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Characterizing slopes for torus knots

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A slope $\frac{p}{q}$ is called a characterizing slope for a given knot K_0 in S^3 if whenever the $\frac{p}{q}$–surgery on a knot K in S^3 is homeomorphic to the $\frac{p}{q}$–surgery on K_0 via an orientation preserving homeomorphism, then $K = K_0$. In this paper we try to find characterizing slopes for torus knots $T_{r,s}$. We show that any slope $\frac{p}{q}$ which is larger than the number $30(r^2-1)(s^2-1)/67$ is a characterizing slope for $T_{r,s}$. The proof uses Heegaard Floer homology and Agol–Lackenby's 6–theorem. In the case of $T_{5,2}$, we obtain more specific information about its set of characterizing slopes by applying further Heegaard Floer homology techniques.

57M27, 57R58; 57M50

1 Introduction

A long-standing conjecture due to Gordon stated that if for some nontrivial slope $\frac{p}{q} \neq \frac{1}{0}$, the $\frac{p}{q}$–surgery on a knot $K \subset S^3$ is homeomorphic to the $\frac{p}{q}$–surgery on the unknot in S^3 via an orientation preserving homeomorphism, then K must be the unknot. This conjecture was originally proved by Kronheimer, Mrowka, Ozsváth and Szabó using monopole Floer homology [18], and there were also proofs via Heegaard Floer homology; see Ozsváth and Szabó [27; 32]. It is natural to ask whether there are other knots in S^3 which admit a similar Dehn surgery characterization as the unknot. In [25] Ozsváth and Szabó showed that the trefoil knot and the figure 8 knot are two such knots. In the proof of these results, one uses the fact that the unknot is the only genus zero knot, and the trefoil knot and the figure 8 knot are the only genus one fibered knots.

For a given knot $K_0 \subset S^3$, we call a slope $\frac{p}{q}$ *a characterizing slope for* K_0 if whenever the $\frac{p}{q}$–surgery on a knot K in S^3 is homeomorphic to the $\frac{p}{q}$–surgery on K_0 via an orientation preserving homeomorphism, then $K = K_0$. In this terminology, the results cited above say that for each of the unknot, the trefoil knot and the figure 8 knot, every nontrivial slope is a characterizing slope.

On the other hand there are infinitely many knots in S^3 which have nontrivial non-characterizing slopes, including some genus two fibered knots. Osoinach [24] found

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examples of infinitely many knots in S^3 on which the 0–surgery yields the same manifold. In one case, the knots he constructed are all genus 2 fibered knots, one of which is the connected sum of two copies of the figure 8 knot. Teragaito [39] constructed infinitely many knots on which the 4–surgery yields the same Seifert fiber space over the base orbifold $S^2(2,6,7)$. One of these knots is 9_{42}, again a genus 2 fibered knot. The following two examples show that some torus knots also have nontrivial noncharacterizing slopes.

Example 1.1 Consider the 21–surgery on the torus knots $T_{5,4}$ and $T_{11,2}$. By Moser [21], the resulting oriented manifolds are the lens spaces $L(21,16)$ and $L(21,4)$. Here our orientation on the lens space $L(p,q)$ is induced from the orientation on S^3 by $\frac{p}{q}$–surgery on the unknot. Since $16\cdot 4 = 64 \equiv 1 \pmod{21}$, $L(21,16) \cong L(21,4)$, where here (and throughout the paper) $\cong$ stands for an orientation-preserving homeomorphism. Similarly, the $(n^3+6n^2+10n+4)$–surgery on $T_{n^2+3n+1,n+3}$ and $T_{n^2+5n+5,n+1}$ gives rise to homeomorphic oriented lens spaces.

Example 1.2 Let K be the $(59,2)$–cable of $T_{6,5}$, denoted $K = C_{59,2} \circ T_{6,5}$, and consider the 119–surgery on $T_{24,5}$ and K. The resulting manifold of the surgery on K is the same as the $\frac{119}{4}$ surgery on $T_{6,5}$, which is the lens space $L(119,100)$. The 119–surgery on $T_{24,5}$ is the lens space $L(119,25)$. Since $25\cdot 100 \equiv 1 \pmod{119}$, $L(119,100) \cong L(119,25)$. There are infinitely many such pairs of torus and cable knots, of which the first few are listed below:

p	119	697	4059
knots	$T_{24,5}, C_{59,2}\circ T_{6,5}$	$T_{29,24}, C_{349,2}\circ T_{29,6}$	$T_{140,29}, C_{2029,2}\circ T_{35,29}$

In general it is a challenging problem to determine, for a given knot, its set of characterizing slopes. Our aim in this paper is to focus on torus knots and try to find their sets of characterizing slopes. In this regard, Rasmussen [35] proved, using a result of Baker [2], that $(4n+3)$ is a characterizing slope for $T_{2n+1,2}$. It was also known, as a consequence of a result of Greene [16], that rs is a characterizing slope for $T_{r,s}$. Our first result is the following theorem.

Theorem 1.3 *For a torus knot $T_{r,s}$ with $r > s > 1$, a nontrivial slope $\frac{p}{q}$ is a characterizing slope if it is larger than the number $30(r^2-1)(s^2-1)/67$.*

The proof of Theorem 1.3 uses a genus bound on K from Heegaard Floer homology, as well as the 6–theorem of Agol [1] and Lackenby [19]. A more careful study involving Heegaard Floer homology should definitely give a smaller bound, which is expected to be of order rs. For example, in the special case of $T_{5,2}$, we have the following theorem.

Theorem 1.4 *For the torus knot $T_{5,2}$, its set of characterizing slopes includes the set of slopes*

$$\left\{\tfrac{p}{q} > 1, |p| \geq 33\right\} \cup \left\{\tfrac{p}{q} < -6, |p| \geq 33, |q| \geq 2\right\} \cup \left\{\tfrac{p}{q}, |q| \geq 9\right\}$$
$$\cup \left\{\tfrac{p}{q}, |q| \geq 3, 2 \leq |p| \leq 2|q|-3\right\} \cup \left\{9, 10, 11, \tfrac{19}{2}, \tfrac{21}{2}, \tfrac{28}{3}, \tfrac{29}{3}, \tfrac{31}{3}, \tfrac{32}{3}\right\}.$$

Theorem 1.4 is essentially saying that besides some finite set of slopes $\frac{p}{q}$ with $-47 \leq p \leq 32$ and $1 \leq q \leq 8$, only negative integer slopes could possibly be nontrivial noncharacterizing slopes for $T_{5,2}$. We suspect that for $T_{5,2}$ every nontrivial slope is a characterizing slope.

Throughout this paper our notation is consistent. A slope $\frac{p}{q}$ of a knot in S^3 is always parameterized with respect to the standard meridian–longitude coordinates so that a meridian has slope $\frac{1}{0}$ and a longitude 0, and p, q are always assumed to be relatively prime. Our definition for the (m,n)–cabled knot K on a knot K' in S^3 is standard, ie m, n are a pair of relatively prime integers with $|n| > 1$ and K can be embedded in the boundary torus of a regular neighborhood of K' having slope $\frac{m}{n}$ for the knot K'. In particular the (m,n)–cabled knot on the unknot is called a torus knot, which we denote by $T_{m,n}$. Given a knot $K \subset S^3$, $S^3_{p/q}(K)$ will denote the oriented 3–manifold obtained by $\frac{p}{q}$–surgery on S^3 along K, with the orientation induced from that of $S^3 - K$, whose orientation is in turn induced from a fixed orientation of S^3. Similarly if L is a nullhomologous knot in a rational homology sphere Y, $Y_{p/q}(L)$ will denote the oriented 3–manifold obtained by the $\frac{p}{q}$–surgery on Y along L, with $\frac{p}{q}$ parameterized by the standard meridian–longitude coordinates of L. For a knot K in S^3, $g(K)$ will denote the genus of K. For two slopes $\frac{p}{q}$ and $\frac{m}{n}$ of a knot, $\Delta(\frac{p}{q}, \frac{m}{n})$ will denote the "distance" between the two slopes, which is equal to $|pn - qm|$.

We conclude this section by raising the following two questions, whose solutions might not be out of reach.

Question 1.5 Can every nontrivial slope be realized as a noncharacterizing slope for some knot in S^3?

Question 1.6 If K_0 is a hyperbolic knot and $\frac{p}{q}$ is a slope with $|p| + |q|$ sufficiently large, must $\frac{p}{q}$ be a characterizing slope for K_0?

The paper is organized as follows. In Section 2, we prove Theorem 1.3 by using a genus bound from Heegaard Floer homology and Agol–Lackenby's 6–theorem. The rest of the paper is dedicated to the proof of Theorem 1.4. In Section 3, we give some necessary background on Heegaard Floer homology, including Ozsváth–Szabó's rational surgery formula. We prove an explicit formula for $HF_{\mathrm{red}}(S^3_{p/q}(K))$ (Proposition 3.5), which is of independent interest. In Section 4, we deduce information about knot Floer homology under the Dehn surgery condition, $S^3_{p/q}(K)\cong S^3_{p/q}(T_{5,2})$ for $\frac{p}{q}>1$, or $\frac{p}{q}<-6$ and $|q|>2$, and conclude that K is either a genus 2 fibered knot or a genus 1 knot. In Section 5, we finish the proof of Theorem 1.4 by applying the results obtained in the early sections. The proof also applies a result of Lackenby and Meyerhoff [20].

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2 Proof of Theorem 1.3

Throughout this section, $T_{r,s}$ is the torus knot given in Theorem 1.3 with $r>s>1$. In order to prove Theorem 1.3, we need a little bit knowledge on Heegaard Floer homology. Recall that a rational homology sphere Y is an *L–space* if the rank of its Heegaard Floer homology $\widehat{HF}(Y)$ is equal to the order of $H_1(Y;\mathbb{Z})$. For example, lens spaces are L–spaces. If a knot $K\subset S^3$ admits an L–space surgery with positive slope, then $S^3_{p/q}(K)$ is an L–space if and only if $\frac{p}{q}\ge 2g(K)-1$ [32].

Proposition 2.1 *Suppose that for a knot $K\subset S^3$ and a slope $\frac{p}{q}\ge rs-r-s$, we have*

$$S^3_{p/q}(K)\cong S^3_{p/q}(T_{r,s}).$$

Then K is a fibered knot and

$$2g(K)-1\le \frac{(r^2-1)(s^2-1)}{24}.$$

Proof Since $\frac{p}{q}\ge rs-r-s=2g(T_{r,s})-1$, $S^3_{p/q}(T_{r,s})$ is an L–space. As K admits an L–space surgery, according to Ozsváth and Szabó [30] its Alexander polynomial should be

$$\Delta_K(T)=(-1)^k+\sum_{i=1}^{k}(-1)^{k-i}(T^{n_i}+T^{-n_i})$$

for some positive integers $0<n_1<n_2<\cdots<n_k=g(K)$. Moreover, K is fibered; see the first author [22]. We can compute

$$\Delta''_K(1)=2\sum_{i=1}^{k}(-1)^{k-i}n_i^2\ge 2(2n_k-1)=2(2g(K)-1).$$

Let $\lambda(Y)$ be the Casson–Walker invariant (see Walker [40]) for a rational homology sphere Y, normalized so that $\lambda(S^3_{+1}(T_{3,2}))=1$. The surgery formula (see Boyer and Lines [4, Theorem 2.8])

$$\lambda(Y_{p/q}(L))=\lambda(Y)+\lambda(L(p,q))+\tfrac{q}{2p}\Delta_L''(1)$$

is well known, where $L\subset Y$ is a nullhomologous knot. Applying the formula to our case, we conclude that

$$\Delta_K''(1)=\Delta_{T_{r,s}}''(1), \tag{2-1}$$

hence $2(2g(K)-1)\le\Delta_{T_{r,s}}''(1)=\frac{(r^2-1)(s^2-1)}{12}$. We get our conclusion. □

Lemma 2.2 *Suppose that K is a hyperbolic knot in S^3 and $S^3_{p/q}(K)$ is not a hyperbolic manifold, then*

$$|p|\le\tfrac{36}{3.35}(2g(K)-1)<10.75(2g(K)-1).$$

Proof Following [1], let C be the maximal horocusp of S^3-K with embedded interior. Note that ∂C is a Euclidean torus. For any slope α on ∂C, let $l_C(\alpha)$ be the Euclidean length of the geodesic loop in the homology class of α in ∂C. Let $\lambda\subset\partial C$ be the canonical longitude of K, then it follows from [1, Theorem 5.1] that

$$l_C(\lambda)\le 6(2g(K)-1).$$

Since $S^3_{p/q}(K)$ is not hyperbolic, the 6–theorem [19; 1] implies that $l_C(\frac{p}{q})\le 6$ (Note that by the geometrization theorem due to Perelman [33; 34], for closed 3–manifolds, the term hyperbolike, as used in [1], is equivalent to hyperbolic). Let θ be the angle between the two geodesic loops in the homology classes λ, $\frac{p}{q}$. As in the proof of [1, Theorem 8.1]

$$|p|=\Delta\Big(\lambda,\frac{p}{q}\Big)=\frac{l_C(\lambda)l_C(\frac{p}{q})\sin\theta}{\mathrm{Area}(\partial C)}\le\frac{l_C(\lambda)l_C(\frac{p}{q})}{\mathrm{Area}(\partial C)}\le\frac{36(2g(K)-1)}{3.35},$$

where we use Cao and Meyerhoff's result [7] that $\mathrm{Area}(\partial C)\ge 3.35$. □

Corollary 2.3 *If $S^3_{p/q}(K)\cong S^3_{p/q}(T_{r,s})$ holds for a knot K and $|\frac{p}{q}|>\frac{30(r^2-1)(s^2-1)}{67}$, then K is not hyperbolic.*

Proof Since $r>s>1$, we have $\frac{p}{q}>30(r^2-1)(s^2-1)/67>(r-1)(s-1)>rs-r-s$. By Proposition 2.1 we have $2g(K)-1\le(r^2-1)(s^2-1)/24$. If K is hyperbolic, Lemma 2.2 implies that

$$\Big|\frac{p}{q}\Big|\le|p|\le\frac{36}{3.35}\cdot\frac{(r^2-1)(s^2-1)}{24}=\frac{30(r^2-1)(s^2-1)}{67},$$

which is a contradiction. □

Proposition 2.4 *Suppose that for a general torus knot $T_{m,n} \subset S^3$, we have that $S^3_{p/q}(T_{m,n}) \cong S^3_{p/q}(T_{r,s})$ for a slope $\frac{p}{q} \notin \{rs \pm 1, rs \pm \frac{1}{2}\}$. Then $T_{m,n} = T_{r,s}$.*

Proof By (2-1) we have

$$(r^2-1)(s^2-1) = (m^2-1)(n^2-1). \tag{2-2}$$

If the manifold $S^3_{p/q}(T_{m,n}) \cong S^3_{p/q}(T_{r,s})$ is reducible, then by [21] the slope p/q is $rs = mn$. Using (2-2) we easily see that $T_{m,n} = T_{r,s}$. (In [21], the slope $\frac{p}{q}$ is represented by the pair (q, p).)

If the manifold $S^3_{p/q}(T_{m,n}) \cong S^3_{p/q}(T_{r,s})$ is a lens space, it follows from [21] that the slope $\frac{p}{q}$ satisfies

$$\Delta(\tfrac{p}{q}, rs) = |p - rsq| = \Delta(\tfrac{p}{q}, mn) = |p - mnq| = 1. \tag{2-3}$$

From (2-3), we have $rsq = mnq$ or $rsq = mnq \pm 2$. If $rs = mn$, then using (2-2) we get $T_{m,n} = T_{r,s}$. If $rsq = mnq \pm 2$, then $|q| = 1, 2$ and $\frac{p}{q} = rs \pm \frac{1}{q}$.

So by [21], we may now assume that the manifold $S^3_{p/q}(T_{m,n}) \cong S^3_{p/q}(T_{r,s})$ is a Seifert fiber space whose base orbifold is S^2 with three cone points, and the orders

$$\{r, s, |p - rsq|\} = \{|m|, |n|, |p - mnq|\},$$

so $\{r, s\} \cap \{|m|, |n|\} \neq \varnothing$. Without loss of generality, we may assume $r = |m|$, then $s = |n|$ by (2-2). So $|p - rsq| = |p - mnq|$. If $T_{m,n} \neq T_{r,s}$, then $mn = -rs$ and so we must have $p = 0$. But clearly $S^3_0(T_{r,s}) \ncong S^3_0(T_{r,-s})$. Hence $T_{m,n}$ must be $T_{r,s}$. □

Proposition 2.5 *If $S^3_{p/q}(K) \cong S^3_{p/q}(T_{r,s})$ holds for a knot K and*

$$\frac{p}{q} > rs + \frac{3}{7}\max\{r, s\} \quad \textit{with either } |p| > \frac{30(r^2-1)(s^2-1)}{67} \textit{ or } |q| \geq 3,$$

then K is not a satellite knot.

Proof We may assume $p, q > 0$. As $\frac{p}{q} > rs + 1$, $S^3_{p/q}(T_{r,s})$ is a Seifert fiber space whose base orbifold is S^2 with three cone points of orders $r, s, p - qrs$.

If K is a satellite knot, let $R \subset S^3 - K$ be the "innermost" essential torus, then R bounds a solid torus $V \supset K$ in S^3. Let K' be the core of V. The "innermost" condition means that K' is not satellite, so it is either hyperbolic or a torus knot. Note that $S^3_{p/q}(T_{r,s})$ is irreducible and does not contain incompressible torus. So by Gabai [10] we know that $V_{p/q}(K)$ (the p/q–surgery on V along K) is a solid torus, K is a braid in V, and $S^3_{p/q}(K) \cong S^3_{p/(qw^2)}(K')$, where $w > 1$ is the winding number of K in V and $\gcd(p, qw^2) = 1$.

If $K' = T_{m,n}$ is a torus knot, since K' admits a positive L–space surgery, we can choose $m, n > 0$. Then $S^3_{p/(qw^2)}(K')$ is a Seifert fiber space whose base orbifold is S^2 with three cone points of orders $m, n, |p - qmnw^2|$. Now that we have $\{r, s, p - qrs\} = \{m, n, |p - qmnw^2|\}$ and $w > 1$, if $\{r, s\} = \{m, n\}$, then $S^3_{p/q}(T_{r,s}) \cong S^3_{p/(qw^2)}(T_{r,s})$, which is not possible. So we may assume $m = r$, $n = p - qrs$ and $|p - qmnw^2| = s$. We get

$$p - qr(p - qrs)w^2 = \pm s,$$

hence

$$p = \frac{sq^2r^2w^2 \mp s}{qrw^2 - 1} \le s\left(qr + \frac{qr + 1}{4qr - 1}\right) \le sqr + \frac{3}{7}s,$$

where we use the fact that $r \ge 2, w \ge 2$. This contradicts the assumption that $\frac{p}{q} > rs + \frac{3}{7}\max\{r, s\}$.

If K' is hyperbolic, $S^3_{p/(qw^2)}(K')$ is a nonhyperbolic surgery. Note that $g(K') \le g(K)$. The argument in Corollary 2.3 implies $p \le 30(r^2 - 1)(s^2 - 1)/67$. Hence we should have $q \ge 3$, then $qw^2 \ge 12$. It follows from [20] that $S^3_{p/(qw^2)}(K')$ is hyperbolic, a contradiction. □

Proof of Theorem 1.3 Suppose that for a knot $K \subset S^3$ we have $S^3_{p/q}(K) = S^3_{p/q}(T_{r,s})$ for some slope $\frac{p}{q} > 30(r^2 - 1)(s^2 - 1)/67$. Note that a knot $K \subset S^3$ is either a hyperbolic knot, or a torus knot, or a satellite knot. Corollary 2.3 implies that K is not hyperbolic. By [25], we may assume $T_{r,s} \ne T_{3,2}$, so $(r-1)(s-1) \ge 4$. So

$$\frac{p}{q} > \frac{30(r^2 - 1)(s^2 - 1)}{67} > (r + 1)(s + 1) > rs + \frac{3}{7}\max\{r, s\}.$$

Proposition 2.5 then implies that K is not satellite, so we can apply Proposition 2.4 to conclude that $K = T_{r,s}$. □

3 Preliminaries on Heegaard Floer homology

The rest of this paper is devoted to the special case of $T_{5,2}$. In order to study this case, we need to understand the Heegaard Floer homology in more detail. In this section, we will give the necessary background on Heegaard Floer homology.

3.1 Heegaard Floer homology and correction terms

Heegaard Floer homology, introduced by Ozsváth and Szabó [29], is an invariant for closed oriented 3–manifolds Y equipped with Spin^c structures $\mathfrak{s}$, taking the form of a collection of related homology groups denoted $\widehat{HF}(Y, \mathfrak{s})$, $HF^{\pm}(Y, \mathfrak{s})$ and $HF^{\infty}(Y, \mathfrak{s})$.

Remark For simplicity, throughout this paper we will use $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients for Heegaard Floer homology.

There is an absolute Maslov $\mathbb{Z}/2\mathbb{Z}$–grading on the Heegaard Floer homology groups. When $\mathfrak{s}$ is torsion, there is an absolute Maslov $\mathbb{Q}$–grading on $HF^+(Y,\mathfrak{s})$. Let $J\colon \mathrm{Spin}^c(Y) \to \mathrm{Spin}^c(Y)$ be the conjugation on $\mathrm{Spin}^c(Y)$, then

$$HF^+(Y,\mathfrak{s}) \cong HF^+(Y, J\mathfrak{s}) \tag{3-1}$$

as ($\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Q}$) graded groups.

There is a U–action on HF^+, and the isomorphism (3-1) respects the U–action.

For a rational homology three-sphere Y with Spin^c structure $\mathfrak{s}$, $HF^+(Y,\mathfrak{s})$ can be decomposed as the direct sum of two groups: the first group is the image of $HF^\infty(Y,\mathfrak{s})$ in $HF^+(Y,\mathfrak{s})$, which is isomorphic to $\mathcal{T}^+ = \mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$, supported in the even $\mathbb{Z}/2\mathbb{Z}$–grading, and its minimal absolute $\mathbb{Q}$–grading is an invariant of $(Y,\mathfrak{s})$, denoted by $d(Y,\mathfrak{s})$, the *correction term* (see Ozsváth and Szabó [26]); the second group is the quotient modulo the above image and is denoted by $HF_{\mathrm{red}}(Y,\mathfrak{s})$. Altogether, we have

$$HF^+(Y,\mathfrak{s}) = \mathcal{T}^+ \oplus HF_{\mathrm{red}}(Y,\mathfrak{s}).$$

The correction term satisfies

$$d(Y,\mathfrak{s}) = d(Y, J\mathfrak{s}), \quad d(-Y,\mathfrak{s}) = -d(Y,\mathfrak{s}). \tag{3-2}$$

Let $L(p,q)$ be the lens space obtained by $\frac{p}{q}$–surgery on the unknot. The correction terms for lens spaces can be computed inductively as follows:

$$\begin{aligned} d(S^3,0) &= 0,\\ d(-L(p,q),i) &= \frac{1}{4} - \frac{(2i+1-p-q)^2}{4pq} - d(-L(q,r),j), \end{aligned}$$

where $0 \le i < p+q$, r and j are the reductions modulo q of p and i, respectively.

3.2 The knot Floer complex

Given a nullhomologous knot $K \subset Y$, Ozsváth and Szabó [28] and Rasmussen [36] defined the knot Floer homology. For knots in S^3, the knot Floer homology is a finitely generated bigraded group

$$\widehat{HFK}(K) = \bigoplus_{d,s\in\mathbb{Z}} \widehat{HFK}_d(K,s),$$

where d is the Maslov grading and s is the Alexander grading. The Euler characteristic of the knot Floer homology is the Alexander polynomial. More precisely, suppose

$$\Delta_K(T)=\sum_{s\in\mathbb{Z}} a_s T^s \tag{3-3}$$

is the normalized Alexander polynomial of K, then

$$a_s=\sum_{d\in\mathbb{Z}}(-1)^d \dim \widehat{HFK}_d(K,s).$$

Knot Floer homology is closely related to the topology of knots. It detects the Seifert genus of a knot [27], and it determines whether the knot is fibered; see Ghiggini [14] and the first author [22].

More information is contained in the knot Floer chain complex

$$C=CFK^\infty(Y,K)=\bigoplus_{i,j\in\mathbb{Z}} C\{(i,j)\}.$$

The differential $\partial\colon C\to C$ satisfies

$$\partial^2=0,\quad \partial C\{(i_0,j_0)\}\subset C\{i\le i_0,\ j\le j_0\}.$$

Moreover, $H_*(C\{(i,j)\})\cong \widehat{HFK}_{*-2i}(Y,K,j-i)$, and there is a natural chain complex isomorphism $U\colon C\{(i,j)\}\to C\{(i-1,j-1)\}$ which decreases the Maslov grading by 2. By [36], we can always assume

$$C\{(i,j)\}\cong \widehat{HFK}(Y,K,j-i).$$

There are quotient chain complexes

$$A_k^+=C\{i\ge 0 \text{ or } j\ge k\},\quad k\in\mathbb{Z},$$

and $B^+=C\{i\ge 0\}\cong CF^+(Y)$. They satisfy that $H_*(A_k^+)\cong H_*(A_{-k}^+)$. As in Ozsváth and Szabó [31], there are chain maps

$$v_k,h_k\colon A_k^+\to B^+.$$

Here v_k is the natural vertical projection, and h_k is more or less a horizontal projection. Let $H^T(A_k^+)$ be the image of the induced map $H_*(C)\to H_*(A_k^+)$, then $H^T(A_k^+)\cong \mathcal{T}^+$. Let $H_{\mathrm{red}}(A_k^+)=H_*(A_k^+)/H^T(A_k^+)$.

When $Y=S^3$, $H_*(B^+)\cong\mathcal{T}^+$. The homogeneous map $(v_k)_*\colon H^T(A_k^+)\to H_*(B^+)$ is U–equivariant, so it is equal to U^{V_k} for some nonnegative integer V_k. Similarly, $(h_k)_*$ is equal to U^{H_k} for some nonnegative integer H_k. The numbers V_k,H_k satisfy that

$$V_k=H_{-k},\quad V_k\ge V_{k+1}\ge V_k-1. \tag{3-4}$$

Convention 3.1 The groups A_k^+ will be relatively $\mathbb{Z}$–graded groups. For our convenience, we choose an absolute grading on A_k^+, such that $\mathbf{1} \in H^T(A_k^+)$ has grading 0.

For any positive integer d, define

$$\mathcal{T}_d = \langle \mathbf{1}, U^{-1}, \dots, U^{1-d} \rangle \subset \mathcal{T}^+,$$
$$\mathcal{T}_0 = 0 \subset \mathcal{T}^+.$$

Suppose $K \subset S^3$ has Alexander polynomial (3-3), let

$$t_i(K) = \sum_{j=1}^{\infty} j a_{i+j}, \quad i = 0, 1, 2, \dots .$$

Then the coefficients a_s can be recovered by the formula

$$a_s = t_{s-1} - 2t_s + t_{s+1}, \quad s = 1, 2, \dots .$$

Lemma 3.2 *Suppose $K \subset S^3$.*

(1) *For any $k \geq 0$, we have*

$$\ker(v_k)_* \cong \mathcal{T}_{V_k} \oplus H_{\mathrm{red}}(A_k^+),$$
$$\chi(\ker(v_k)_*) = t_k(K).$$

(2) *Suppose $g = g(K)$ is the Seifert genus of K, then $\widehat{HFK}(K, g) \cong \ker(v_{g-1})_*$ and*

$$g - 1 = \max\{k \mid \ker(v_k)_* \neq 0\}. \tag{3-5}$$

Proof From the definition of V_k and $H_{\mathrm{red}}(A_k^+)$, it is clear that

$$\ker(v_k)_* \cong \mathcal{T}_{V_k} \oplus H_{\mathrm{red}}(A_k^+).$$

The short exact sequence of chain complexes

$$0 \longrightarrow C\{i < 0, j \geq k\} \longrightarrow A_k^+ \xrightarrow{v_k} B^+ \longrightarrow 0$$

induces an exact triangle between the homology groups. Since $(v_k)_*$ is always surjective, its kernel is the homology of $C\{i < 0, j \geq k\}$, whose Euler characteristic is t_k. In particular,

$$\ker(v_{g-1})_* \cong C\{(-1, g-1)\} \cong \widehat{HFK}(K, g) \neq 0$$

and $\ker(v_k)_* = 0$ when $k \geq g$. □

3.3 The rational surgery formula

The basic philosophy of knot Floer homology is, if we know all the information about it, then we can compute the Heegaard Floer homology of all the surgeries on K. Let us give more details about the surgery formula below, following [32].

Let

$$\mathbb{A}_i^+ = \bigoplus_{s\in\mathbb{Z}} \big(s, A^+_{\lfloor (i+ps)/q \rfloor}(K)\big), \quad \mathbb{B}_i^+ = \bigoplus_{s\in\mathbb{Z}} (s, B^+).$$

Define maps

$$v_{\lfloor (i+ps)/q \rfloor}\colon \big(s, A^+_{\lfloor (i+ps)/q \rfloor}(K)\big) \to (s, B^+),$$
$$h_{\lfloor (i+ps)/q \rfloor}\colon \big(s, A^+_{\lfloor (i+ps)/q \rfloor}(K)\big) \to (s+1, B^+).$$

Adding these up, we get a chain map

$$D^+_{i,p/q}\colon \mathbb{A}_i^+ \to \mathbb{B}_i^+,$$
$$D^+_{i,p/q}\{(s,a_s)\}_{s\in\mathbb{Z}} = \{(s,b_s)\}_{s\in\mathbb{Z}},$$

where

$$b_s = v_{\lfloor (i+ps)/q \rfloor}(a_s) + h_{\lfloor (i+p(s-1))/q \rfloor}(a_{s-1}).$$

In [32], when Y is an integer homology sphere there is an affine identification of $\mathrm{Spin}^c(Y_{p/q}(K))$ with $\mathbb{Z}/p\mathbb{Z}$, such that Theorem 3.3 below holds for each i between 0 and $p-1$. This identification can be made explicit by the procedure in [32, Sections 4 and 7]. For our purpose in this paper, we only need to know that such an identification exists. We will use this identification throughout this paper.

We first recall the rational surgery formula [32, Theorem 1.1].

Theorem 3.3 (Ozsváth and Szabó) *Suppose that Y is an integer homology sphere, $K\subset Y$ is a knot. Let $\mathbb{X}^+_{i,p/q}$ be the mapping cone of $D^+_{i,p/q}$, then there is a graded isomorphism of groups*

$$H_*(\mathbb{X}^+_{i,p/q}) \cong HF^+(Y_{p/q}(K), i).$$

The absolute $\mathbb{Q}$–grading on $\mathbb{X}^+_{i,p/q}$ can be determined as follows. We first require that $D^+_{i,p/q}$ drops the absolute $\mathbb{Q}$–grading by 1. Then an appropriate absolute $\mathbb{Q}$–grading on $\mathbb{B}_i^+$ will determine the absolute grading of the mapping cone. We choose the absolute $\mathbb{Q}$–grading on (s, B^+) such that the corresponding absolute grading on the mapping cone gives us the right grading on $HF^+(Y_{p/q}(\text{unknot}), i) = HF^+(Y\#L(p,q), i)$. The absolute $\mathbb{Z}/2\mathbb{Z}$–grading can also be determined in this way.

More concretely, when $Y=S^3$ and $p,q>0$, the computation of $HF^+(S^3_{p/q}(\text{unknot}),i)$ using Theorem 3.3 (see the first author and Wu [23]) shows that the absolute $\mathbb{Q}$–grading on $\mathbb{B}^+_i$ is determined as follows.

Let

$$s_i=\begin{cases}0, & \text{if } V_{\lfloor i/q\rfloor}\ge H_{\lfloor (i+p(-1))/q\rfloor},\\ -1, & \text{if } V_{\lfloor i/q\rfloor}< H_{\lfloor (i+p(-1))/q\rfloor}.\end{cases}$$

Then

$$\begin{aligned}\mathrm{gr}(s_i,\mathbf{1})&=d(L(p,q),i)-1,\\ \mathrm{gr}(s+1,\mathbf{1})&=\mathrm{gr}(s,\mathbf{1})+2\lfloor\tfrac{i+ps}{q}\rfloor\quad\text{for any } s\in\mathbb{Z}.\end{aligned}\tag{3-6}$$

When $Y=S^3$, we can use the above theorem to compute

$$HF^+(S^3_{p/q}(K),i)\cong\mathcal{T}^+\oplus HF_{\mathrm{red}}(S^3_{p/q}(K),i).$$

For the part that is isomorphic to $\mathcal{T}^+$, we only need to know its absolute $\mathbb{Q}$–grading, which is encoded in the correction term. We recall the following proposition from [23].

Proposition 3.4 *Suppose $K\subset S^3$, and $p,q>0$ are relatively prime integers. Then for any $0\le i\le p-1$ we have*

$$d(S^3_{p/q}(K),i)=d(L(p,q),i)-2\max\{V_{\lfloor i/q\rfloor},H_{\lfloor (i-p)/q\rfloor}\}.$$

Using (3-4) and $-\lfloor (i-p)/q\rfloor=\lfloor (p+q-1-i)/q\rfloor$, we have $H_{\lfloor (i-p)/q\rfloor}=V_{\lfloor (p+q-1-i)/q\rfloor}$. To simplify the notation, let

$$\delta_i=\delta_i(K)=\max\{V_{\lfloor i/q\rfloor},H_{\lfloor (i-p)/q\rfloor}\}=\max\{V_{\lfloor i/q\rfloor},V_{\lfloor (p+q-1-i)/q\rfloor}\}.$$

If $s_i=0$, then $\delta_i=V_{\lfloor i/q\rfloor}$. Since $i\ge 0$, it follows from (3-4) that

$$V_{\lfloor i/q\rfloor}\le V_0=H_0\le H_{\lfloor i/q\rfloor}.$$

If $s_i=-1$, then $\delta_i=H_{\lfloor (i-p)/q\rfloor}$. Since $i\le p-1$, we have

$$V_{\lfloor (i-p)/q\rfloor}\ge V_0=H_0\ge H_{\lfloor (i-p)/q\rfloor}.$$

In any case, δ_i can be written as

$$\delta_i=\min\{V_{\lfloor (i+ps_i)/q\rfloor},H_{\lfloor (i+ps_i)/q\rfloor}\}.\tag{3-7}$$

Let $(D_{i,p/q})^T_*$ be the restriction of $(D^+_{i,p/q})_*$ on

$$H^T(\mathbb{A}^+_i) = \bigoplus_{s\in\mathbb{Z}} \big(s, H^T(A^+_{\lfloor (i+ps)/q \rfloor}(K))\big).$$

When $Y = S^3$ and $p/q > 0$, then the map $(D_{i,p/q})^T_*\colon H^T(\mathbb{A}^+_i) \to H_*(\mathbb{B}_i)$ is onto [23, Lemma 2.8], and hence $(D^+_{i,p/q})_*\colon H_*(\mathbb{A}^+_i) \to H_*(\mathbb{B}_i)$ is surjective. Therefore, $HF^+(S^3_{p/q}(K), i)$ is isomorphic to $\ker(D^+_{i,p/q})_*$.

The reduced part $HF_{\mathrm{red}}(S^3_{p/q}(K), i)$ comes in two parts. One part is the contribution of

$$H_{\mathrm{red}}(\mathbb{A}^+_i) = \bigoplus_{s\in\mathbb{Z}} \big(s, H_{\mathrm{red}}(A^+_{\lfloor (i+ps)/q \rfloor}(K))\big).$$

The other part is a subgroup of $\ker(D_{i,p/q})^T_*$, which can be computed from the integers V_k, H_k.

Proposition 3.5 *Suppose that K is a knot in S^3, $p, q > 0$ are relatively prime integers. Under the identification in Theorem 3.3, the group*

$$H_{\mathrm{red}}(\mathbb{A}^+_i) \oplus \Big(\bigoplus_{s\in\mathbb{Z}} (s, \mathcal{T}_{\min\{V_{\lfloor (i+ps)/q \rfloor}, H_{\lfloor (i+ps)/q \rfloor}\}})\Big) \big/ (s_i, \mathcal{T}_{\delta_i}) \tag{3-8}$$

is identified with $HF_{\mathrm{red}}(S^3_{p/q}(K), i)$. Here it follows from (3-7) *that $(s_i, \mathcal{T}_{\delta_i})$ is a summand in the direct sum in* (3-8).

Proof Since the sequence $V_k = H_{-k}$ $(k \in \mathbb{Z})$ is nonincreasing, we have

$$\begin{aligned} H_{\lfloor (i+p(s-1))/q \rfloor} &\geq H_0 = V_0 \geq V_{\lfloor (i+ps)/q \rfloor} \text{if } s > 0, \\ H_{\lfloor (i+p(s-1))/q \rfloor} &\leq H_0 = V_0 \leq V_{\lfloor (i+ps)/q \rfloor} \text{if } s < 0. \end{aligned} \tag{3-9}$$

Given $\xi \in \mathcal{T}^+$, define

$$\rho(\xi) = \{(s, \xi_s)\}_{s\in\mathbb{Z}}$$

as follows:

$$\begin{aligned} &\text{If } V_{\lfloor i/q \rfloor} \geq H_{\lfloor (i+p(-1))/q \rfloor}, && \text{let } \xi_{-1} = U^{V_{\lfloor i/q \rfloor} - H_{\lfloor (i+p(-1))/q \rfloor}} \xi, \quad \xi_0 = \xi. \\ &\text{If } V_{\lfloor i/q \rfloor} < H_{\lfloor (i+p(-1))/q \rfloor}, && \text{let } \xi_{-1} = \xi, \quad \xi_0 = U^{H_{\lfloor (i+p(-1))/q \rfloor} - V_{\lfloor i/q \rfloor}} \xi. \end{aligned}$$

For other s, using (3-9), let

$$\xi_s = \begin{cases} U^{H_{\lfloor (i+p(s-1))/q \rfloor} - V_{\lfloor (i+ps)/q \rfloor}} \xi_{s-1}, & \text{if } s > 0, \\ U^{V_{\lfloor (i+p(s+1))/q \rfloor} - H_{\lfloor (i+ps)/q \rfloor}} \xi_{s+1}, & \text{if } s < -1. \end{cases}$$

As in the proof of [23, Proposition 1.6], ρ maps $\mathcal{T}^+$ injectively into $\ker(D_{i,p/q})^T_*$ and $U\rho(\mathbf{1}) = 0$. Hence $\rho(\mathcal{T}^+)$ is the part of $\ker(D_{i,p/q})^T_*$ which is isomorphic to $\mathcal{T}^+$.

Suppose $\eta = \{(s, \eta_s)\}_{s\in\mathbb{Z}} \in \ker(D_{i,p/q})_*^T$. Let $\zeta = \eta - \rho(\eta_{s_i}) \in \ker(D_{i,p/q})_*^T$, then $\zeta_{s_i} = 0$. Using (3-9), we can check $\zeta_s \in \mathcal{T}_{\min\{V_{\lfloor (i+ps)/q\rfloor}, H_{\lfloor (i+ps)/q\rfloor}\}}$ for any $s \neq s_i$. So ζ is contained in the group (3-8). On the other hand, the group (3-8) is clearly in the kernel of $(D^+_{i,p/q})_*$, so our conclusion holds. □

Corollary 3.6 *Suppose $K \subset S^3$, and $p, q > 0$ are relatively prime integers. Then the rank of $HF_{\mathrm{red}}(S^3_{p/q}(K))$ is equal to*

$$q\left(\dim H_{\mathrm{red}}(A_0^+) + V_0 + 2\sum_{k=1}^{\infty}(\dim H_{\mathrm{red}}(A_k^+) + V_k)\right) - \sum_{i=0}^{p-1}\delta_i.$$

Proof By Proposition 3.5, the rank of $HF_{\mathrm{red}}(S^3_{p/q}(K))$ can be computed by

$$\begin{aligned}
&\sum_{i=0}^{p-1}\dim H_{\mathrm{red}}(\mathbb{A}_i^+) + \sum_{i=0}^{p-1}\sum_{s\in\mathbb{Z}}\min\{V_{\lfloor (i+ps)/q\rfloor}, H_{\lfloor (i+ps)/q\rfloor}\} - \sum_{i=0}^{p-1}\delta_i \\
&\qquad = \sum_{n\in\mathbb{Z}}(\dim H_{\mathrm{red}}(A^+_{\lfloor n/q\rfloor}) + \min\{V_{\lfloor n/q\rfloor}, V_{\lfloor (q-1-n)/q\rfloor}\}) - \sum_{i=0}^{p-1}\delta_i \\
&\qquad = q\left(\sum_{s\in\mathbb{Z}}\dim H_{\mathrm{red}}(A_s^+) + V_0 + 2\sum_{k=1}^{\infty}V_k\right) - \sum_{i=0}^{p-1}\delta_i.
\end{aligned}$$

Since $\dim H_{\mathrm{red}}(A_s^+(K)) = \dim H_{\mathrm{red}}(A_{-s}^+(K))$, we get our conclusion. □

4 The genus bound in the $T_{5,2}$ case

The purpose of this section is to show the following theorem.

Theorem 4.1 *Suppose $K \subset S^3$, $\frac{p}{q} \in \mathbb{Q}$, and $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$. Then one of the following two cases happens:*

(1) *K is a genus $(n+1)$ fibered knot for some $n \geq 1$ with*

(4-1) $$\Delta_K(T) = (T^{n+1} + T^{-(n+1)}) - 2(T^n + T^{-n}) + (T^{n-1} + T^{1-n}) + (T + T^{-1}) - 1.$$

(2) *K is a genus 1 knot with $\Delta_K(T) = 3T - 5 + 3T^{-1}$.*

Moreover, if

$$\tfrac{p}{q} \in \{\tfrac{p}{q} > 1\} \cup \{\tfrac{p}{q} < -6, |q| \geq 2\},$$

then the number n in the first case must be 1.

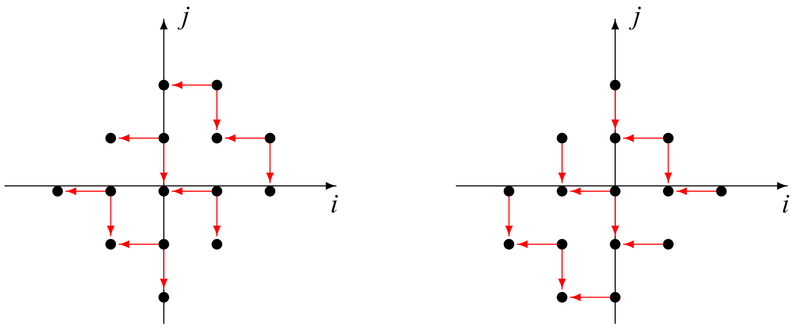

Figure 1: The knot Floer complexes of $T_{5,2}$ and $T_{5,-2}$, where here a black dot stands for a copy of $\mathbb{F}$, and the arrows indicate the differential

Suppose $S^3_0(K) = S^3_0(T_{5,2})$, then K is a genus 2 fibered knot by Gabai [9]. Clearly, it must have the same Alexander polynomial as $T_{5,2}$. From now on we consider the case $\frac{p}{q} \neq 0$.

Proposition 4.2 *Let K_0 be either $T_{5,2}$ or $T_{5,-2}$. Suppose $K \subset S^3$ is a knot with $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$ for $\frac{p}{q} > 0$, then*

$$t_s(K) = \dim H_{\mathrm{red}}(A^+_s(K)) + V_s(K) = \dim \ker(v_s)_*$$

for any $s \geq 0$.

Proof The knot Floer chain complexes of $T_{5,2}$ and $T_{5,-2}$ are illustrated in Figure 1. It is easy to check that all $H_*(A^+_s(K_0))$ are supported in the even $\mathbb{Z}/2\mathbb{Z}$–grading. So $HF_{\mathrm{red}}(S^3_{p/q}(K_0))$ is supported in the even grading by Proposition 3.5. Now it follows from Proposition 3.5 that $\ker(v_s(K))_*$ is supported in the even grading for all $s \geq 0$. By Lemma 3.2, $t_s(K) = \chi(\ker(v_s)_*) = \dim \ker(v_s)_* = \dim H_{\mathrm{red}}(A^+_s(K)) + V_s(K)$. $\square$

Proposition 4.3 *Conditions are as in Proposition 4.2. Then either K is a genus $(n+1)$ fibered knot for some $n \geq 1$, and $\Delta_K(T)$ is given by* (4-1)*, or K is a genus 1 knot with $\Delta_K(T) = 3T - 5 + 3T^{-1}$.*

Proof By Proposition 3.4,

$$\sum_{i=0}^{p-1} \delta_i(K) = \sum_{i=0}^{p-1} \delta_i(K_0). \tag{4-2}$$

By Corollary 3.6 and Proposition 4.2, we also have

$$t_0(K) + 2\sum_{k>0} t_k(K) = t_0(K_0) + 2\sum_{k>0} t_k(K_0) = 3, \tag{4-3}$$

and $t_k(K) \geq 0$, so $t_0(K)$ is equal to either 1 or 3.

If $t_0(K) = 1$, then by (4-3) there is exactly one $n > 0$ such that $\dim\ker(v_n)_* = t_n = 1$, and $\dim\ker(v_k)_* = t_k$ is 0 for $k > 0$ and $k \neq n$. By Lemma 3.2, $g(K) = n+1$ and (4-1) holds. Lemma 3.2 also implies that $\widehat{HFK}(K, n+1) \cong \ker(v_n)_* \cong \mathbb{F}$, so K is fibered [22].

If $t_0(K) = 3$, then $t_s(K) = 0$ for all $s > 0$. It follows from Lemma 3.2 and Proposition 4.2 that $g(K) = 1$. Clearly, in this case $\Delta_K(T) = 3T - 5 + 3T^{-1}$. □

4.1 The case when $K_0 = T_{5,2}$ and $\frac{p}{q} > 1$

In this subsection we will consider the case when $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$, where $K_0 = T_{5,2}$, $p > q > 0$, and $t_0(K) = t_n(K) = 1$. We can compute $V_0(K_0) = V_1(K_0) = 1$, and $V_k(K_0) = 0$ when $k \geq 2$. Using (3-4) and (4-2), we see that $V_0(K) > 0$. Since $V_k \leq t_k$, $V_0(K) = 1 \geq V_1(K)$ and $V_k(K) = 0$ when $k \geq 2$.

Now we have $V_k(K) \leq V_k(K_0)$ for any $k \geq 0$, so $\delta_i(K) \leq \delta_i(K_0)$ for all $0 \leq i \leq p-1$. In light of (4-2) we must have $\delta_i(K) = \delta_i(K_0)$ for all $0 \leq i \leq p-1$. If $V_1(K) = 0$, since $p > q$, we have $\delta_q(K) = \max\{V_1(K), V_{\lfloor (p-1)/q \rfloor}(K)\} = 0$, but $\delta_q(K_0) = \max\{V_1(T_{5,2}), V_{\lfloor (p-1)/q \rfloor}(T_{5,2})\} = 1$, a contradiction. This shows that $V_1(K) = 1$ hence $t_1(K) = 1$. By Proposition 4.3, we have $g(K) = 2$ and $\Delta_K(T) = (T^2 + T^{-2}) - (T + T^{-1}) + 1$.

4.2 The case when $K_0 = T_{5,-2}$ and $\frac{p}{q} > 6$

In this subsection we will consider the case when $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$, where $K_0 = T_{5,-2}$, $p > 6q > 0$, and $t_0(K) = t_n(K) = 1$.

For the knot K_0, we have $V_k(K_0) = 0$ when $k \geq 0$. It then follows from (4-2) that $V_k(K) = 0$ when $k \geq 0$. By Proposition 3.4, we have

$$d(S^3_{p/q}(K), i) = d(S^3_{p/q}(K_0), i) = d(L(p,q), i), \quad i = 0, 1, \ldots, p-1. \tag{4-4}$$

By Lemma 3.2, we see that $H_{\mathrm{red}}(A^+_k(K)) \cong \mathbb{F}$ when $k = 0, \pm n$, and $H_{\mathrm{red}}(A^+_k(K)) = 0$ for all other k. Following Convention 3.1, $H_{\mathrm{red}}(A^+_k(K))$ is absolutely graded. We assume $H_{\mathrm{red}}(A^+_0(K)) \cong \mathbb{F}_{(d_0)}$, $H_{\mathrm{red}}(A^+_{\pm n}(K)) \cong \mathbb{F}_{(d_n)}$, where $\mathbb{F}_{(d)}$ means a copy of $\mathbb{F}$ supported in grading d.

By Proposition 3.5,

$$HF_{\mathrm{red}}(S^3_{p/q}(K), i) \cong H_{\mathrm{red}}(\mathbb{A}^+_i)$$

for any $i \in \mathbb{Z}/p\mathbb{Z}$.

Lemma 4.4 (1) *Suppose that $\frac{p}{2} > q \geq 2$, and let $N_q = \{0, 1, \dots, q-1\}$. If $\phi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is an affine isomorphism such that $\phi(N_q) = N_q$, then either ϕ is the identity or $\phi(i) \equiv q - 1 - i \pmod{p}$.*

(2) *Suppose that $\frac{p}{6} > q \geq 3$, and let*

$$N_q^- = \{p-q, \dots, p-1\}, \quad N_q^+ = \{q, q+1, \dots, 2q-1\}.$$

If $\phi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is an affine isomorphism such that

$$\phi(N_q) \subset N_q^- \cup N_q^+,$$

then either $\phi(N_q) = N_q^-$ or $\phi(N_q) = N_q^+$.

Proof (1) The result is obvious if $q = 2$, now we assume $q > 2$. Suppose that $\phi(i) \equiv ai + b \pmod{p}$. For any $i \in N_q$, let $D_i = \{j - i \mid j \in N_q,\ j \neq i\} \subset \mathbb{Z}/p\mathbb{Z}$. Then the number a satisfies that a is contained in all but exactly one D_i. Noticing that $D_0 \cap D_{q-1} = \varnothing$, so a must be contained in $D_1 \cap D_2 \cap \dots \cap D_{q-2} = \{-1, 1\}$. If $a = 1$, then $\phi = \mathrm{id}$. If $a = -1$, then $\phi(i) \equiv q - 1 - i \pmod{p}$.

(2) Suppose there exists such an affine isomorphism $\phi(i) \equiv ai + b \pmod{p}$ for some integers a, b. Without loss of generality, we can suppose $0 < a \leq \frac{p}{2}$.

Let $D^{\pm} \subset \mathbb{Z}/p\mathbb{Z}$ be the set of the differences between any element in $N_q^{\pm}$ and any element in $N_q^{\mp}$, and let $D \subset \mathbb{Z}/p\mathbb{Z}$ be the set of the differences between any two elements in $N_q^{\pm}$. Then

$$D^+ = \{q+1, q+2, \dots, 3q-1\},$$
$$D^- = \{p-3q+1, \dots, p-q-1\},$$
$$D = \{1, 2, \dots, q-1\} \cup \{p-q+1, \dots, p-1\}.$$

If $a \leq q$, then $a \notin D^+ \cup D^-$. So $\phi(N_q)$ must be either $N^-(q)$ or $N^+(q)$.

If $a > q$, we also have $a \leq \frac{p}{2} < p - 3q$, so $a \notin D^- \cup D$. This means that if $0 \leq i \leq q-2$, then $\phi(i) \in N_q^-, \phi(i+1) \in N_q^+$, which forces $q \leq 2$, a contradiction. □

Given a rational homology sphere Y and a Spin^c structure $\mathfrak{s}$, let $\widetilde{HF}_{\mathrm{red}}(Y, \mathfrak{s})$ be the group $HF_{\mathrm{red}}(Y, \mathfrak{s})$, with the absolute grading shifted down by $d(Y, \mathfrak{s})$.

Lemma 4.5 *Suppose $p > 3q > 0$, then the conjugation in $\mathrm{Spin}^c(S^3_{p/q}(K))$ is*

$$J(i) \equiv q - 1 - i \pmod{p}.$$

Proof When $p > 3q$, we can see that

$$\widetilde{HF}_{\mathrm{red}}(S^3_{p/q}(K_0), i) \cong \begin{cases} \mathbb{F}_{(0)} & \text{when } 0 \le i \le q-1, \\ \mathbb{F}_{(2)} & \text{when } q \le i \le 2q-1 \text{ or } p-q \le i \le p-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4-5}$$

The conjugation J on $\mathrm{Spin}^c(S^3_{p/q}(K_0))$ is an affine involution on $\mathbb{Z}/p\mathbb{Z}$. By (3-1), we have $J(N_q) = N_q$. Since $p > 2$, J is not the identity map. If $q \ge 2$, it follows from Lemma 4.4 that $J(i) \equiv q-1-i \pmod p$ for $S^3_{p/q}(K_0)$. If $q = 1$, by (4-5) we have

$$J(0) = 0, \quad J(1) = -1,$$

so we must have $J(i) \equiv -i \pmod p$. Since the identification of $\mathrm{Spin}^c(S^3_{p/q}(K))$ with $\mathbb{Z}/p\mathbb{Z}$ is purely homological, $J(i) \equiv q-1-i \pmod p$ should also be true for $S^3_{p/q}(K)$. □

In fact, the above lemma is true for any $p, q > 0$ and $\frac{p}{q}$ surgery on any knot in homology spheres. This can be proved by examining the proof of [26, Proposition 4.8].

Lemma 4.6 *Suppose that $\frac{p}{6} > q > 0$, and $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$. Then*

$$HF^+(S^3_{p/q}(K), i) \cong HF^+(S^3_{p/q}(K_0), i)$$

as $\mathbb{Q}$–graded groups for any $i \in \mathbb{Z}/p\mathbb{Z}$, and the isomorphism respects the U–action.

Proof Since $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$, there is an affine isomorphism $\phi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ such that

$$HF^+(S^3_{p/q}(K), i) \cong HF^+(S^3_{p/q}(K_0), \phi(i))$$

as $\mathbb{Q}$–graded $\mathbb{F}[U]$–modules for any $i \in \mathbb{Z}/p\mathbb{Z}$.

If $q = 1$, by (4-4) we have

$$d(S^3_{p/q}(K), i) = d(S^3_{p/q}(K_0), i) = d(L(p,1), i) = -\frac{1}{4} + \frac{(2i-p)^2}{4p}.$$

It is easy to check that the only affine isomorphisms $\phi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ satisfying that $d(S^3_{p/q}(K), i) = d(S^3_{p/q}(K_0), \phi(i))$ are $\phi(i) = i$ and $\phi(i) = J(i)$. Our conclusion holds by (3-1).

If $q = 2$,

$$d(L(p,2), i) = -\frac{1}{4} + \frac{(2i-p-1)^2}{8p} + \begin{cases} -\frac{1}{4} & i \text{ even,} \\ \frac{1}{4} & i \text{ odd.} \end{cases}$$

We can check that $d(L(p,2),i)$ attains its maximal value if and only if $i = 0,1$. If there is an affine isomorphism $\phi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ such that $d(L(p,2),i) = d(L(p,2),\phi(i))$, then ϕ must be either id or J. We get our conclusion as in the last paragraph.

Now consider the case when $q \geq 3$. The group $H_{\mathrm{red}}(A_0^+) \cong \mathbb{F}_{(d_0)}$ contributes to each $\widetilde{HF}_{\mathrm{red}}(S^3_{p/q}(K),i)$ when $i \in N_q$. Comparing (4-5), we must have

$$\widetilde{HF}_{\mathrm{red}}(S^3_{p/q}(K),i) \cong \mathbb{F}_{(d_0)},$$

and either $\phi(N_q) = N_q$ or $\phi(N_q) \subset N_q^+ \cup N_q^-$. If $\phi(N_q) = N_q$, then Lemma 4.4 implies that $\phi = \mathrm{id}$ or J, hence our conclusion holds. If $\phi(N_q) \subset N_q^+ \cup N_q^-$, then Lemma 4.4 implies that $\phi(N_q) = N_q^+$ or $\phi(N_q) = N_q^-$. However, as an isomorphism of Spin^c structures induced by a homeomorphism, ϕ must commute with J. Since $J(N_q^{\pm}) = N_q^{\mp}$ and $J(N_q) = N_q$, we get a contradiction. □

Proposition 4.7 *Suppose that $\frac{p}{6} > q \geq 2$, and $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$. Then $g(K) = 2$ and $\Delta_K(T) = (T^2 + T^{-2}) - (T + T^{-1}) + 1$.*

Proof The group $H_{\mathrm{red}}(A_k^+)$ is nontrivial only for $k = 0, \pm n$, and the group $H_{\mathrm{red}}(A_0^+)$ contributes to $H_{\mathrm{red}}(\mathbb{A}_i^+)$ for each $i \in N_q$. The group $H_{\mathrm{red}}(A_n^+)$ contributes to $H_{\mathrm{red}}(\mathbb{A}_i^+)$ if and only if there exists an $s \in \mathbb{Z}$ such that $\lfloor \frac{i+ps}{q} \rfloor = n$ for some $s \in \mathbb{Z}$. For any $j \in \{0,1,\dots,q-1\}$, let $\mathfrak{i}(j) \in \{0,1,\dots,p-1\}$ satisfy that $\mathfrak{i}(j) + ps = nq + j$ for some $s \in \mathbb{Z}$. Then $\mathfrak{i}(0), \mathfrak{i}(1), \dots, \mathfrak{i}(q-1)$ are consecutive numbers in $\mathbb{Z}/p\mathbb{Z}$. By Lemma 4.6 and (4-5), they should be either the numbers in N_q^+ or the numbers in N_q^-.

If these numbers are in N_q^+, then $\mathfrak{i}(j) = q + j$, and there exists a nonnegative integer m such that $n = mp + 1$. Then $H_{\mathrm{red}}(\mathbb{A}^+_{\mathfrak{i}(j)})$ is supported in $(mq, H_{\mathrm{red}}(A^+_{mp+1}))$, and its absolute grading (see (3-6)) is given by

$$d_n + 1 + \mathrm{gr}(mq, \mathbf{1}) = d_n + d(L(p,q), q+j) + \sum_{s=0}^{mq-1} 2\left\lfloor \frac{q+j+ps}{q} \right\rfloor,$$

where $(mq, \mathbf{1})$ is the lowest element in $(mq, H_*(B^+)) \subset H_*(\mathbb{B}^+_{p/q,i})$. Comparing with (4-5), we have

$$2 = d_n + \sum_{s=0}^{mq-1} 2\left\lfloor \frac{q+j+ps}{q} \right\rfloor \quad \text{for any } j \in \{0,1,\dots,q-1\}.$$

This is impossible if $m \geq 1, q \geq 2$. In fact, there exists $\overline{s} \in \{0, 1, \ldots, q-1\}$ such that $q \mid (q+1+p\overline{s})$, which implies that

$$\sum_{s=0}^{mq-1} 2\left\lfloor \frac{q+ps}{q} \right\rfloor < \sum_{s=0}^{mq-1} 2\left\lfloor \frac{q+1+ps}{q} \right\rfloor \quad \text{when } m \geq 1, q \geq 2.$$

So if $q \geq 2$ we must have $m = 0$, which implies $g(K) = n+1 = mp+2 = 2$.

If these numbers are in N_q^-, then $\mathrm{i}(j) = p-q+j$, and there exists a nonnegative integer m such that $n = mp-1$. We can get a contradiction as before. □

4.3 Proof of Theorem 4.1

If $p/q > 0$, let $K_0 = T_{5,2}$. If $p/q < 0$, let $K_0 = T_{5,-2}$. Then the $-p/q$ surgery on the mirror of K is homeomorphic to $S^3_{-p/q}(K_0)$ via an orientation preserving homeomorphism. So we may always assume $p/q > 0$ and $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$ for $K_0 = T_{5,2}$ or $T_{5,-2}$.

If $t_0(K) = 3$, the proof of Proposition 4.3 shows that $g(K) = 1$ and $\Delta_K(T) = 3T - 5 + 3T^{-1}$. If $t_0(K) = 1$, then $g(K) = n+1$ and $\Delta_K(T)$ is given by (4-1).

If $t_0(K) = 1$, $K_0 = T_{5,2}$ and $\frac{p}{q} > 1$, then the result in Section 4.1 shows that K is a genus 2 fibered knot with $\Delta_K(T) = (T^2 + T^{-2}) - (T + T^{-1}) + 1$.

If $t_0(K) = 1$, $K_0 = T_{5,-2}$ and $\frac{p}{q} > 6$, then Proposition 4.7 implies K is a genus 2 fibered knot with $\Delta_K(T) = (T^2 + T^{-2}) - (T + T^{-1}) + 1$ unless $|q| = 1$.

This finishes the proof of Theorem 4.1.

Remark We have the following addendum to Theorem 4.1:

(a) If p is even, then case (2) of Theorem 4.1 cannot happen and in case (1) of Theorem 4.1, the number n must be odd.

(b) If p is divisible by 3, then case (2) cannot happen and in case (1), the number n is not divisible by 3.

Proof For a knot K in S^3, let M_K be its exterior and $\{\mu, \lambda\}$ be the standard meridian–longitude basis on ∂M_K. For an integer $j > 1$, let $f_j \colon \widetilde{M}^j_K \to M_K$ be the unique j–fold free cyclic covering. On $\partial \widetilde{M}^j_K$ we choose the basis $\{\widetilde{\mu}_j, \widetilde{\lambda}_j\}$ such that $f_j(\widetilde{\mu}_j) = \mu^j$ and $f(\widetilde{\lambda}_j) = \lambda$. We also use $M_K(p/q)$ to denote $S^3_{p/q}(K)$, and use $\widetilde{M}^j_K(p/q)$ to denote the Dehn filling of $\widetilde{M}^j_K$ with slope p/q with respect to the basis $\{\widetilde{\mu}_j, \widetilde{\lambda}_j\}$. Note that $\widetilde{M}^j_K(1/0)$ is the unique j–fold cyclic branched cover of S^3

branched over K. By Burde and Zieschang [6, Theorem 8.21], if no root of $\Delta_K(T)$ is a j^{th} root of unity, then the order of the first homology of $\widetilde{M}_K^j(1/0)$ is

$$|H_1(\widetilde{M}_K^j(1/0))| = \left|\prod_{i=1}^{j} \Delta(\xi_i)\right|,$$

where $\xi_1, \dots, \xi_j$ are the j roots of $x^j = 1$, in which case it follows from [6, Proposition 8.19] that

$$|H_1(\widetilde{M}_K^j(p/q))| = |\mathbb{Z}/p\mathbb{Z} \oplus H_1(\widetilde{M}_K^j(1/0))| = |p| \cdot \left|\prod_{i=1}^{j} \Delta(\xi_i)\right|.$$

(a) Suppose p is even. Let $\widetilde{p} = p/2$. Then $\widetilde{M}_K^2(\widetilde{p}/q)$ is the unique free double cover of $M_K(p/q)$. If $\Delta_K(T) = 3(T + T^{-1}) - 5$ or if

$$\Delta_K(T) = (T^{(n+1)} + T^{-(n+1)}) - 2(T^n + T^{-n}) + (T^{(n-1)} + T^{-(n-1)}) + (T + T^{-1}) - 1$$

for n even, then we have that $|H_1(\widetilde{M}_K^2(\widetilde{p}/q))| = |\widetilde{p}||\Delta_K(-1)| = 11|\widetilde{p}|$. But we have $|H_1(\widetilde{M}_{T_{5,2}}^2(\widetilde{p}/q))| = 5|\widetilde{p}|$. Hence case (2) cannot happen and n must be odd.

(b) Suppose p is divisible by 3. The argument is similar. Let $\widetilde{p} = p/3$. Then $\widetilde{M}_K^3(\widetilde{p}/q)$ is the unique free 3–fold cyclic cover of $M_K(p/q)$. If we have that $\Delta_K(T) = 3(T + T^{-1}) - 5$ or

$$\Delta_K(T) = (T^{(n+1)} + T^{-(n+1)}) - 2(T^n + T^{-n}) + (T^{(n-1)} + T^{-(n-1)}) + (T + T^{-1}) - 1$$

for n divisible by 3, then $|H_1(\widetilde{M}_K^3(\widetilde{p}/q))| = 64|\widetilde{p}|$. But $|H_1(\widetilde{M}_{T_{5,2}}^3(\widetilde{p}/q))| = |\widetilde{p}|$. Hence case (2) cannot happen and n is not divisible by 3. □

5 Finishing the proof of Theorem 1.4

Proposition 5.1 *Suppose that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ and $|q| \geq 9$, then $K = T_{5,2}$.*

Proof Since $|q| \geq 9$, the result of [20] implies that K is not hyperbolic.

If K is a torus knot, then the computation of $\Delta_K(T)$ in Theorem 4.1 implies that $K = T_{5,2}$ or $T_{5,-2}$, and now it is easy to see that $K = T_{5,2}$.

If K is a satellite knot, as in the proof of Proposition 2.5, we may assume the companion knot K' is not a satellite knot. Since $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ does not contain any incompressible tori and $|q| \geq 9 > 1$, it follows from Gabai [10; 11] that K is a $(\pm a, b)$–cable of K' for some integers a, b with $a > 0, b > 1$, and $S^3_{p/q}(K) \cong S^3_{p/(qb^2)}(K')$. Again, by [20] K' is not hyperbolic, so K' is a torus knot.

Let us recall the formula for the Alexander polynomial of a satellite knot [6, Proposition 8.23]. If S is a satellite knot with pattern knot P and companion knot C and the winding number w of S in a regular neighborhood of C, then the Alexander polynomials of S, C and P satisfy the relation

$$\Delta_S(T) = \Delta_C(T^w) \cdot \Delta_P(T). \tag{5-1}$$

In our present case, we see immediately that $\Delta_K(T)$ is monic since both the pattern knot and the companion knot of K have monic Alexander polynomials. So $\Delta_K(T)$ is given by (4-1). Now if $a = 1$, then the pattern knot is trivial, so $\Delta_K(T) = \Delta_{K'}(T^b)$. The right hand side of (4-1) is not of this form, so this case does not happen.

Now we have $a > 1$, and $K' = T_{\pm c,d}$ for $c, d > 1$. The pattern knot is $T_{\pm a,b}$, whose Alexander polynomial has the form

$$T^{g_{a,b}} - T^{g_{a,b}-1} + \text{lower-order terms}, \quad \text{where } g_{a,b} = \tfrac{(a-1)(b-1)}{2}.$$

Similarly,

$$\Delta_{K'}(T^b) = T^{bg_{c,d}} - T^{bg_{c,d}-b} + \text{lower-order terms}.$$

Hence by (5-1)

$$\Delta_K(T) = T^{bg_{c,d}+g_{a,b}} - T^{bg_{c,d}+g_{a,b}-1} + \text{lower-order terms},$$

which could be equal to the right hand side of (4-1) only when $n = 1$. However, in this case the degrees of the highest terms of the two polynomials do not match, so this case does not happen. □

Lemma 5.2 *Suppose that K is a satellite knot or a torus knot, $g(K) \le 2$, K is fibered when $g(K) = 2$, and $S^3_{p/q}(K)$ is homeomorphic to $S^3_{p/q}(T_{5,2})$ (not necessarily orientation preserving) for a nontrivial slope $p/q \ne 0$. Then $K = T_{5,2}$.*

Proof If K is a torus knot, then from $g(K) \le 2$ we know that K is one of $T_{5,\pm 2}$ or $T_{3,\pm 2}$. Now it is easy to see that $K = T_{5,2}$.

So suppose that K is a satellite knot, with companion knot C, pattern knot P and winding number w recalled as above. Since $S^3_{p/q}(K)$ does not contain any incompressible tori, it follows from [10] and Culler, Gordon, Luecke and Shalen [38] that $w > 1$ and P is a w–braid in the solid torus.

By [6, Proposition 2.10], $g(K) \ge wg(C) + g(P) \ge w > 1$. So K is fibered of genus 2, $g(C) = 1$, $g(P) = 0$, $w = 2$.

The companion knot C must also be fibered (see, eg Hirasawa, Murasugi and Silver [17]). Therefore C is either the trefoil knot or the figure 8 knot, and K is a

$(\pm 1, 2)$ cable on C. It follows from Gordon [15, Lemma 7.2 and Corollary 7.3] that either $p/q = \pm 2$ and $S^3_{\pm 2}(K) = S^3_{\pm 1/2}(C) \# L(2,1)$ or $p/q = (\pm 2q + \varepsilon)/q$ and $S^3_{(\pm 2q+\varepsilon)/q}(K) = S^3_{(\pm 2q+\varepsilon)/(4q)}(C)$, where ε is 1 or -1. It follows that C cannot be the figure 8 knot since all nonintegral surgery on the figure 8 knot yields hyperbolic manifolds. So C is the trefoil knot. As $S^3_{\pm 1/2}(C)$ can never be a lens space of order 5, we have $p/q = (\pm 2q + \varepsilon)/q$ and

$$S^3_{(\pm 2q+\varepsilon)/q}(K) = S^3_{(\pm 2q+\varepsilon)/(4q)}(C).$$

Now $S^3_{(\pm 2q+\varepsilon)/q}(T_{5,2})$ is Seifert fibered over $S^2(2, 5, |\pm 2q + \varepsilon - 10q|)$ while $S^3_{(\pm 2q+\varepsilon)/(4q)}(C)$ is Seifert fibered over $S^2(2, 3, |\pm 2q + \varepsilon - 24q|)$ or $S^2(2, 3, |\pm 2q + \varepsilon + 24q|)$ depending on whether C is right-hand or left-hand trefoil. Hence $|\pm 2q + \varepsilon - 10q| = 3$ and $|\pm 2q + \varepsilon - 24q| = 5$ (or $|\pm 2q + \varepsilon + 24q| = 5$), which is not possible. □

Corollary 5.3 *Suppose that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ with $|p| \geq 33$ and $g(K) \leq 2$, then $K = T_{5,2}$.*

Proof By Lemma 2.2, K is not a hyperbolic knot. Theorem 4.1 implies that K is fibered if $g(K) = 2$. Our conclusion follows from Lemma 5.2. □

Lemma 5.4 *Suppose that $K \subset S^3$ is a hyperbolic fibered knot. Then $S^3_{p/q}(K)$ is hyperbolic if $|q| \geq 3$ and $1 \leq |p| \leq 2|q| - 3$.*

Proof By Gabai and Oertel [13, Theorem 5.3], there is an essential lamination in the complement of K with a degenerate slope γ_0 such that γ_0 is either the trivial slope or an integer slope of K. Also by Gabai [12, Theorem 8.8] (or Roberts [37, Corollary 7.2]), if γ_0 is an integer slope, then $|\gamma_0| \geq 2$. Furthermore by [41, Theorem 2.5] combined with the geometrization theorem of Perelman, $S^3_{p/q}(K)$ is hyperbolic if $\Delta(p/q, \gamma_0) > 2$. Hence if $\gamma_0 = 1/0$, then $S^3_{p/q}(K)$ is hyperbolic for $|q| \geq 3$. So we may assume that γ_0 is an integer with $|\gamma_0| \geq 2$. Now $\Delta(p/q, \gamma_0) = |p - q\gamma_0| \geq |q\gamma_0| - |p| \geq 3$ by our condition on p/q and thus $S^3_{p/q}(K)$ is hyperbolic. □

Corollary 5.5 *Suppose that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ with $|q| \geq 3$ and $2 \leq |p| \leq 2|q| - 3$, then $K = T_{5,2}$.*

Proof By Theorem 4.1, K is either a fibered knot or a genus one knot. Note that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ is Seifert fibered over S^2 with three singular fibers. So if K is a genus one hyperbolic knot, then it follows from Boyer, Gordon and the second author [3, Theorem 1.5] that $|p| \leq 3$, and by the Remark after the proof of Theorem 4.1,

$|p| \neq 2$ or 3. Hence by our assumption on p/q, K is not a genus one hyperbolic knot. So by Lemma 5.2, we may assume that K is not a genus one knot. Hence K is a fibered knot. Now the proof proceeds similarly to that of Proposition 5.1, using Lemma 5.4 instead of [20]. We only need to note that if K is a fibered satellite knot, then any companion knot of K is also fibered. □

Proof of Theorem 1.4 Suppose that $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ for some slope p/q in

$$\left\{\tfrac{p}{q} > 1, |p| \geq 33\right\} \cup \left\{\tfrac{p}{q} < -6, |p| \geq 33, |q| \geq 2\right\} \cup \left\{\tfrac{p}{q}, |q| \geq 9\right\}$$
$$\cup \left\{\tfrac{p}{q}, |q| \geq 3, 2 \leq |p| \leq 2|q| - 3\right\} \cup \left\{9, 10, 11, \tfrac{19}{2}, \tfrac{21}{2}, \tfrac{28}{3}, \tfrac{29}{3}, \tfrac{31}{3}, \tfrac{32}{3}\right\}.$$

If $\frac{p}{q} > 1$ and $|p| \geq 33$, by Theorem 4.1 we know $g(K) \leq 2$. Now by Corollary 5.3 we have $K = T_{5,2}$.

If $\frac{p}{q} < -6$, $|q| \geq 2$ and $|p| \geq 33$, the argument is as in the preceding case.

If $|q| \geq 9$, then $K = T_{5,2}$ by Proposition 5.1.

If $|q| \geq 3$ and $2 \leq |p| \leq 2|q| - 3$, then $K = T_{5,2}$ by Corollary 5.5.

If $\frac{p}{q} = 9$ or 11, then $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ is a lens space of order 9 or 11 respectively. By [2, Theorem 1.6], K is not a hyperbolic knot. As K is either a genus two fibered knot or a genus one knot by Theorem 4.1, we must have $K = T_{5,2}$ by Lemma 5.2.

If $\frac{p}{q} = 10$, then $K = T_{5,2}$ as remarked in Section 1 (just before Theorem 1.3).

If $\frac{p}{q} = \frac{19}{2}, \frac{21}{2}, \frac{29}{3}$, or $\frac{31}{3}$, then $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ is a lens space. By the cyclic surgery theorem of Culler, Gordon, Luecke and Shalen [8], K is not a hyperbolic knot. Hence it follows from Theorem 4.1 and Lemma 5.2 that $K = T_{5,2}$.

If $\frac{p}{q} = \frac{28}{3}$ or $\frac{32}{3}$, then $S^3_{p/q}(K) \cong S^3_{p/q}(T_{5,2})$ is a spherical space form. By Boyer and the second author [5, Corollary 1.3], K is not a hyperbolic knot. Again by Theorem 4.1 and Lemma 5.2 we have $K = T_{5,2}$. □

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