i)$ be one which has the minimal number of boundary components. By Lemma 14, we see that F is not a planar surface, and by Lemma 13, $\widehat{F}$ is incompressible in $M(r_i)$. Since $M(r_i)$ is assumed to be Seifert fibred, $\widehat{F}$ is isotopic to either a horizontal surface (transverse to all the fibres) or to a vertical surface (consisting of fibres). But as $M(r_i)$ has non-empty boundary, the closed surface $\widehat{F}$ cannot be isotopic to a horizontal surface. So $\widehat{F}$ is vertical and thus is an incompressible torus. Hence F has genus one. Now as in the proof of Lemma 13, $\partial_- W_m$ is a connected closed surface in X which has genus 2 in our current case and which must be compressible in X since M does not contain any closed essential surfaces. Compressing $\partial_- W_m$ in X, we get either one or two tori in X. Each such torus is either compressible and thus bounds a solid torus in X or is incompressible and thus parallel to one of boundary tori of ∂M. Therefore X contains at most two boundary components of M. A similar discussion works for X'. Therefore M has at most five boundary components. But this contradicts to our assumption that $n \geqslant 6$, thereby completing the proof of Corollary 4. □

References

[1] A. Casson and C. Gordon, Reducing Heegaard splittings, Topology Appl. 27 (1987) 275–283.
[2] M. Culler, C. Gordon, J. Luecke and P. Shalen, Dehn surgery on knots, Ann. Math. 125 (1987) 237–300.
[3] M. Domergue and H. Short, Surfaces incompressible dans les varieties obtenues par chirurgie a partir dun noeud de S^3, C. R. Acad. Sci. Paris 300 (1985) 669–672.
[4] C. Gordon and R. Litherland, Incompressible surfaces in branched coverings, in: J. Morgan and H. Bass, eds., The Smith Conjecture (1984) 139–151.
[5] A. Hatcher, On the boundary curves of incompressible surfaces, Pacific J. Math. 99 (1982) 373–377.
[6] A. Hatcher and W. Thurston, Incompressible surfaces in 2-bridge knot complements, Invent. Math. 79 (1985) 225–246.
[7] W. Jaco, Adding a 2-handle to a 3-manifold: an application to property R, Proc. Amer. Math. Soc. 92 (1984) 288–292.
[8] K. Johannson, On surfaces in one-relator 3-manifolds, in: Low-Dimensional Topology and Kleinian Groups, London Math. Soc. Lecture Ser. 112 (1986).
[9] Lei Fengchun, A general handle addition lemma, Math. Z. 221 (1996) 221–216.
[10] W. Menasco, Closed incompressible surfaces in alternating knot and link complements, Topology 23 (1984) 37–44.
[11] J. Przytycki, Incompressibility of surfaces after Dehn surgery, Michigan Math. J. 30 (1983) 289–308.
[12] D. Rolfsen, Knots and Links (Publish or Perish Inc., 1976).
[13] M. Scharlemann, Outermost forks and a theorem of Jaco, in: Proc. Rochester Conf., Amer. Math. Soc. Contemp. Math. Series 44 (1985) 189–193.
[14] Y. Wu, A generalization of the handle addition theorem, Proc. Amer. Math. Soc. 114 (1992) 237–242.